Soft Theorem and its Classical Limit

Ashoke Sen

Harish-Chandra Research Institute, Allahabad, India

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Units: $\hbar = \mathbf{c} = \mathbf{M_{pl}} = 1$

In this talk we focus on soft graviton theorem

– however similar results exist for soft photon theorems as well.

ArXiV references for this talk:

1706.00759, 1707.06803, 1801.07719, 1804.09193, 1806.01872

What is soft graviton theorem?

Take a general coordinate invariant quantum theory of gravity coupled to matter fields

Consider an S-matrix element involving

– arbitrary number N of external particles of finite momentum $p_1, \cdots p_N$

– M external gravitons carrying small momentum $k_1, \cdots k_M$.

Soft graviton theorem: Expansion of this amplitude in power series in $k_1, \cdots k_M$ in terms of the amplitude without the soft gravitons.

Plan

1. Results in quantum theory: D>4

2. Classical limit: D>4

3. D=4

Results in quantum theory

– many explicit results in field theory and string theory

Weinberg; . . .

White; Cachazo, Strominger; Bern, Davies, Di Vecchia, Nohle; Elvang, Jones, Naculich; . . .

Klose, McLoughlin, Nandan, Plefka, Travaglini; Saha

Bianchi, Guerrieri; Di Vecchia, Marotta, Mojaza; . . .

– general arguments based on asymptotic symmetry (mostly in D=4, some in even D)

Strominger; He, Lysov, Mitra, Strominger; Strominger, Zhiboedov; Campiglia, Laddha; . . .

In D=4 there are also problems since the S-matrix is IR divergent

Bern, Davies, Nohle; Cachazo, Yuan; He, Kapec, Raclariu, Strominger

Under some assumptions one can give a completely general derivation of soft graviton theorem

A.S.; Laddha, A.S.; Chakrabarti, Kashyap, Sahoo, A.S., Verma

– generic theory

– generic number of dimensions

– arbitrary mass and spin of elementary / composite finite momentum external states

e.g. gravitons, photons, electrons, massive string states, nuclei, molecules, planets, stars, black holes

Main ingredient: Graviton coupling with zero or one derivative is fixed completely by general coordinate invariance.

Assumptions

1. The scattering is described by a general coordinate invariant one particle irreducible (1PI) effective action

– tree amplitudes computed from this give the full quantum results

2. The vertices do not contribute powers of soft momentum in the denominator

– breaks down in D=4

In D=4 the results will be valid only at tree level

– to be partially rectified at the end

Note: In string theory we do not compute amplitudes from Feynman diagrams

– but we could, using string field theory

– can introduce 1PI effective action exactly as in an ordinary quantum field theory

The existence of such a 1PI effective action is sufficient for our analysis

We do not need its explicit form

Strategy

1. Assume a general form of the gauge invariant 1PI effective action of the theory

2. Expand the action in powers of all fields, including the metric fluctuations, around the extremum of the action

– assumed to have zero cosmological constant

3. Require the gauge fixing terms to be manifestly Lorentz invariant.

General form of the action + gauge fixing terms

$$\sum_{n\geq 2}\int\prod_{i=1}^{n}\frac{d^D p_i}{(2\pi)^D}\,(2\pi)^D\delta^{(D)}(p_1+\cdots+p_n)\,V^{(n)}_{\alpha_1\cdots\alpha_n}(p_1,\cdots p_n)\,\phi_{\alpha_1}(p_1)\cdots\phi_{\alpha_n}(p_n)$$

$\{\phi_\alpha\}$: set of all the fields (in momentum space)

$V^{(n)}$: fixed for a given theory

This action is Lorentz invariant but not general coordinate invariant since we have gauge fixed.

This action is used to compute vertices and propagators of finite energy external states but <u>not</u> of <u>soft gravitons</u>.

4. To calculate the coupling of the soft graviton $S_{\mu\nu}$ to the rest of the fields, we covariantize the gauge fixed action.

a. Replace the background metric $\eta_{\mu\nu}$ by $\eta_{\mu\nu} + 2\, S_{\mu\nu}$

b. Replace all space-time derivatives by covariant derivatives computed with the metric $\eta_{\mu\nu} + 2\, S_{\mu\nu}$

$\Rightarrow$ coupling of soft graviton determined from the 'hard vertices' $V^{(n)}$.

A technical point

We are normally familiar with covariantization in position space, e.g.

$$\partial_{\mathbf{a}} \Rightarrow \mathbf{E_a}{}^{\mu}(\partial_\mu + \mathbf{i}\,\omega_\mu^{\mathbf{bc}}\,\Sigma^{\mathbf{bc}})$$

$\mathbf{E_a}{}^{\mu} = \delta_{\mathbf{a}}^{\mu} - \mathbf{S_a}{}^{\mu}$**: inverse vielbein**

$\omega_\mu^{\mathbf{bc}}$**: spin connection ,** $\Sigma^{\mathbf{bc}}$**: Generator of spin**

We need to translate this into appropriate operations in momentum space e.g. by replacing ∂_μ **by** $\mathbf{p}_\mu$

Consider a functional

$$\int \mathbf{d^D p_1} \cdots \mathbf{d^D p_N}\, \phi_{\alpha_1}(\mathbf{p_1}) \cdots \phi_{\alpha_N}(\mathbf{p_N})$$
$$\delta^{(D)}(\mathbf{p_1} + \cdots \mathbf{p_N})\, \mathbf{F}^{\alpha_1 \cdots \alpha_N}(\mathbf{p_1}, \ldots \mathbf{p_N})$$

Covariantization produces an additional term

$$\int \mathbf{d^D k} \int \mathbf{d^D p_1} \cdots \mathbf{d^D p_N}\, \phi_{\alpha_1}(\mathbf{p_1}) \cdots \phi_{\alpha_N}(\mathbf{p_N})\, \delta^{(D)}(\mathbf{p_1} + \cdots \mathbf{p_N} + \mathbf{k})$$
$$\sum_{i=1}^{N} \left[-\delta_{\beta_i}^{\alpha_i} \mathbf{S}_\mu^{\ \nu}(\mathbf{k}) \mathbf{p}_{i\nu} \frac{\partial}{\partial \mathbf{p}_{i\mu}} + i\, \mathbf{k}^b \mathbf{S}_\mu^{\ a}(\mathbf{k}) (\Sigma_{ab})_{\beta_i}^{\ \alpha_i} \frac{\partial}{\partial \mathbf{p}_{i\mu}} \right.$$
$$\left. -\frac{1}{2} \delta_{\beta_i}^{\alpha_i} \left\{ \mathbf{k}_\mu \mathbf{S}_\nu^{\ \rho}(\mathbf{k}) + \mathbf{k}_\nu \varepsilon_\mu^{\ \rho} - \mathbf{k}^\rho \varepsilon_{\mu\nu} \right\} \mathbf{p}_{i\rho} \frac{\partial^2}{\partial \mathbf{p}_{i\mu} \partial \mathbf{p}_{i\nu}} \right]$$
$$\mathbf{F}^{\alpha_1 \cdots \alpha_{i-1} \beta_i \alpha_{i+1} \cdots \alpha_N}(\mathbf{p_1}, \ldots, \mathbf{p_N}) + \mathcal{O}(\mathbf{k}^\mu \mathbf{k}^\nu)\,.$$

This procedure misses terms involving Riemann tensor computed from the metric $\eta_{\mu\nu} + 2\,\mathsf{S}_{\mu\nu}$

– begins contributing at the subsubleading order

– must be added explicitly for subsubleading order calculations.

5. Once we have determined the action, we analyze various amplitudes by representing them as sum over Feynman diagrams.

We need to compute only tree amplitudes with this action since we begin with 1PI effective action.

Notations

We denote the amplitude without the soft graviton by

$$\epsilon_{1,\alpha_1}(p_1)\ldots\epsilon_{N,\alpha_N}(p_N)\,\Gamma^{\alpha_1\ldots\alpha_N}(p_1,\ldots,p_N)$$

ϵ_{i,α_i}: polarisation tensor of i-th external state

p_i: momentum of i-th external state, counted as positive if ingoing

$\Gamma^{\alpha_1\cdots\alpha_N}$ includes the $\delta^{(D)}(p_1+\cdots p_N)$ factor.

Final result for the amplitude with

– same set of finite energy external states

– a single soft graviton with polarization ε and momentum k

to subsubleading order in k:

$$\prod_{j=1}^{N} \epsilon_{j,\alpha_j}(p_j) \left[\left\{ S^{(0)} \Gamma \right\}^{\alpha_1 \ldots \alpha_N} + \left\{ S^{(1)} \Gamma \right\}^{\alpha_1 \ldots \alpha_N} + \left\{ S^{(2)} \Gamma \right\}^{\alpha_1 \ldots \alpha_N} \right]$$

$$\{S^{(0)}\Gamma\}^{\alpha_1\ldots\alpha_N} \equiv \sum_{i=1}^{N} (p_i \cdot k)^{-1}\, \varepsilon_{ab}\, p_i^a\, p_i^b\, \Gamma^{\alpha_1\ldots\alpha_N}$$

$$\left\{S^{(1)}\Gamma\right\}^{\alpha_1\ldots\alpha_N} \equiv \sum_{i=1}^{N} (p_i \cdot k)^{-1}\, \varepsilon_{ab}\, p_i^a\, k_c \left(p_i^b \frac{\partial}{\partial p_{ic}} - p_i^c \frac{\partial}{\partial p_{ib}}\right) \Gamma^{\alpha_1\cdots\alpha_N}$$
$$+ i \sum_{i=1}^{N} (p_i \cdot k)^{-1}\, \varepsilon_{ab}\, p_i^a\, k_c\, (\Sigma^{cb})_{\gamma}^{\ \alpha_i}\, \Gamma^{\alpha_1\cdots\alpha_{i-1}\gamma\alpha_{i+1}\cdots\alpha_N}$$

Σ^{cb}: spin angular momentum

Note: $S^{(0)}$ and $S^{(1)}$ do not depend on $V^{(n)}$

$$\left\{ S^{(2)}\Gamma \right\}^{\alpha_1\dots\alpha_N} = \frac{1}{2}\sum_{i=1}^{N} (p_i \cdot k)^{-1} \epsilon_{i,\alpha} \varepsilon_{ac} k_b k_d$$

$$\left[\left\{ p_i^b \frac{\partial}{\partial p_{ia}} - p_i^a \frac{\partial}{\partial p_{ib}} \right\} \delta_\beta^{\ \alpha_i} + i\,(\Sigma^{ab})_\beta^{\ \alpha_i} \right]$$

$$\left[\left\{ p_i^d \frac{\partial}{\partial p_{ic}} - p_i^c \frac{\partial}{\partial p_{id}} \right\} \delta_\gamma^{\ \beta} + i\,(\Sigma^{cd})_\gamma^{\ \beta} \right] \Gamma^{\alpha_1\cdots\alpha_{i-1}\gamma\alpha_{i+1}\cdots\alpha_N}$$

$$+\frac{1}{2}\left(\varepsilon^{ab}k^c k^d \ - \ \varepsilon^{ad}k^b k^c \ - \ \varepsilon^{bc}k^d k^a \ + \ \varepsilon^{cd}k^a k^b\right)$$

$$\sum_{i=1}^{N} (p_i \cdot k)^{-1} \left(M_{(i)}\right)^{\alpha_i}_{\gamma;acbd}(-p_i)\, \Gamma^{\alpha_1\cdots\alpha_{i-1}\gamma\alpha_{i+1}\cdots\alpha_N}$$

$M_{(i)}$: Theory dependent non-universal term

– agrees with all known results in QFT and string theory

Multiple soft gravitons to subleading order

Chakrabarti, Kashyap, Sahoo, A.S, Verma

Take the same set of finite energy particles and M soft gravitons with polarisation $\{\varepsilon_r\}$ and momenta $\{k_r\}$:

Result:

$$\left\{\prod_{i=1}^{N}\epsilon_{i,\alpha_i}(p_i)\right\}\Bigg[\left\{\prod_{r=1}^{M}S_r^{(0)}\right\}\Gamma^{\alpha_1\cdots\alpha_N}+\sum_{s=1}^{M}\left\{\prod_{\substack{r=1\\r\neq s}}^{M}S_r^{(0)}\right\}\left[S_s^{(1)}\Gamma\right]^{\alpha_1\cdots\alpha_N}$$

$$+\sum_{\substack{r,u=1\\r<u}}^{M}\left\{\prod_{\substack{s=1\\s\neq r,u}}^{M}S_s^{(0)}\right\}\left\{\sum_{j=1}^{N}\{p_j\cdot(k_r+k_u)\}^{-1}\,\mathcal{M}(p_j;\varepsilon_r,k_r,\varepsilon_u,k_u)\right\}\Gamma^{\alpha_1\cdots\alpha_N}\Bigg]$$

$S_r^{(0)}, S_r^{(1)}$: Soft factors defined earlier for r-th soft graviton

$\mathcal{M}$: ‘contact term’

$$
\begin{aligned}
&\mathcal{M}(p_i;\varepsilon_1,k_1,\varepsilon_2,k_2)\\
&= (p_i\cdot k_1)^{-1}(p_i\cdot k_2)^{-1}\Big\{-k_1\cdot k_2\; p_i\cdot\varepsilon_1\cdot p_i\; p_i\cdot\varepsilon_2\cdot p_i\\
&+2\,p_i\cdot k_2\; p_i\cdot\varepsilon_1\cdot p_i\; p_i\cdot\varepsilon_2\cdot k_1+2\,p_i\cdot k_1\; p_i\cdot\varepsilon_2\cdot p_i\; p_i\cdot\varepsilon_1\cdot k_2\\
&-2\,p_i\cdot k_1\; p_i\cdot k_2\; p_i\cdot\varepsilon_1\cdot\varepsilon_2\cdot p_i\Big\}\\
&+(k_1\cdot k_2)^{-1}\Big\{-(k_2\cdot\varepsilon_1\cdot\varepsilon_2\cdot p_i)(k_2\cdot p_i)-(k_1\cdot\varepsilon_2\cdot\varepsilon_1\cdot p_i)(k_1\cdot p_i)\\
&+(k_2\cdot\varepsilon_1\cdot\varepsilon_2\cdot p_i)(k_1\cdot p_i)+(k_1\cdot\varepsilon_2\cdot\varepsilon_1\cdot p_i)(k_2\cdot p_i)\\
&-\varepsilon_1^{\gamma\delta}\varepsilon_{2\gamma\delta}(k_1\cdot p_i)(k_2\cdot p_i)-2(p_i\cdot\varepsilon_1\cdot k_2)(p_i\cdot\varepsilon_2\cdot k_1)\\
&+(p_i\cdot\varepsilon_2\cdot p_i)(k_2\cdot\varepsilon_1\cdot k_2)+(p_i\cdot\varepsilon_1\cdot p_i)(k_1\cdot\varepsilon_2\cdot k_1)\Big\}\,.
\end{aligned}
$$

– agrees with results of explicit calculation for two soft gravitons and some finite energy gravitons

Klose, McLoughlin, Nandan, Plefka, Travaglini; Saha

Classical limit

Laddha, A.S.

(Also Weinberg; Strominger, Zhiboedov; Pasterski, Strominger, Zhiboedov; Pate, Raclariu, Strominger)

We take the limit in which

1. Energy of each finite energy external state becomes large (compared to M_{pl})

2. The total energy carried by the soft particles is small compared to the energies of the finite energy particles.*

*Finite energy particles: Those counted in $\sum_{i=1}^{N} \cdots$

In this limit the soft theorem simplifies in many ways

1. We can make the replacement

$$-i\left\{p_i^b \frac{\partial}{\partial p_{ia}} - p_i^a \frac{\partial}{\partial p_{ib}}\right\} \delta_\beta^{\ \alpha} + (\Sigma^{ab})_\beta^{\ \alpha} \quad \Rightarrow \quad J_i^{ab}\, \delta_\beta^{\ \alpha}$$

where J_i^{ab} is the classical angular momentum of the i-th external particle

2. The contact term $\mathcal{M}$ can be ignored compared to the other terms

– has more powers of p_i in the denominator than the non-contact terms.

In this limit the multiple soft theorem takes the form

$$\left\{\prod_{i=1}^{M} S_{gr}(\varepsilon_r, k_r)\right\} \Gamma^{\alpha_1\cdots\alpha_n} \qquad S_{gr} = S^{(0)} + S^{(1)} + S^{(2)}$$

$$S^{(0)} \equiv \sum_{i=1}^{N} (p_i \cdot k)^{-1}\, \varepsilon_{ab}\, p_i^a\, p_i^b$$

$$S^{(1)} = i \sum_{i=1}^{N} (p_i \cdot k)^{-1}\, \varepsilon_{ab}\, p_i^a\, k_c\, J_i^{cb}$$

$$S^{(2)} = -\frac{1}{2} \sum_{i=1}^{N} (p_i \cdot k)^{-1} \varepsilon_{ac} k_b k_d J_i^{ab}\, J_i^{cd} + \text{non-universal terms}$$

$\Rightarrow$ S_{gr} is large in the classical limit.

While working to subsubleading order we can work with large impact parameter so that the non-universal terms are small compared to the J J term

Amplitude: $\Gamma_{soft} \equiv \left\{ \prod_{r=1}^{M} S_{gr}(\epsilon_r, k_r) \right\} \Gamma$

Probability of producing M soft gravitons of

- **polarisation** ε**,**
- **energy between** ω **and** $\omega(1+\delta)$
- **within a small solid angle** Ω **around a unit vector** $\hat{n}$

$$\frac{1}{M!} \, |\Gamma_{soft}|^2 \times \left\{ \frac{1}{(2\pi)^{D-1}} \, \frac{1}{2\omega} \, \omega^{D-2} \, (\omega\,\delta) \, \Omega \right\}^M = |\Gamma|^2 \, A^M / M! \, ,$$

$$A \;\; \equiv \;\; |S_{gr}(\varepsilon, k)|^2 \, \frac{1}{(2\pi)^{D-1}} \, \frac{1}{2\omega} \, \omega^{D-2} \, (\omega\,\delta) \, \Omega \, , \quad k = -\omega(1, \hat{n})$$

Note: A is large in the classical limit

$$|\Gamma|^2 \, A^M / M!$$

is maximised at

$$\frac{\partial}{\partial M} \ln \left\{ |\Gamma|^2 \, A^M / M! \right\} = 0$$

Assuming that M is large,

$$\Rightarrow \frac{\partial}{\partial M} (M \ln A - M \ln M + M) = 0$$

$$\Rightarrow M = A$$

In the classical limit M is large since A is large

Probability distribution of M is sharply peaked

Note: the value of M does not change if we allow soft radiation in other bins.

$$\text{no. of gravitons} = A = \frac{1}{2^D \pi^{D-1}} |S_{gr}(\varepsilon, k)|^2 \omega^{D-2}\, \Omega\, \delta$$

Total energy radiated in this bin

$$A\, \omega = \frac{1}{2^D \pi^{D-1}} |S_{gr}(\varepsilon, k)|^2 \omega^{D-1}\, \Omega\, \delta$$

This can be related to the radiative part of the metric field

$\Rightarrow$ gives a prediction for the low frequency radiative part of the metric field during classical scattering

(up to overall phase and gauge transformation)

Define

$$\tilde{\mathbf{h}}_{\alpha\beta}(\omega,\vec{\mathbf{x}}) \equiv \int \frac{\mathbf{dt}}{\mathbf{2}\pi} \mathbf{e}^{\mathbf{i}\omega\mathbf{t}} (\mathbf{g}_{\alpha\beta} - \eta_{\alpha\beta})/\mathbf{2},$$

$$\tilde{\mathbf{e}}_{\alpha\beta}(\omega,\vec{\mathbf{x}}) \equiv \tilde{\mathbf{h}}_{\alpha\beta}(\omega,\vec{\mathbf{x}}) - \frac{\mathbf{1}}{\mathbf{2}}\,\eta_{\alpha\beta}\,\tilde{\mathbf{h}}_{\gamma}^{\ \gamma}(\omega,\vec{\mathbf{x}})$$

$$\mathbf{R} \equiv |\vec{\mathbf{x}}|, \quad \hat{\mathbf{n}} = \vec{\mathbf{x}}/|\vec{\mathbf{x}}|\,,$$

$$\mathcal{N} \equiv \mathbf{e}^{\mathbf{i}\omega\mathbf{R}} \left(\frac{\omega}{\mathbf{2}\pi\mathbf{iR}}\right)^{\mathbf{(D-2)/2}} \frac{\mathbf{1}}{\mathbf{2}\omega}, \quad \mathbf{k} \equiv -\omega(\mathbf{1},\hat{\mathbf{n}})\,.$$

Then

$$\varepsilon^{\alpha\beta}\,\tilde{\mathbf{e}}_{\alpha\beta}(\omega,\vec{\mathbf{x}}) = \mathcal{N}\,\mathbf{S}_{\rm gr}(\varepsilon,\mathbf{k})\,,$$

Note: $\mathbf{S_{gr}}$ is determined in terms of initial and final particle trajectories and spin

– does not require knowledge of the forces operating on the systems during the scattering.

Test:

1. Consider a classical scattering satisfying the desired conditions

2. Calculate radiative part of the gravitational field

3. Compare with the prediction of the soft theorem

– need to compute $J_i^{\mu\nu}$

If in the far past / future the object has trajectory

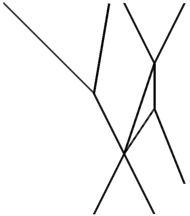
$$x^\mu = c_i^\mu + m_i^{-1}\, p_i^\mu \tau$$

then

$$J_i^{\mu\nu} = (x_i^\mu p_i^\nu - x_i^\nu p_i^\mu) + \text{spin} = (c_i^\mu p_i^\nu - c_i^\nu p_i^\mu) + \text{spin}$$

Examples studied:

1. Multiple inelastic scattering:



The D-momentum is assumed to be conserved at each vertex.

Total gravitational radiation = sum of contributions from each leg

– perfect agreement with soft theorem to subsubleading order.

2. Coulomb scattering of a light particle by a heavy particle with impact parameter $>>$ the Schwarzschild radius of the heavy particle.

a. Assume that the scattering is dominated by the Coulomb interaction.

b. Compute gravitational radiation due to the energy momentum tensor of the particles and the electromagnetic field.

Result agrees perfectly with soft graviton theorem to subsubleading order.

D=4

The S-matrix suffers from IR divergence, making soft factor ill-defined.

However we can still use the radiative part of the gravitational field during classical scattering to define soft factor.

Naive guess: Soft factor defined this way is still given by the same formulæ:

$$S^{(0)} \equiv \sum_{i=1}^{N} (p_i \cdot k)^{-1} \, \varepsilon_{ab} \, p_i^a \, p_i^b$$

$$S^{(1)} = i \sum_{i=1}^{N} (p_i \cdot k)^{-1} \, \varepsilon_{ab} \, p_i^a \, k_c \, J_i^{cb}$$

$$S^{(2)} = -\frac{1}{2} \sum_{i=1}^{N} (p_i \cdot k)^{-1} \varepsilon_{ac} k_b k_d J_i^{ab} \, J_i^{cd} + \text{non-universal terms}$$

Due to long range force on the initial / final trajectories due to other particles, the trajectory of the i-th particle takes the form:

$$\mathbf{x}_\mathbf{i}^\mu = \mathbf{c}_\mathbf{i}^\mu + \mathbf{m}_\mathbf{i}^{-1}\,\mathbf{p}_\mathbf{i}^\mu\,\tau + \mathbf{b}_\mathbf{i}^\mu \ln|\tau|$$

for some constants $\mathbf{b}_\mathbf{i}^\mu$.

$$\mathbf{J}_\mathbf{i}^{\mu\nu} = (\mathbf{x}_\mathbf{i}^\mu \mathbf{p}_\mathbf{i}^\nu - \mathbf{x}_\mathbf{i}^\nu \mathbf{p}_\mathbf{i}^\mu) = (\mathbf{c}_\mathbf{i}^\mu \mathbf{p}_\mathbf{i}^\nu - \mathbf{c}_\mathbf{i}^\nu \mathbf{p}_\mathbf{i}^\mu) + (\mathbf{b}_\mathbf{i}^\mu \mathbf{p}_\mathbf{i}^\nu - \mathbf{b}_\mathbf{i}^\nu \mathbf{p}_\mathbf{i}^\mu) \ln|\tau|$$

Due to the $\ln|\tau|$ term, the soft factors do not have well defined $|\tau| \to \infty$ limit

Next guess: The soft expansion has a $\ln \omega^{-1}$ term at the subleading order, given by $S^{(1)}$ with $\ln|\tau|$ replaced by $\ln \omega^{-1}$.

$\omega \equiv k_0$

$$S^{(1)} = i \sum_{i=1}^{N} (p_i \cdot k)^{-1} \varepsilon_{\mu\nu} \, p_i^{\mu} \, k_{\rho} \, J_i^{\rho\nu}$$

$$= i \sum_{i=1}^{N} (p_i \cdot k)^{-1} \varepsilon_{\mu\nu} \, p_i^{\mu} \, k_{\rho} \, (b_i^{\rho} p_i^{\nu} - b_i^{\nu} p_i^{\rho}) \ln \omega^{-1}$$

$$+ \text{ finite}$$

This has been tested by studying explicit examples of gravitational radiation during scattering in D=4.

Example 1: Coulomb scattering of a light particle by a heavy charged particle at large impact parameter

Ignore effect of gravitation on the scattering but compute gravitational radiation from the scatterers and electromagnetic field.

To subleading order the result is in perfect agreement with soft theorem with $\ln|\tau|$ replaced by $\ln\omega^{-1}$

Example 2: Scattering of a light particle by a Schwarzschild black hole at large impact parameter Peters

In this case gravity wave is sourced by the particles and the gravitational field.

Again to subleading order the result is in perfect agreement with soft theorem with $\ln|\tau|$ replaced by $\ln\omega^{-1}$.

Assuming the validity of the $\ln|\tau| \Rightarrow \ln\omega^{-1}$ rule, we can write down the classical soft graviton factor for general gravitational scattering

$$
\begin{aligned}
S_{gr} &= \sum_i \frac{\varepsilon_{\mu\nu} p_i^\mu p_i^\nu}{p_i.k} \\
&\quad + \frac{i}{8\pi} \ln\omega^{-1} \sum_i \frac{\varepsilon_{\mu\nu} p_i^\nu k_\rho}{p_i.k} \sum_{\substack{j\neq i \\ \eta_i\eta_j=1}} \frac{p_j.p_i}{\{(p_j.p_i)^2 - m_i^2 m_j^2\}^{3/2}} \\
&\qquad \times (p_j^\rho p_i^\mu - p_j^\mu p_i^\rho) \left\{ 2(p_j.p_i)^2 - 3m_i^2 m_j^2 \right\} + \text{finite}\,.
\end{aligned}
$$

η_i: 1 for incoming and -1 for outgoing

Can we see this from the analysis of the S-matrix?

Sahoo, A.S., work in progress

What do the $\ln\omega^{-1}$ terms translate to in the time dependence of gravitational field?

Laddha, A.S.

$$\mathbf{h_{ij} = A_{ij} + B_{ij}\, t^{-1}} \quad \textbf{for large t}$$

$$\mathbf{A_{ij}} = \omega \sum_{\mathbf{i}} \frac{\varepsilon_{\mu\nu} \mathbf{p_i^\mu p_i^\nu}}{\mathbf{p_i . k}}$$

– **memory effect**

$$\mathbf{B_{ij}} = \frac{1}{8\pi} \sum_{\mathbf{i}} \frac{\varepsilon_{\mu\nu} \mathbf{p_i^\nu k_\rho}}{\mathbf{p_i . k}} \sum_{\substack{\mathbf{j \neq i} \\ \eta_{\mathbf{i}}\eta_{\mathbf{j}}=1}} \frac{\mathbf{p_j . p_i}}{\{(\mathbf{p_j . p_i})^2 - \mathbf{m_i^2 m_j^2}\}^{3/2}}$$
$$\times (\mathbf{p_j^\rho p_i^\mu} - \mathbf{p_j^\mu p_i^\rho}) \left\{ 2(\mathbf{p_j . p_i})^2 - 3\mathbf{m_i^2 m_j^2} \right\}$$

– **a tail in the gravitational wave-form**

Binary black hole merger

A massive system $\Rightarrow$ another massive system + massless gravitational radiation

$\sum_{i=1}^{N}$ runs over the initial and final massive object and finite frequency gravitons in the final state

Result:

$$B_{ij} = -2\,G\,M\,A_{ij}$$

M: mass of the final system

$$\Rightarrow \quad h_{ij} = A_{ij}\left(1 - 2\,G\,M\,t^{-1}\right)$$

Conclusions

1. Up to subleading order we have universal soft graviton theorem in all dimensions > 4, for all mass and spin of external states.

2. At the subsubleading order there still exists a soft theorem but it is not universal

3. Classical limit of soft theorem determines the low frequency radiative part of the gravitational field during classical scattering

4. The 'classical soft theorem' is valid also in D=4, but at the subleading order there is a term $\propto$ the log of the soft energy, determined from soft theorem

Future

Recent interest in soft theorem began by noting its connection to asymptotic symmetries

Soft theorems hold also in dimensions > 4 where the role of asymptotic symmetries is less understood

On the other hand soft theorem in four dimensions undergo modification due to long range interactions

This perhaps indicates that we need better understanding of asymptotic symmetries, which may then tell us something useful about quantum gravity