Modern methods in scattering amplitudes and their applications
C/K duality, the double copy and their applications: **Lecture 3 – Loop level**

Last time we set out to understand the generalized unitarity method and we concluded that it is a reorganization of Feynman diagramatics which determines a presentation of the integrand of L-loop m-point scattering amplitudes from their residues and poles. The poles are at the poles of all propagators that can appear in Feynman diagrams for the amplitude we care about and the residues at the poles where all but k propagators are at their poles is

$$\mathcal{C}^{\mathrm{N}^k\mathrm{MC}} = \sum_{\text{states}} \mathcal{A}^{\text{tree}}_{m(1)} \cdots \mathcal{A}^{\text{tree}}_{m(p)} \,. \tag{1}$$

Generalize

$\mathcal{A}^{\text{tree}}_{m(i)}$ are tree-level $m(i)$-multiplicity amplitudes corresponding to the blobs. <**drawn picture**>

Up to here the construction is general field theories; colored or not. The difference between theories is what we take for the states and for the tree amplitudes. The sum runs over all the physical states of the regularized theory that can appear in loops, given the external states of the amplitude. These are the various physical states of the gluon as well as the other physical states of all the other fields. For example, in N=4 sYM one sums over the 2 gluon states, 2x4 gluino stats and 6 scalar states.

They can be done "by hand" – by just listing all contributions and adding them up – or more automatically in theories with sufficient supersymmetry. There is actually a fair bit of technology on how to actually carry out these sums, especially in 4 and 6 dimensions, but to explain it properly and see some examples it takes some time. Notes online contain references. So in case you want to – or need to – do calculations in the immediate future, I refer you to the original literature. Sometimes the sum happens to be trivial, even in theories with many fields.

The algorithm is:

1) compute MC of amplitude and equate with MC of ansatz

2) compute NMC of amplitude and equate with NMC of ansatz

...

n) solve the full system.

It is in principle straightforward, though it can be tedious and at times scary depending on the size of the ansatz. Result is the amplitude presented as a sum over cubic graphs.

One may take shortcuts and start with some NkMC with $k \geq 2$; before GU became mainstream, this was the approach of choice.

Example: 1-loop – If to do explicitly need to define spinor products.

Evaluating the quadruple cut:

$$\mathcal{C}_4^{1234} = i s_{12} s_{23} A(1234)\text{color}(1234) \tag{2}$$
$$\mathcal{C}_4^{1342} = i s_{12} s_{13} A(1342)\text{color}(1342) \tag{3}$$
$$\mathcal{C}_4^{1423} = i s_{14} s_{42} A(1423)\text{color}(1423) \;. \tag{4}$$

In double-copy theories the generalized cuts are inherited from the single copies. This is, in fact, how the KLT relations were originally used to construct gravity amplitudes. Gravity cuts were constructed from the cuts of YM theory and then they were matched onto an ansatz. So, schematically, if

$$\mathcal{M}_n = \sum_{i,j} K_{ij} \mathcal{A}_n(i)\mathcal{A}_n(j) \tag{5}$$

where i, j label permutations of the n external lgs and K is some momentum-dependent kernel; we saw example of this las time, for the 4-point amplitude:

$$\begin{aligned} M_3^{\rm tr}(1,2,3) &= \;\; iA_3^{\rm tr}(1,2,3)A_3^{\rm tr}(1,2,3) \\ M_4^{\rm tr}(1,2,3,4) &= -is_{12}A_4^{\rm tr}(1,2,3,4)A_4^{\rm tr}(1,2,4,3) \\ M_5^{\rm tr}(1,2,3,4,5) &= \;\; is_{12}s_{34}A_5^{\rm tr}(1,2,3,4,5)A_5^{\rm tr}(2,1,4,3,5) + (2 \leftrightarrow 3) \end{aligned} \tag{6}$$

Then generalized cuts are

$$\mathcal{C}^{{\rm N}^k{\rm MC}} = \sum_{\rm states} \mathcal{M}^{\rm tree}_{m(1)} \cdots \mathcal{M}^{\rm tree}_{m(p)} \tag{7}$$

$$= \sum_{{\rm states}_1} \sum_{{\rm states}_2} (K_{i_1 j_1} \mathcal{A}_{m(1)}(i_1)\mathcal{A}_{m(1)}(j_1)) \cdots (K_{i_p m j_p} \mathcal{A}_{m(p)}(i_p)\mathcal{A}_{m(p)}(i_p)) \tag{8}$$

$$= \sum K_{i_1 j_1} \dots K_{i_p j_p} \sum_{{\rm states}_1} \mathcal{A}^{L,tree}_{m(1)}(i_1) \dots \mathcal{A}^{L,tree}_{m(p)}(i_p) \sum_{{\rm states}_2} \mathcal{A}^{R,tree}_{m(1)}(j_1)) \cdots \mathcal{A}^{R,tree}_{m(p)}(i_p) \tag{9}$$

For any 2-copy theory this structure holds, just A and K have different expressions. **The point** is that the two sums are gauge theory cuts. So, if one has some expression for the gauge theory amplitude, one can compute the gravity cuts by taking the cuts of the gauge theory expression and assembling them this way. **This makes use of simplifications carried out in the construction of the g.t. amplitude**

Example: 1-loop gravity form 1-loop YM.

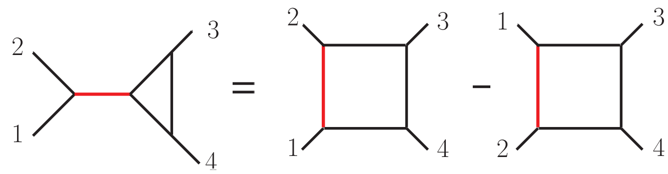
The easiest way to see the result is to look at the maximal cut and use the M3 KLT:

$$\begin{aligned} \mathcal{C}_4^{\mathcal{M}} &= \sum_{\rm states} \mathcal{M}(q_1,1,q_2)\mathcal{M}(q_2,2,q_3)\mathcal{M}(q_3,3,q_4)\mathcal{M}(q_4,4,q_1) \\ &= (\sum_{\rm states} \mathcal{M}(q_1,1,q_2)\mathcal{M}(q_2,2,q_3)\mathcal{M}(q_3,3,q_4)\mathcal{M}(q_4,4,q_1))^2 \\ &= (is_{12}s_{23}A(1234))^2 \propto s_{12}s_{13}s_{23}M_4^{\rm tr}(1,2,3,4) \end{aligned} \tag{10}$$

The same can be seen from the computation of the N2MC.

How does generalize unitarity mix with c/k duality? The idea is that, in general, the solution to matching cuts and ansatze is not unique and one can adjust the free coefficients such that numerators have additional properties. Alternatively, one can work backwards – which is in a sense more efficient. Solve the kinematic Jacobi relations – subject to whatever additional constraints one wants to impose – and then match that onto cuts. For example, at 1 loop,

If one demands e.g. that the expression of the amplitudes has no triangles, then lhs is 0 and kinematic numerators of boxes are all equal – which is what we found. Also, from this perspective, direct 1-loop double-copy would give a gravity amplitude with numerators which nsFigure



are squared of the YM one – which is also what we found by independent computation using KLT on the cuts.

One can however relax the no triangle requirement and find more complicated expressions for the same amplitude – though no less correct. Their double-copy would still be correct, and will contain triangles and such. Such a result is closer to e.g. $\mathcal{N}=1$ and $\mathcal{N}=2$ sYM 1-loop 4-gluon amplitudes expressions.

When using GU and/or c/k duality it is useful to try to constrain the ansatz for numerators. Of course, there is no guarantee that random constraints yield a solution. However, the construction tells you when the constraints imposed are too strong by just failing to produce a solution to cut constraints. Also, when using c/k duality, it is useful to start with a subset of the kinematic Jacobi relations, which reduce the number of independent numerators, and verify at the end whether the rest are satisfied. There is a little bit of a tradeoff: if there are very few independent numerators then the others may be expressed in terms of them in complicated ways; if more numerators are assumed independent, then the ansatz may become large.

Three-loop example

Besides the duality and unitarity constraints, it is beneficial to impose other constraints to simplify the structure of the ansatze we might make. Such constraints depend on the problem at hand. Sometimes such constraints could be too strong and no solution exists; then one needs to relax them. For $\mathcal{N}=4$ sYM amplitudes at 4 points natural constraints are:

1. One-loop tadpole, bubble and triangle subgraphs do not appear in any graph.
2. A one-loop n-gon subgraph carries no more than $n-4$ powers of loop momentum for that loop.
3. Numerators carry the same relabeling symmetries as the graphs.

For the four-point amplitude whose structure is we can add an additional constraint:

4. After extracting an overall factor of stA_4^{tree}, the numerators are polynomials in D-dimensional Lorentz products of the independent loop and external momenta.

Modifications may be needed at higher points (kill 4); for less supersymmetry modifications are definitely needed in assumptions 1 and 2; kill assumption 4.

These constraints together with c/k duality determine the amplitude up to one coefficient, which is determined from one maximal cut.

At 3 loops: 12 graphs without bubbles and triangles. There are 17 including bubbles and triangles. Define numerators

$$\begin{aligned} n^{(x)} &= stA_4^{\text{tree}}(1,2,3,4)\, N^{(x)}\,, \\ N^{(x)} &\equiv N^{(x)}(p_1,p_2,p_3,\ell_5,\ell_6,\ell_7)\,. \end{aligned} \tag{11}$$

$stA_4^{\text{tree}}(1,2,3,4)$ is fully crossing symmetric; thus N and n have the same symmetry properties and powercounting says that N has degree 2 in invariants.

There are many duality relations – naively 1/3 x 12 x 10 = 40. Consider a subset corresponding to one edge in each graph except 5th. Because of absence of triangles some of them relate 2 graphs; others relate 3.

$$\begin{aligned}
N^{(\mathrm{a})} &= N^{(\mathrm{b})}(p_1,p_2,p_3,\ell_5,\ell_6,\ell_7)\,, \\
N^{(\mathrm{b})} &= N^{(\mathrm{d})}(p_1,p_2,p_3,\ell_5,\ell_6,\ell_7)\,, \\
N^{(\mathrm{c})} &= N^{(\mathrm{a})}(p_1,p_2,p_3,\ell_5,\ell_6,\ell_7)\,, \\
N^{(\mathrm{d})} &= N^{(\mathrm{h})}(p_3,p_1,p_2,\ell_7,\ell_6,p_{1,3}-\ell_5+\ell_6-\ell_7) + N^{(\mathrm{h})}(p_3,p_2,p_1,\ell_7,\ell_6,p_{2,3}+\ell_5-\ell_7)\,, \\
N^{(\mathrm{f})} &= N^{(\mathrm{e})}(p_1,p_2,p_3,\ell_5,\ell_6,\ell_7)\,, \\
N^{(\mathrm{g})} &= N^{(\mathrm{e})}(p_1,p_2,p_3,\ell_5,\ell_6,\ell_7)\,, \\
N^{(\mathrm{h})} &= -N^{(\mathrm{g})}(p_1,p_2,p_3,\ell_5,\ell_6,p_{1,2}-\ell_5-\ell_7) - N^{(\mathrm{i})}(p_4,p_3,p_2,\ell_6-\ell_5,\ell_5-\ell_6+\ell_7-p_{1,2},\ell_6)\,, \\
N^{(\mathrm{i})} &= N^{(\mathrm{e})}(p_1,p_2,p_3,\ell_5,\ell_7,\ell_6) - N^{(\mathrm{e})}(p_3,p_2,p_1,-p_4-\ell_5-\ell_6,-\ell_6-\ell_7,\ell_6)\,, \\
N^{(\mathrm{j})} &= N^{(\mathrm{e})}(p_1,p_2,p_3,\ell_5,\ell_6,\ell_7) - N^{(\mathrm{e})}(p_2,p_1,p_3,\ell_5,\ell_6,\ell_7)\,, \\
N^{(\mathrm{k})} &= N^{(\mathrm{f})}(p_1,p_2,p_3,\ell_5,\ell_6,\ell_7) - N^{(\mathrm{f})}(p_2,p_1,p_3,\ell_5,\ell_6,\ell_7)\,, \\
N^{(\mathrm{l})} &= N^{(\mathrm{g})}(p_1,p_2,p_3,\ell_5,\ell_6,\ell_7) - N^{(\mathrm{g})}(p_2,p_1,p_3,\ell_5,\ell_6,\ell_7)\,,
\end{aligned} \tag{12}$$

BCJjacobi

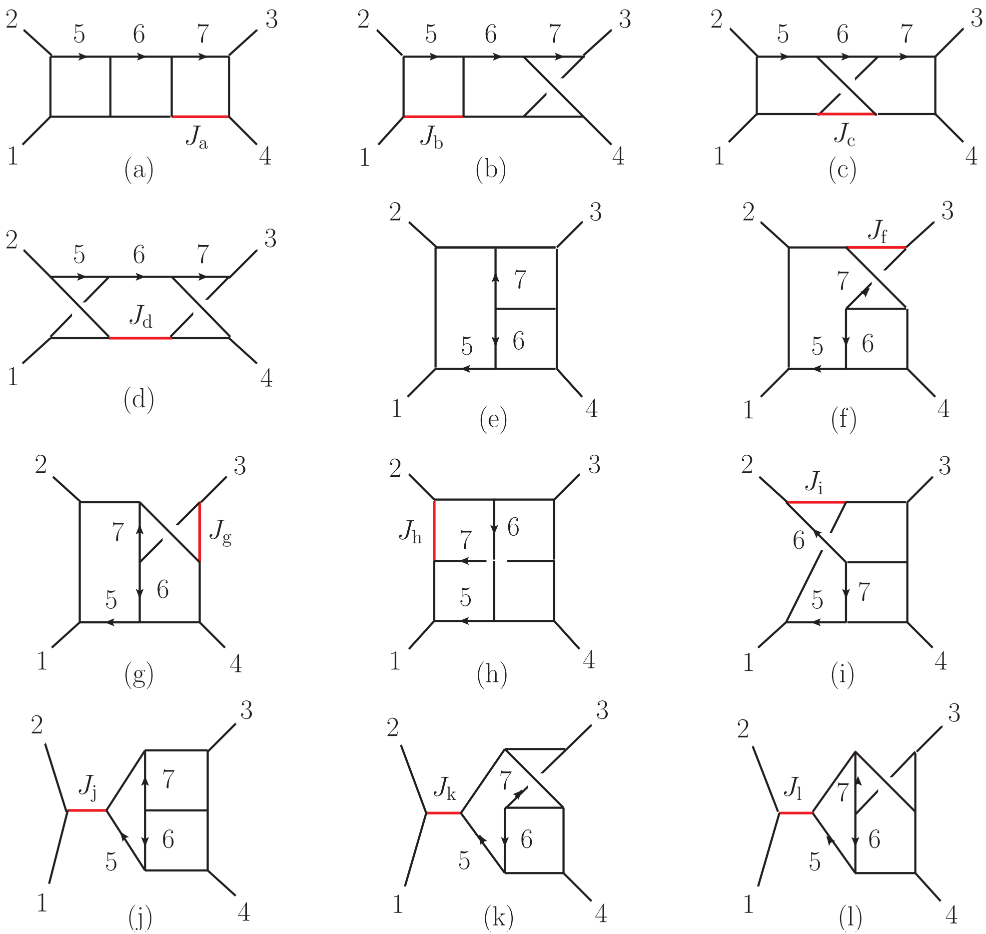


Figure 1: The diagrams for constructing the $\mathcal{N}=4$ super-Yang-Mills and $\mathcal{N}=8$ supergravity three-loop four-point amplitudes. The shaded (red) lines indicate the application of the duality relation. The external momenta are outgoing and the arrows indicate the directions of the labeled loop momenta. Diagram (e) is the master diagram. ThreeLoopD

Here $p_{i,j} \equiv p_i + p_j$; canonical $(p_1, p_2, p_3, \ell_5, \ell_6, \ell_7)$ on lhs suppressed, of the numerators on the left-hand side of the equations (12). Jacobi-s are done on highlighted lines; one for each graph. Graphs with triangles have 0 numerators; their appearance in Jacobi relations leads to 2-term relations.

This system can be used to express any numerator factor in terms of combinations of the numerator $N^{(\mathrm{e})}$, with various different arguments. **This is convenient choice but not the only choice. Others – graphs g or f. Another option is to choose pairs of remaining graphs.**

It is important to stress that these are not all the kinematic Jacobi relations; it is a convenient subset which, once solved together with the cuts, provides a set of numerators which one is supposed to further plug in the remaining relations to check them.

At this stage one can make ansatz for $N^{(e)}$, derive the other numerators from eqs above and impose cuts sequentially. Note though that this is a planar graph; the planar 3-loop amplitude

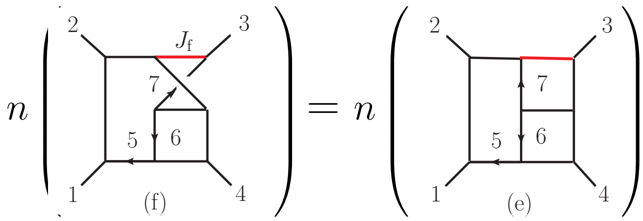


Figure 2: Examples of BCJ kinematic numerator relation at three loops for $\mathcal{N}=4$ sYM theory. In the two term relations one of the three numerators in a Jacobi triplet of diagrams vanishes. ThreeLoopB

has been known for quite some time and the numerator that appears there is

$$N^{(e)}_{\rm rr} = s(\ell_5+p_4)^2\,. \tag{13}$$ RungRuleNu

A variant of this can be derived immediately form the method of maximal cuts we discussed; or from some shortcuts to it. So it is hard to argue that this term should not be present. There may however be more terms that are not detected by max cuts. Such terms can be thought of as generalized gauge transformations, which change numerators without changing the amplitude.

To find a modification of $N^{(e)}$ we demand maximal cuts and that the auxiliary constraints are obeyed: **(1)** $N^{(e)}$ has mass dimension four and possesses the symmetry of the graph **(2)** no loop momentum for any box subgraph in (e) appears in it (ruling out ℓ_6 and ℓ_7); **(3)** $N^{(e)}$ is at most linear in the pentagon loop momenta ℓ_5. To impose this one it is easier to relax it ti quadratic dependence and then require that the quadratic terms cancel out. This is because to make sure max cuts are not messed up by etra terms it is useful to use inverse propagators to make ansatz for $N^{(e)}$ and propagators with ℓ_5 necessarily contain ℓ_5^2

Symmetry condition:

$$\{p_1\leftrightarrow p_2, p_3\leftrightarrow p_4, \ell_5\to p_1+p_2-\ell_5\}\,. \tag{14}$$

The most general polynomial consistent with these constraints is

$$N^{(e)} = s(\ell_5+p_4)^2 + (\alpha s+\beta t)\ell_5^2 + (\gamma s+\delta t)(\ell_5-p_1)^2 + (\alpha s+\beta t)(\ell_5-p_1-p_2)^2\,, \tag{15}$$ ThreeLoopA

First term is the max cut contribution adjusted to look like the known contribution to planar amplitudes. All extra terms vanish on maximal cuts, so they have the structure observed at tree level to be local generalized gauge transformations.

Demanding now that ℓ_5^2 terms cancel out leads to 2 constraints – one from $s\ell_6^2$ and one from $t\ell_6^2$:

$$\gamma = -1-2\alpha \quad \text{and} \quad \delta=-2\beta\,,$$

leading to

$$N^{(e)} = s(\tau_{45}+\tau_{15}) + (\alpha s+\beta t)(s+\tau_{15}-\tau_{25})\,, \tag{16}$$ CleanAnsat

where we use the notation,

$$\tau_{ij}\equiv 2p_i\cdot\ell_j\,,\quad (i\le 4, j\ge 5)\,, \tag{17}$$ tauDef

This is all about graph e; the numerators of all the other graphs however should obey the same constraints.

A graph closely related to e is j; the reason to look at is is that ℓ_5 is the same between the two graphs and in e it is a pentagon momentum but in j is it a box momentum. So, in j, terms linear in ℓ_5 should cancel. Using N_e above and Jacobi relations leads to

$$N^{(\mathrm{j})} = s(1+2\alpha-\beta)(\tau_{15}-\tau_{25}) + \beta s(t-u)\,. \tag{18}$$

Njtemp

Demanding that ℓ_5 cancels out implies

$$\beta = 1+2\alpha\ .$$

leading to

$$\begin{aligned} N^{(\mathrm{e})} &= s(\tau_{45}+\tau_{15}) + (\alpha(t-u)+t)(s+\tau_{15}-\tau_{25})\,, && (19)\\ N^{(\mathrm{j})} &= (1+2\alpha)(t-u)s\,, && (20)\end{aligned}$$

Simplified

Simplified

So we are down to one parameter. It turns out that it also can be fixed by the constraints we imposed; or, i can also be fixed by computing some cut. Since we are still left with loop momenta, we may guess that there are some further constraints coming from the requirement that boxes have no loop momenta. So let's look at graphs with only boxes, such as graph a. Solving for its numeraor is a little lengthy and gives a 6 term expression

$$\begin{aligned} N^{(\mathrm{a})} = {} & N^{(\mathrm{e})}(p_1,p_2,p_4,-p_3+\ell_5-\ell_6+\ell_7,\ell_5-\ell_6,-\ell_5)\\ & + N^{(\mathrm{e})}(p_2,p_1,p_4,-p_3-\ell_5+\ell_7,-\ell_5,\ell_5-\ell_6)\\ & - N^{(\mathrm{e})}(p_4,p_1,p_2,\ell_6-\ell_7,\ell_6,\ell_5-\ell_6) - N^{(\mathrm{e})}(p_4,p_2,p_1,\ell_6-\ell_7,\ell_6,-\ell_5)\\ & - N^{(\mathrm{e})}(p_3,p_1,p_2,\ell_7,\ell_6,\ell_5-\ell_6) - N^{(\mathrm{e})}(p_3,p_2,p_1,\ell_7,\ell_6,-\ell_5)\,. \end{aligned} \tag{21}$$

DualityEqu

which, after $N^{(e)}$ is plugged in, becomes

$$N^{(\mathrm{a})} = s^2 + (1+3\alpha)\Big((\tau_{16}-\tau_{46})s - 2(\tau_{17}+\tau_{37})s + (\tau_{16}-2\tau_{17}-\tau_{26}+2\tau_{27})t + 4ut\Big)\,. \tag{22}$$

Demand no loop momentum sets $\alpha = -1/3$ and

$$N^{(\mathrm{e})} = s(\tau_{45}+\tau_{15}) + \frac{1}{3}(t-s)(s+\tau_{15}-\tau_{25})\,, \tag{23}$$

NumeratorE

Next step: construct all numerators, check all D-dimensional unitarity cuts and all Jacobi relations and all graph symmetries. Everything checks out. Squaring these numerators leads to gravity; its cuts must also be checked, etc.

A (more) constructive version of the method of maximal cuts starts from the observation that each numerator can be written as a sum of products of the inverse propagators of the graph with coefficients which are polynomials in other momentum invariants and dot products with polarization vectors, etc. It is worth mentioning that this decomposition is not unique and depends on one's choice of solution to momentum conservation. The terms with no inverse propagators are those with the strongest singularity; those with a single inverse propagator are those with next-to-strongest, etc. The NkMC fix the terms with k inverse propagators; these terms also correspond to graphs with higher-point vertices, corresponding to the canceled propagators.

So, starting with no ansatz, the construction is: I am phrasing it for 4-point amplitudes. For higher-points the algorithm is essentially the same, but has a couple more subtleties which I can describe if you ask.

1) evaluate MC's: they are local and are the numerators of the graphs with only cubic vertices

2) evaluate NMC; they are nonlocal, having a term corresponding to the off shell propagator. Subtract the NMC of the cubic graphs with numerators computed at 1). Result is local. This is the numerator of a graph with on quartic vertex. It corresponds to the 4-point amplitude in the cut.

3) repeat for N2MC, etc.

The point is that each cut uniquely identifies the graph that needs to be added: it is the graph in which all tree amplitudes in the cut are replaced by a vertex of the same multiplicity. Result is the amplitude; this time organized in terms of graphs with vertices of all multiplicities. Simplest first nontrivial example is at 3 loops in N=4 sYM, so I won't go into it. You can read it in the original paper, or we can discuss separately.

This approach gives us a way to double copy when c/k representations are not readily available. They should however exist – in principle – **only at tree level**. This approach was developed due to (technical) difficulties with finding a c/k-satisfying representation for the 5-loop 4-gluon amplitude.

To understand the idea it is useful to step back and think about the practical consequences of c/k at loop level. The main one is that once we have the g.t. amplitudes in the right form computation of cuts is (in principle) no longer necessary.

(1) If this is not available, one may still in principle have g.t. cuts that obey c/k manifestly; if one had tree amplitudes of high-enough multiplicity in a nice, local form, one could in principle find them directly. (relaxing cut conditions to full c/k integrand is still nontrivial; also, "in principle" refers to the fact that simplifications of the product of trees that may be natural may also break the initial manifest c/k duality.) Then, gravity cuts follow straightforwardly, through the usual color→ kinematics replacement. Then one could **either** fit them on an ansatz **or** do the ansatz-less construction in terms of graphs with vertices of all multiplicity.

(2) if none of the above is available, the generalized double-copy is a procedure to correct a naive double-copy when the gauge theory amplitude and cuts are not available in c/k form. It turns out that it is simpler than to take the cuts of g.t. amplitude and massage them into c/k form.

Let us assume we are handed a gauge theory amplitude of the form

$$\mathcal{A} = i^{L-1} g^{n-2+2L} \sum_{\Gamma \in \text{cubic}} \frac{1}{\mathcal{S}_\Gamma} \int \prod_i \frac{d^D l_i}{(2\pi)^D} \frac{n_\Gamma c_\Gamma}{D_\Gamma} \tag{24}$$

and it's naive double copy:

$$\text{stuff} = \sum_{\Gamma \in \text{cubic}} \frac{1}{\mathcal{S}_\Gamma} \int \prod_i \frac{d^D l_i}{(2\pi)^D} \frac{n_\Gamma \tilde{n}_\Gamma}{D_\Gamma} \ , \tag{25}$$

which we want to correct to a gravity amplitude. The thing to notice is that it already has the correct MC and NMC.

Assume that n-s and $\tilde{n}$ do not have the algebraic properties of c. Consider some cut of $\mathcal{A}$, which contains q tree amplitudes with at least 4 external legs (for the amplitude some of these legs may be internal) :

$$\mathcal{C}_{\rm YM} = \sum_{i_1,\dots,i_q} \frac{n_{i_1,i_2,\dots i_q} c_{i_1,i_2,\dots i_q}}{D_{i_1} \dots D_{i_q}} \ . \tag{26}$$

BCJCut

Here i_j (with $j = 1..., q$) runs over the graphs of the j-th tree amplitude factor; D_{i_j} is the product of propagators of that graph. See sec 2.C of `1804.09311` for a more thorough explanation of notation.

The cut numerators $n_{i_1,i_2,\dots i_q}$ also do not obey c/k duality; since they are however tree-level quantities, they are related to ones that do by a shift

$$n_{i_1,i_2,\dots i_q} \quad = \quad n^{\rm BCJ}_{i_1,i_2,\dots i_q} + \Delta_{i_1,i_2,\dots i_q} \ . \tag{27}$$

where Δ is such that the value of the cut does not change, that is

$$\sum_{i_1,\dots,i_q} \frac{\Delta_{i_1,i_2,\dots i_q} c_{i_1,i_2,\dots i_q}}{D_{i_1} \dots D_{i_q}} = 0 \ . \tag{28}$$

gt

In terms of $n^{\rm BCJ}$ the corresponding gravity cut is

$$\mathcal{C}_{\rm G} = \sum_{i_1,\dots,i_q} \frac{n^{\rm BCJ}_{i_1,i_2,\dots i_q} \tilde{n}^{\rm BCJ}_{i_1,i_2,\dots i_q}}{D_{i_1} \dots D_{i_q}} \ . \tag{29}$$

BCJCut

which can now be expressed in terms of $n_{i_1,i_2,\dots i_k}$ and $\Delta_{i_1,i_2,\dots i_k}$.

$$\begin{aligned} \mathcal{C}_{\rm G} &= \sum_{i_1,\dots,i_q} \frac{n_{i_1,i_2,\dots i_q} \tilde{n}_{i_1,i_2,\dots i_q}}{D_{i_1} \dots D_{i_q}} + \mathcal{E}_{\rm G} && (30) \\ &= \sum_{i_1,\dots,i_q} \frac{n_{i_1,i_2,\dots i_q} \tilde{n}_{i_1,i_2,\dots i_q}}{D_{i_1} \dots D_{i_q}} - \sum_{i_1,\dots,i_q} \frac{\Delta_{i_1,i_2,\dots i_q} \tilde{\Delta}_{i_1,i_2,\dots i_q}}{D_{i_1} \dots D_{i_q}} \ , && (31) \end{aligned}$$

CutNDC

Extra

While this may not appear particularly useful; one can argue however, based on gauge invariance, that $\mathcal{E}_G$ should be expressible in terms of bilinears of BCJ discrepancy functions: if

$$c_I + c_J + c_K = 0 \leftrightarrow n_I + n_J + n_K = J_{IJK} \ . \tag{32}$$

Let's see an example – a cut with two 4-point amplitude factors:

$$\mathcal{C}^{4\times4}_{\rm YM}=\sum_{i_1,i_2}^{3}\frac{n_{i_1,i_2}c_{i_1,i_2}}{d^{(1)}_{i_1}d^{(2)}_{i_2}}\,,\tag{33}$$

C4x4

$$\Delta_{i_1,i_2}=n_{i_1,i_2}-n^{\rm BCJ}_{i_1,i_2}=d^{(1)}_{i_1}\alpha^{(1)}_{i_2}+d^{(2)}_{i_2}\alpha^{(2)}_{i_1}\,,\tag{34}$$

Generalize

$$\sum_{i_1=1}^{3}c_{i_1i_2}=0\,,\qquad\qquad \sum_{i_2=1}^{3}c_{i_1i_2}=0\,,\tag{35}$$

To ensure that the cut (33) is invariant Δ :

$$\sum_{i_1,i_2=1}^{3}\frac{\Delta_{i_1,i_2}c_{i_1,i_2}}{d^{(1)}_{i_1}d^{(2)}_{i_2}}=\sum_{i_2=1}^{3}\frac{\alpha^{(1)}_{i_2}}{d^{(2)}_{i_2}}\sum_{i_1=1}^{3}c_{i_1,i_2}+\sum_{i_1=1}^{3}\frac{\alpha^{(2)}_{i_1}}{d^{(1)}_{i_1}}\sum_{i_2=1}^{3}c_{i_1,i_2}=0\,.\tag{36}$$

Thus, the correction term $\mathcal{E}$ which completes the cut of the naive double copy to an N^2 cut of supergravity is

$$\mathcal{E}_{i_1,i_2}=-\sum_{i_1,i_2=1}^{3}\frac{\Delta_{i_1,i_2}\tilde{\Delta}_{i_1,i_2}}{d^{(1)}_{i_1}d^{(2)}_{i_2}}=-\sum_{i_1,i_2=1}^{3}\frac{d^{(1)}_{i_1}d^{(2)}_{i_2}(\alpha^{(1)}_{i_2}\,\tilde{\alpha}^{(2)}_{i_1}+\alpha^{(2)}_{i_1}\,\tilde{\alpha}^{(1)}_{i_2})}{d^{(1)}_{i_1}d^{(2)}_{i_2}}\,,\tag{37}$$

Extra4x4St

where we also used that the sum of the inverse propagators in each four-point amplitude vanishes. The propagators cancel leaving

$$\mathcal{E}^{4\times4}_{\rm G}=-\sum_{i_2=1}^{3}\alpha^{(1)}_{i_2}\sum_{i_1=1}^{3}\tilde{\alpha}^{(2)}_{i_1}-\sum_{i_1=1}^{3}\alpha^{(2)}_{i_1}\sum_{i_2=1}^{3}\tilde{\alpha}^{(1)}_{i_2}\,.\tag{38}$$

To rewrite $\mathcal{E}^{4\times4}_{\rm G}$ in terms of BCJ discrepancy functions we must solve the eqs. (**??**) for this cut. They read

$$J_{\bullet,i_2}\equiv\sum_{i_1=1}^{3}n_{i_1i_2}=d^{(2)}_{i_2}\sum_{i_1}\alpha^{(2)}_{i_1}\,,\qquad\qquad J_{i_1,\bullet}\equiv\sum_{i_2=1}^{3}n_{i_1i_2}=d^{(1)}_{i_1}\sum_{i_2}\alpha^{(1)}_{i_2}\,,\tag{39}$$

LocalJs

Similar formulas hold for the $\tilde{J}$.

We notice here a manifestation of the constraints obeyed by α and J: on the one hand the right-hand side depends on only particular combinations of gauge parameters and on the other existence of solutions to these equations requires that the BCJ discrepancy functions be related to each other.

$$\sum_{i_1=1}^{3}\alpha^{(2)}_{i_1}=\frac{J_{\bullet,1}}{d^{(2)}_1}=\frac{J_{\bullet,2}}{d^{(2)}_2}=\frac{J_{\bullet,3}}{d^{(2)}_3}\,,\qquad\qquad \sum_{i_2=1}^{3}\alpha^{(1)}_{i_2}=\frac{J_{1,\bullet}}{d^{(1)}_1}=\frac{J_{2,\bullet}}{d^{(1)}_2}=\frac{J_{3,\bullet}}{d^{(1)}_3}\,.\tag{40}$$

sol_and_re

On a case by case basis one can check that these relations are obeyed.

We therefore find a simple expression of the extra contribution in terms of discrepancy functions,

$$\mathcal{E}_{\rm G}^{4\times 4} = -\frac{1}{d_1^{(1)} d_1^{(2)}}\Big(J_{\bullet,1}\tilde{J}_{1,\bullet} + J_{1,\bullet}\tilde{J}_{\bullet,1}\Big) = -\frac{1}{9}\sum_{i_1,i_2=1}^{3}\frac{1}{d_{i_1}^{(1)} d_{i_2}^{(2)}}\Big(J_{\bullet,i_2}\tilde{J}_{i_1,\bullet} + J_{i_1,\bullet}\tilde{J}_{\bullet,i_2}\Big)\,. \qquad (41) \quad \boxed{\texttt{Extra4x4}}$$

This expression gives us correct cuts in **all** double-copy theories; based on it we can write formally expressions for the 1-loop amplitudes of any such theory.

See slides for lecture 3 for a clean example of application of this method formula to the derivation of some contact terms at 3 loops.

While this expression appears to be organized in terms of he 9 diagrams contributing to the cut (it's got the propagators), it is in fact a local. The discrepancy functions are such that the ratios are local. So one can add the graph with the 4-point blobs replaced with four-point vertices to the naive double copy. To get the full N2MCs to work we need also the cuts with one 5-point blob. Its expressions, obtained through similar manipulations, is

$$\mathcal{C}_{\rm G}^{5} = \sum_{i=1}^{15}\frac{n_i\tilde{n}_i}{d_i^{(1,1)} d_i^{(1,2)}} + \mathcal{E}_{\rm G}^{5} \qquad \text{with} \qquad \mathcal{E}_{\rm G}^{5} = -\frac{1}{6}\sum_{i=1}^{15}\frac{J_{\{i,1\}}\tilde{J}_{\{i,2\}} + J_{\{i,2\}}\tilde{J}_{\{i,1\}}}{d_i^{(1,1)} d_i^{(1,2)}}\,. \qquad (42) \quad \boxed{\texttt{Extra5}}$$

Similar expressions are available for more complicated cuts as well: many 4-pts blobs and also mixtures of 4 and 5 are nice. Starting at 6 points they are a little cumbersome and not as neat. But the structure – bilinearity in J – remains.

See slides for comments on the construction and implications of the 5-loop 4-graviton amplitude in $\mathcal{N} = 8$ supergravity.