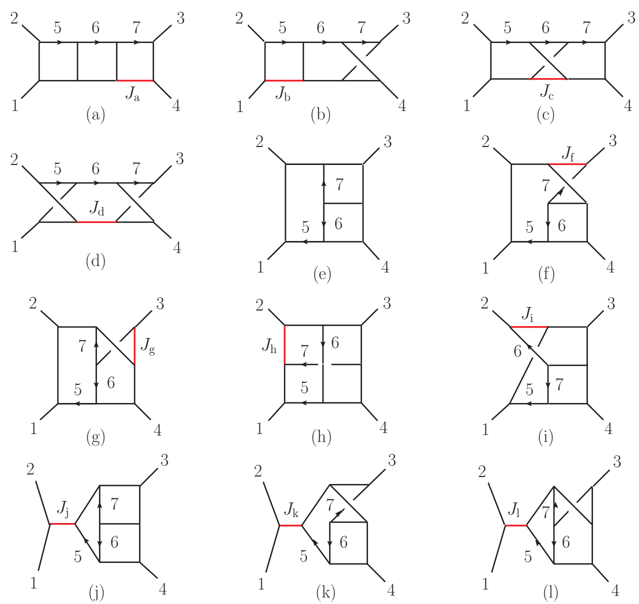

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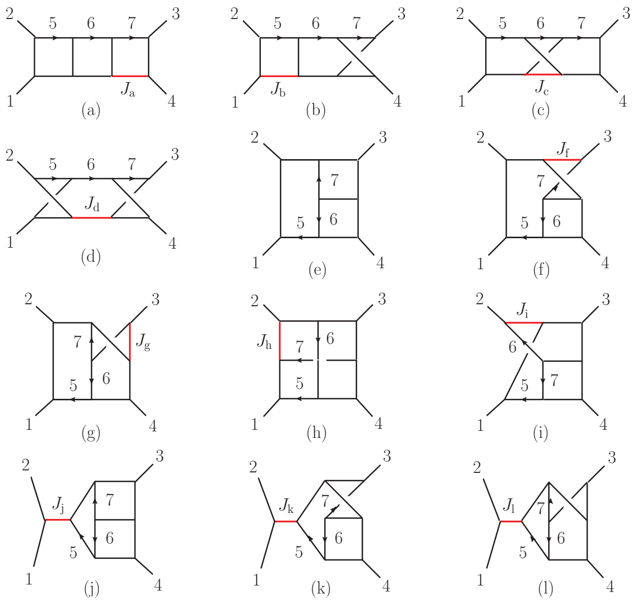

$$
\begin{aligned}
N^{(\mathrm{a})} &= N^{(\mathrm{b})}(p_1, p_2, p_3, \ell_5, \ell_6, \ell_7)\,, \\
N^{(\mathrm{b})} &= N^{(\mathrm{d})}(p_1, p_2, p_3, \ell_5, \ell_6, \ell_7)\,, \\
N^{(\mathrm{c})} &= N^{(\mathrm{a})}(p_1, p_2, p_3, \ell_5, \ell_6, \ell_7)\,, \\
N^{(\mathrm{d})} &= N^{(\mathrm{h})}(p_3, p_1, p_2, \ell_7, \ell_6, p_{1,3} - \ell_5 + \ell_6 - \ell_7) + N^{(\mathrm{h})}(p_3, p_2, p_1, \ell_7, \ell_6, p_{2,3} + \ell_5 - \ell_7)\,, \\
N^{(\mathrm{f})} &= N^{(\mathrm{e})}(p_1, p_2, p_3, \ell_5, \ell_6, \ell_7)\,, \\
N^{(\mathrm{g})} &= N^{(\mathrm{e})}(p_1, p_2, p_3, \ell_5, \ell_6, \ell_7)\,, \\
N^{(\mathrm{h})} &= -N^{(\mathrm{g})}(p_1, p_2, p_3, \ell_5, \ell_6, p_{1,2} - \ell_5 - \ell_7) - N^{(\mathrm{i})}(p_4, p_3, p_2, \ell_6 - \ell_5, \ell_5 - \ell_6 + \ell_7 - p_{1,2}, \ell_6) \\
N^{(\mathrm{i})} &= N^{(\mathrm{e})}(p_1, p_2, p_3, \ell_5, \ell_7, \ell_6) - N^{(\mathrm{e})}(p_3, p_2, p_1, -p_4 - \ell_5 - \ell_6, -\ell_6 - \ell_7, \ell_6)\,, \\
N^{(\mathrm{j})} &= N^{(\mathrm{e})}(p_1, p_2, p_3, \ell_5, \ell_6, \ell_7) - N^{(\mathrm{e})}(p_2, p_1, p_3, \ell_5, \ell_6, \ell_7)\,, \\
N^{(\mathrm{k})} &= N^{(\mathrm{f})}(p_1, p_2, p_3, \ell_5, \ell_6, \ell_7) - N^{(\mathrm{f})}(p_2, p_1, p_3, \ell_5, \ell_6, \ell_7)\,, \\
N^{(\mathrm{l})} &= N^{(\mathrm{g})}(p_1, p_2, p_3, \ell_5, \ell_6, \ell_7) - N^{(\mathrm{g})}(p_2, p_1, p_3, \ell_5, \ell_6, \ell_7)\,, \qquad (
\end{aligned}
$$

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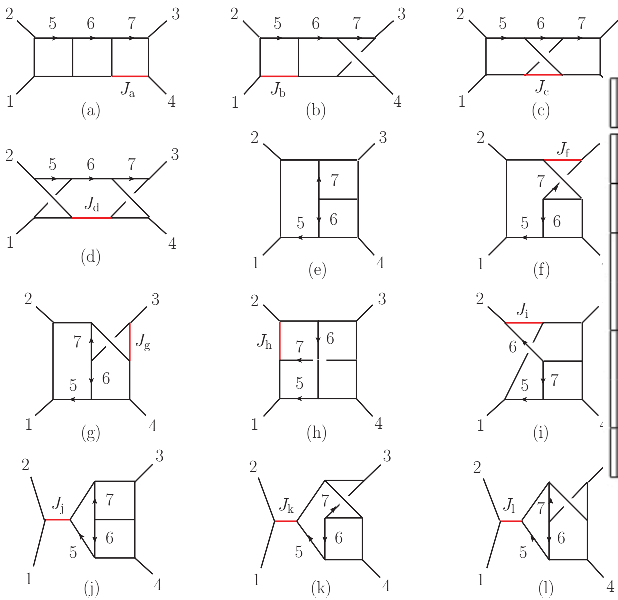

diagram	$\mathcal{N}=4$ Super-Yang-Mills ($\sqrt{\mathcal{N}=8 \text{ supergravity}}$) numerator
(a)–(d)	s^2
(e)–(g)	$\big(s\,(-\tau_{35}+\tau_{45}+t) - t\,(\tau_{25}+\tau_{45}) + u\,(\tau_{25}+\tau_{35}) - s^2\big)/3$
(h)	$\big(s\,(2\tau_{15}-\tau_{16}+2\tau_{26}-\tau_{27}+2\tau_{35}+\tau_{36}+\tau_{37}-u)$ $+t\,(\tau_{16}+\tau_{26}-\tau_{37}+2\tau_{36}-2\tau_{15}-2\tau_{27}-2\tau_{35}-3\tau_{17}) + s^2\big)/3$
(i)	$\big(s\,(-\tau_{25}-\tau_{26}-\tau_{35}+\tau_{36}+\tau_{45}+2t)$ $+t\,(\tau_{26}+\tau_{35}+2\tau_{36}+2\tau_{45}+3\tau_{46}) + u\,\tau_{25} + s^2\big)/3$
(j)-(l)	$s(t-u)/3$

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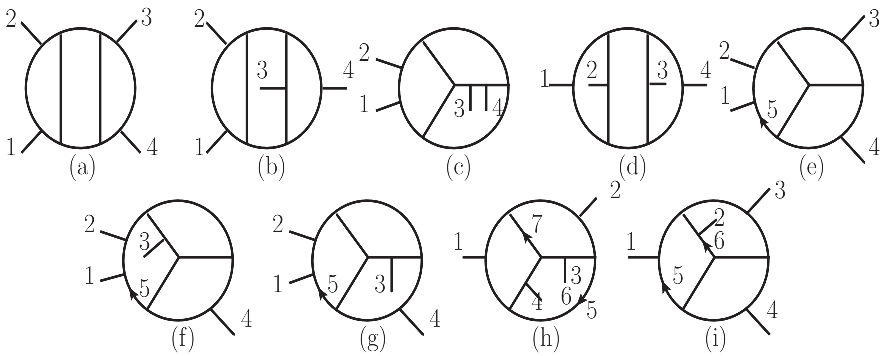
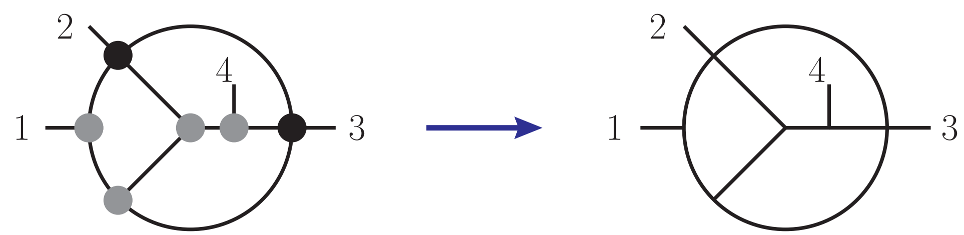
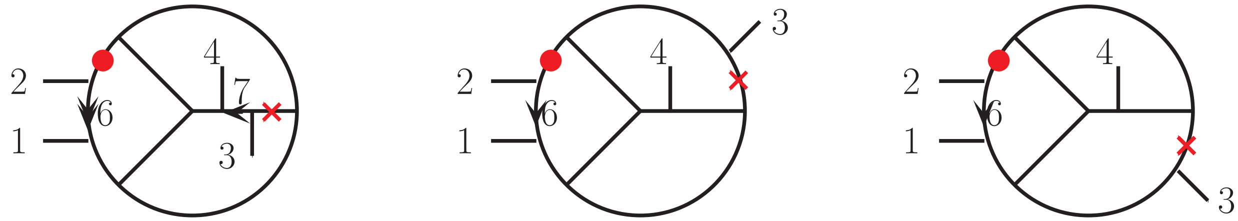
An example: generalized double copy at 3 loops: YM numerators from 0808.4112



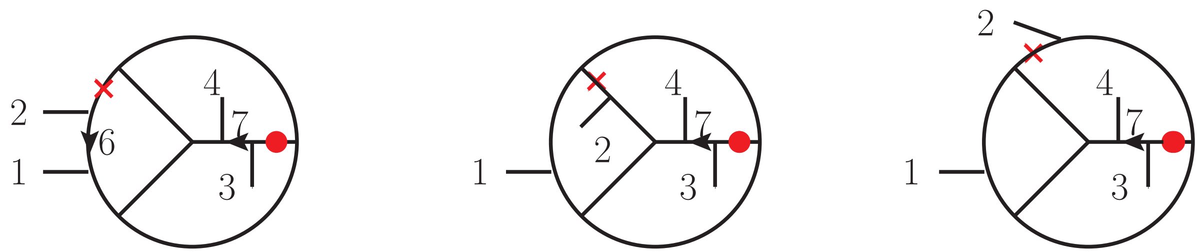
Graph	$\mathcal{N}=4$ sYM numerators.
(a)-(d)	s^2
(e)-(g)	$s(l_5^2+\tau_{45})$
(h)	$s(\tau_{26}+\tau_{36})-t(\tau_{17}+\tau_{27})+st$
(i)	$s(l_5^2+\tau_{45})-t(l_5^2+\tau_{56}+l_6^2)-(s-t)l_6^2/3$

An example of N2 contact term:





$$n_{1,1} = s^2\ , \quad n_{2,1} = s(t + \tau_{26} + \tau_{26})\ , \quad n_{3,1} = s(u - \tau_{26}) \longrightarrow J_{\bullet,1} = s\tau_{26}$$



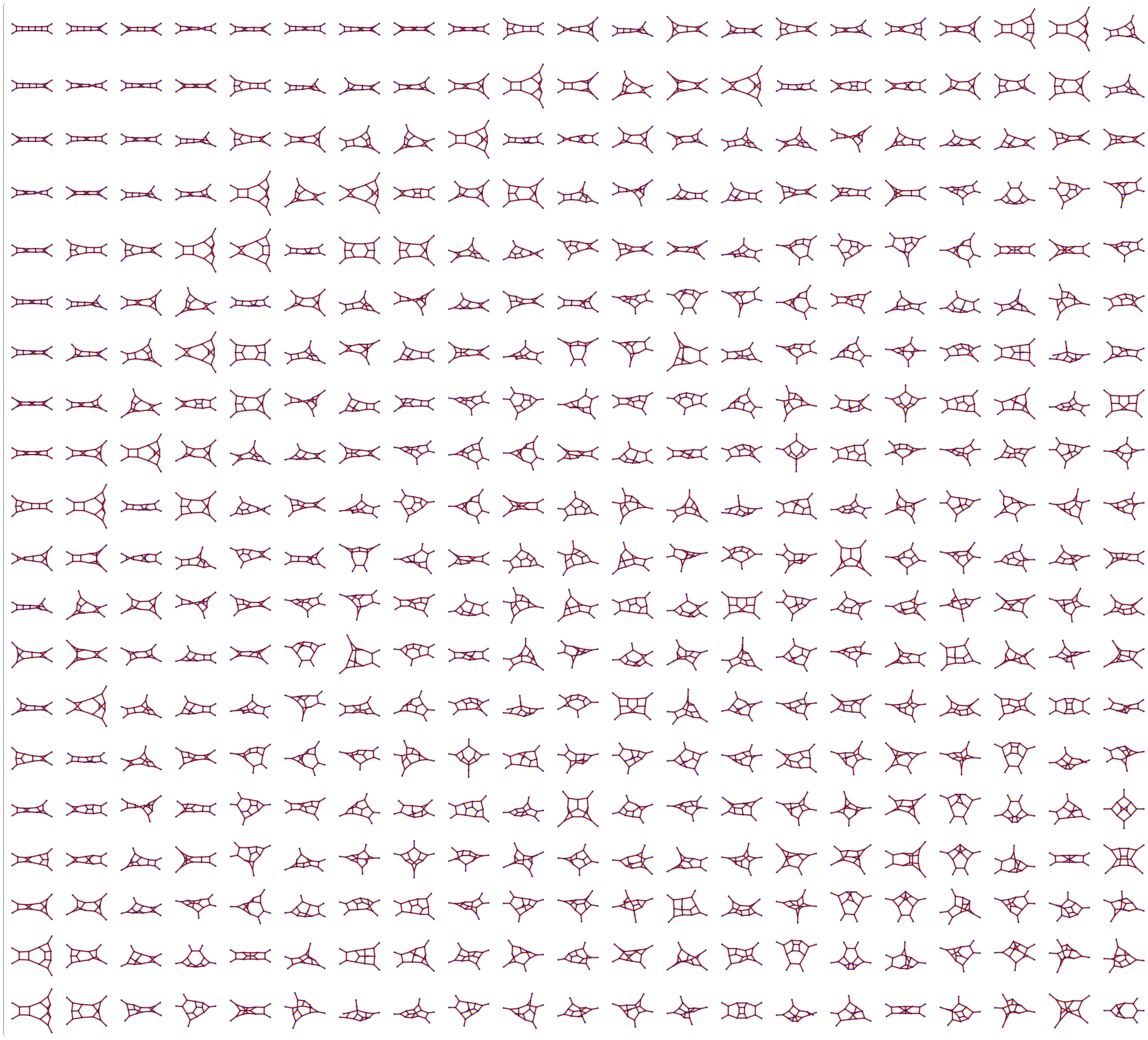
$$n_{1,1} = s^2\ , \quad n_{1,2} = s(t + \tau_{37} + \tau_{27})\ , \quad n_{1,3} = s(u - \tau_{27}) \longrightarrow J_{1,\bullet} = s\tau_{27}$$

$$\mathcal{N}^{\mathcal{N}=8}_{(l)} = -2\frac{J_{\bullet,1}J_{1,\bullet}}{\tau_{26}\tau_{37}} = -2s^2$$

- Reproduces known SG contact term — Bern, Carrasco, Dixon, Johansson, RR
- All other nonzero double-four-point contacts are relabelings of this one
- Five-point contact terms are also present; relevant formulae available

Generalized double-copy allowed construction the 4-point 5-loop integrand of $\mathcal{N}$=8 SG

Bern, Carrasco, Chen, Johansson, Zeng, RR



together with
2-, 3-, 4-, 5-, and
6-collapsed
propagator graphs:

N^0: 752/649
N^2: 9007/1306
N^3: 17479/2457
N^4: 22931/2470
N^5: 20657/1335
N^6: 13071/256

Identification of critical dimension and of the corresponding UV divergence is a nontrivial enterprise

- Separate of UV from IR features
- Expand at large loop momenta/small external momenta
- Identify of all relations between resulting integrals
- `FullSimplify`

$$\mathcal{M}_4^{(5)}\Big|_{\text{leading}} = -\frac{16\times 629}{25}\,\mathcal{K}_{\rm G}\left(\frac{\kappa}{2}\right)^{12}(s^2+t^2+u^2)^2\left(\frac{1}{48}\right.$$

$$+\frac{1}{16}$$

$$\left.\right)$$

Bern, Carrasco, Chen, Edison, Johansson, Parra-Martinez, Zeng, RR

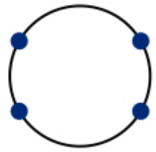
$$D_c = \frac{24}{5}$$

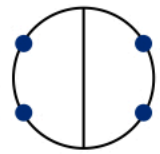
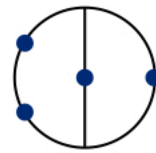
- Divergence corresponds to the D^8R^4 counterterm allowed by E_7 duality; However, implications to *D = 4* properties of $\mathcal{N}$=8 SG are not immediately obvious.

- Puzzle: enhanced cancellations exist in $\mathcal{N}$ = 4, 5 SG at corresponding orders in less supersymmetric theories; why not here?

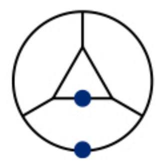
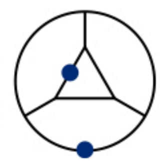
Summary of $\mathcal{N}$=8 SG UV divergences in the loop-dependent critical dimension

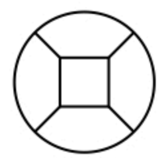
Dc

$\mathcal{M}_4^{(1)}\Big|_{\text{leading}} = -3\,\mathcal{K}_{\mathrm{G}}\left(\frac{\kappa}{2}\right)^4$  , — 8

$\mathcal{M}_4^{(2)}\Big|_{\text{leading}} = -8\,\mathcal{K}_{\mathrm{G}}\left(\frac{\kappa}{2}\right)^6 (s^2+t^2+u^2)\Big(\frac{1}{4}$  $+\frac{1}{4}$  $\Big)$, — 7

$\mathcal{M}_4^{(3)}\Big|_{\text{leading}} = -60\,\mathcal{K}_{\mathrm{G}}\left(\frac{\kappa}{2}\right)^8 stu\Big(\frac{1}{6}$  $+\frac{1}{2}$  $\Big)$, — 6

$\mathcal{M}_4^{(4)}\Big|_{\text{leading}} = -\frac{23}{2}\,\mathcal{K}_{\mathrm{G}}\left(\frac{\kappa}{2}\right)^{10} (s^2+t^2+u^2)^2\Big(\frac{1}{4}$  $+\frac{1}{2}$  $+\frac{1}{4}$  $\Big)$ — 11/2

$\mathcal{M}_4^{(5)}\Big|_{\text{leading}} = -\frac{16\times 629}{25}\,\mathcal{K}_{\mathrm{G}}\left(\frac{\kappa}{2}\right)^{12} (s^2+t^2+u^2)^2\Big(\frac{1}{48}$  $+\frac{1}{16}$  $\Big)$, — 24/5

Observations: 1. none of the vacuum diagrams contain 1-loop triangle subgraphs
2. relative factors given by # of automorphisms of vacuum graphs

If true at higher/all loops, it would provide vast simplifications to direct calculations

Consistency relations across loop orders and dimensions

- On general grounds, one has relations between divergences of *n*-point *L* loop Green's fcts and subdivergences at *(n-2)*-points, *(L+1)*-loop Green's fcts.

- In theories finite in *D = 4*, one may expect relations between divergences of *n*-point *L* loop Green's fcts in *Dc(L)* and subdivergences at *(n-2)*-points, *(L+1)*-loop Green's fcts. in *Dc(L+1)*.

- Amounts to picking out the most divergent subdiagrams/subgraphs

However…

Consistency relations across loop orders and dimensions

- On general grounds, one has relations between divergences of *n*-point *L* loop Green's fcts and subdivergences at *(n-2)*-points, *(L+1)*-loop Green's fcts.

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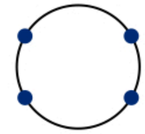
- Amounts to picking out the most divergent subdiagrams/subgraphs

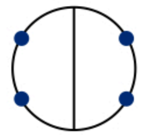
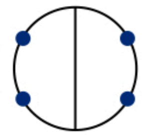
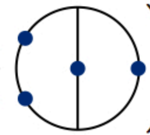
However…

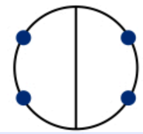
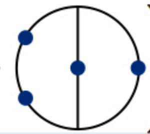
Such relations appear to exist directly between leading UV divergences:

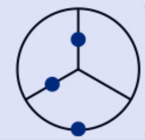
1. For each *L*-loop graph, finds all its *L'*-loop subgraphs
2. Keep the most divergent ones
3. Reduce to a basis of integrals (master integrals)

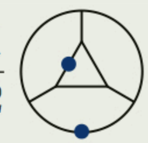
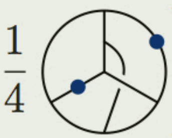
Summary of $\mathcal{N}$=4 SYM UV divergences in the loop-dependent critical dimension

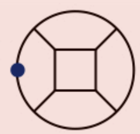
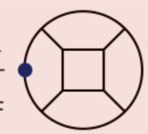
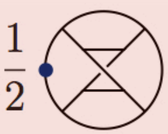
$\mathcal{A}_4^{(1)}\Big|_{\text{leading}} = g^4 \mathcal{K}_{\text{YM}} \left(N_c(\tilde{f}^{a_1 a_2 b}\tilde{f}^{b a_3 a_4} + \tilde{f}^{a_2 a_3 b}\tilde{f}^{b a_4 a_1}) - 3B^{a_1 a_2 a_3 a_4} \right)$  , 8

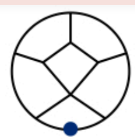
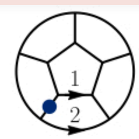
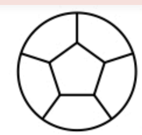
$\mathcal{A}_4^{(2)}\Big|_{\text{leading}} = -g^6 \mathcal{K}_{\text{YM}} \Big[F^{a_1 a_2 a_3 a_4} \Big(N_c^2$  $+ 48 \Big(\frac{1}{4}$  $+ \frac{1}{4}$  $\Big)\Big)$

$+ 48\, N_c\, G^{a_1 a_2 a_3 a_4} \Big(\frac{1}{4}$  $+ \frac{1}{4}$  $\Big)\Big]$, 7

$\mathcal{A}_4^{(3)}\Big|_{\text{leading}} = 2\, g^8\, \mathcal{K}_{\text{YM}}\, N_c F^{a_1 a_2 a_3 a_4} \Big(N_c^2$  $+ 72 \Big(\frac{1}{6}$  $+ \frac{1}{2}$  $\Big)\Big)$, 6

$\mathcal{A}_4^{(4)}\Big|_{\text{leading}} = -6\, g^{10}\, \mathcal{K}_{\text{YM}}\, N_c^2 F^{a_1 a_2 a_3 a_4} \Big(N_c^2$  $+ 48 \Big(\frac{1}{4}$  $+ \frac{1}{2}$  $+ \frac{1}{4}$  $\Big)\Big)$ 11/2

$\mathcal{A}_4^{(5)}\Big|_{\text{leading}} = \frac{144}{5} g^{12} \mathcal{K}_{\text{YM}}\, N_c^3\, F^{a_1 a_2 a_3 a_4} \Big(N_c^2$  $+ 48 \Big(\frac{1}{4}$  $+ \frac{1}{2}$  $+ \frac{1}{4}$  $\Big)\Big)$, 26/5

$\mathcal{A}_4^{(6)}\Big|_{\text{leading}} = -120 g^{14} \mathcal{K}_{\text{YM}}\, F^{a_1 a_2 a_3 a_4} N_c^6 \Big(\frac{1}{2}$  $+ \frac{1}{4}(\ell_1 + \ell_2)^2$  $- \frac{1}{20}$  $\Big)$ 5

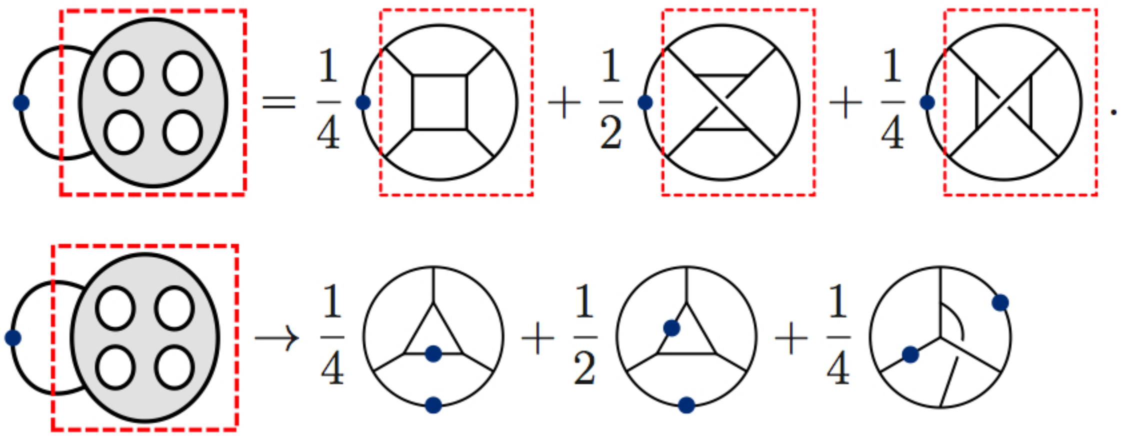
$+ \mathcal{O}(N_c^4)$,

$$F^{a_1 a_2 a_3 a_4} \equiv t\, \tilde{f}^{a_1 a_2 b}\tilde{f}^{b a_3 a_4} + s\, \tilde{f}^{a_2 a_3 b}\tilde{f}^{b a_4 a_1}\,,$$

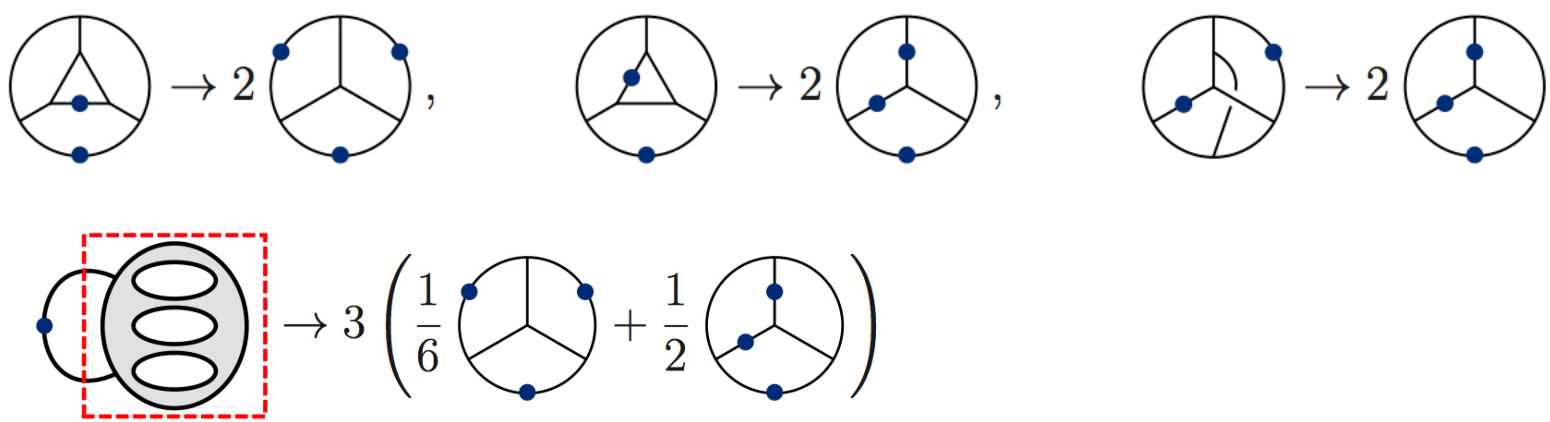
$$G^{a_1 a_2 a_3 a_4} \equiv s\, \delta^{a_1 a_2}\delta^{a_3 a_4} + t\, \delta^{a_4 a_1}\delta^{a_2 a_3} + u\, \delta^{a_1 a_3}\delta^{a_2 a_4}$$

$$B^{a_1 a_2 a_3 a_4} \equiv \tilde{f}^{a_1 b_1 b_2}\tilde{f}^{a_2 b_2 b_3}\tilde{f}^{a_3 b_3 b_4}\tilde{f}^{a_4 b_4 b_1}\,.$$

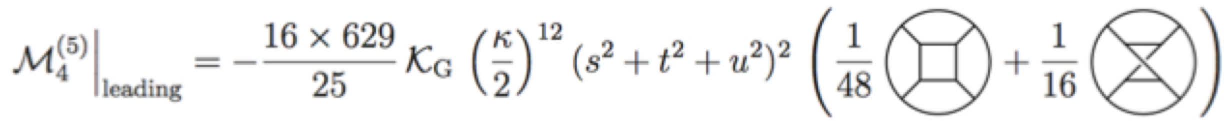
5-loop YM → 4-loop YM

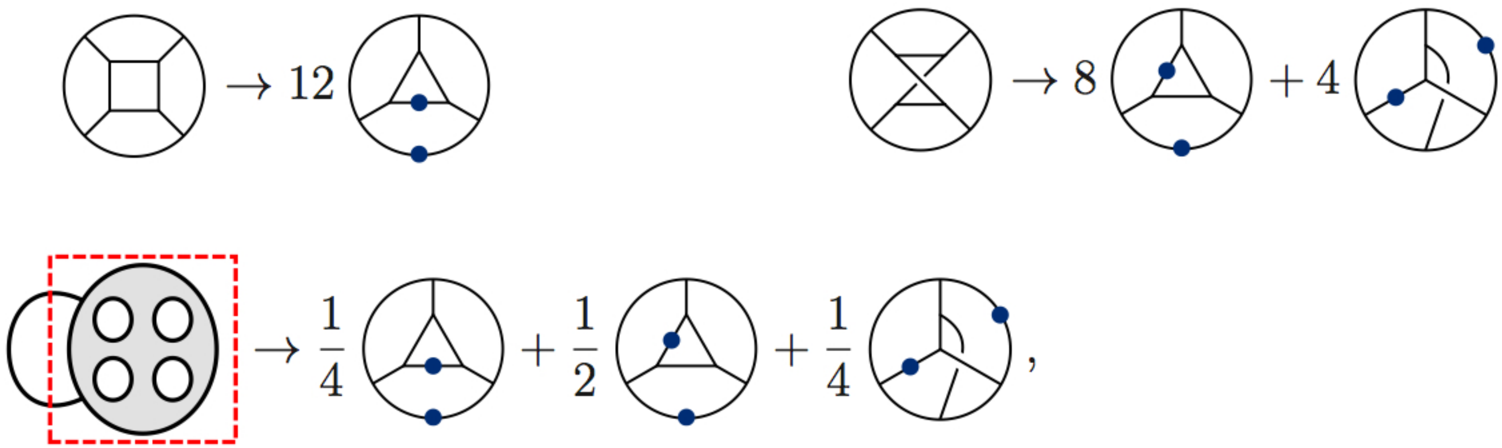


4-loop YM → 3-loop YM

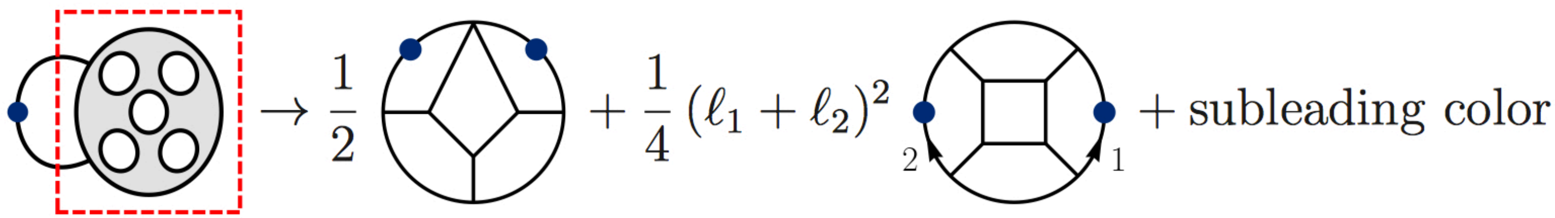


5-loop SG → 4-loop SG

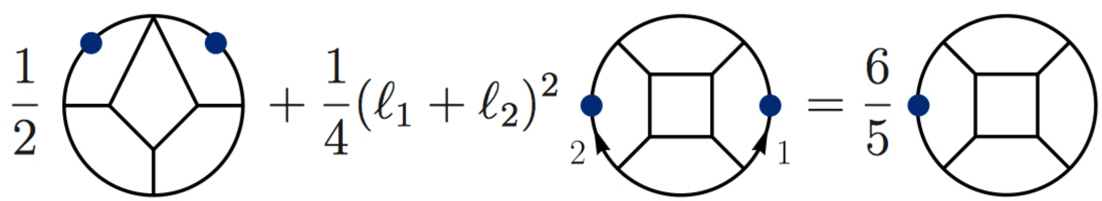




In general, it is necessary to reduce to master (independent) integrals to see these, Relations, though for suitably-chosen basis it may sometimes be avoided



An integral identity:



Possible uses:

- Cross checks of direct calculations at higher loops

- Combine with observation of absence of triangles and constrain an ansatz for the UV divergence at higher loops

- Simplify the extraction of UV divergences by focusing on the integrals that are expected to appear

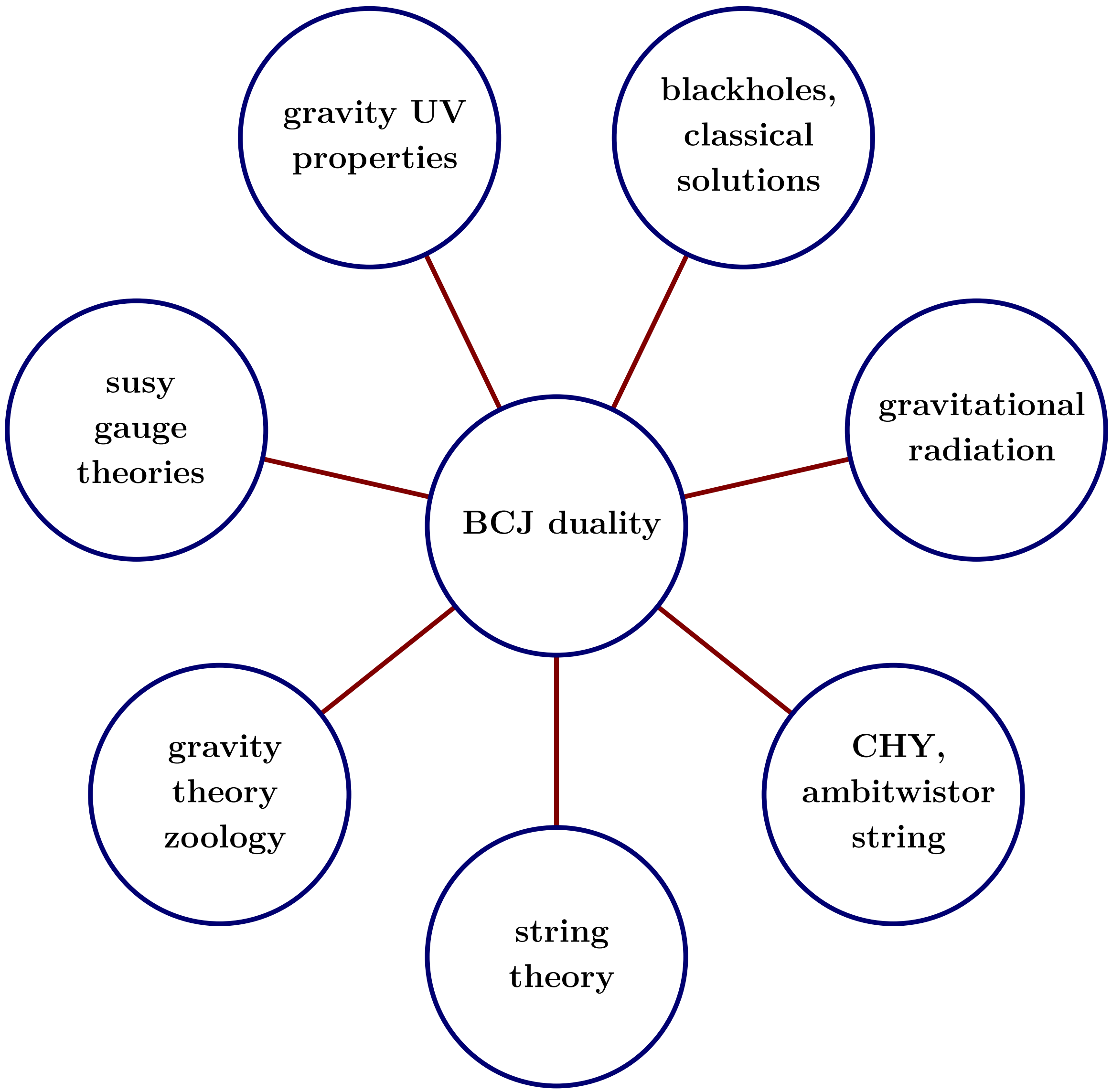
Overview

- Discussed and illustrated: color/kinematics duality
  the double copy & generalized double copy construction
  generalized unitarity as the maximal cut method
  new patterns that emerged from higher-loop calculations

- Many (perhaps all?) gravitational theories enjoy a double copy structure

- So do many nongravitational theories

- While originally developed for the study of UV properties of gravitational theories it has a vast number of other possible applications and relations to other areas of current research, some of which were also mentioned in other lectures


BCJ duality
gravity UV properties
blackholes, classical solutions
gravitational radiation
CHY, ambitwistor string
string theory
gravity theory zoology
susy gauge theories