Max Planck Institute for Mathematics
California Institute of Technology

Supersymmetric Indices and Topology

based on

arXiv:1701.06567

arXiv:1705.01645

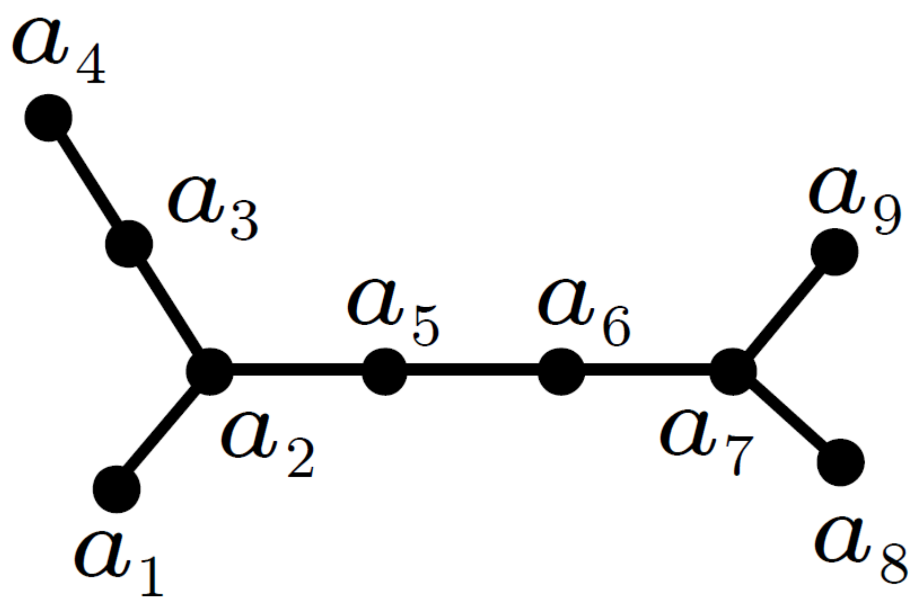
Chapter One

Zed

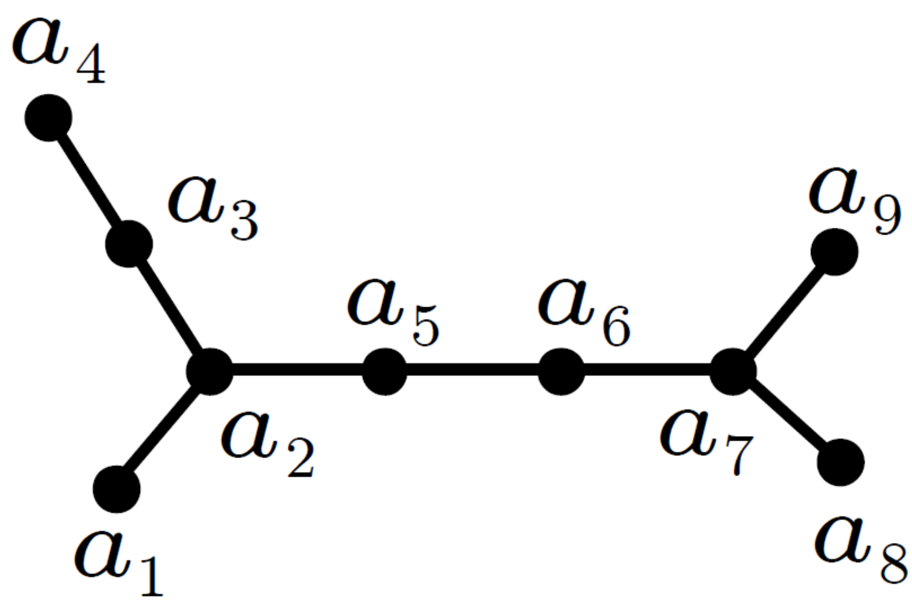
Who's Zed?

Definition: A *conformal field theory* is a table of integrals.

- Brian Greene



$$Z = \int \prod_{v \in \text{Vertices}} \frac{dx_v}{2\pi i x_v} \cdots \prod_{(v_1, v_2) \in \text{Edges}} \left(\cdots \right)$$



$$Q_{ij} = \begin{cases} a_i, & \text{if } i = j \\ 1, & \text{if } i \text{ is connected to } j \text{ by an edge} \\ 0, & \text{otherwise} \end{cases}$$

vertex 

$$\longleftrightarrow \int_{-\infty}^{+\infty} du\; 2(\sinh \pi u)^2\, e^{\frac{\pi i a u^2}{2}}$$

edge

$$\longleftrightarrow \frac{\sin \pi u v}{\sqrt{8i}\, \sinh \pi u \sinh \pi v}$$

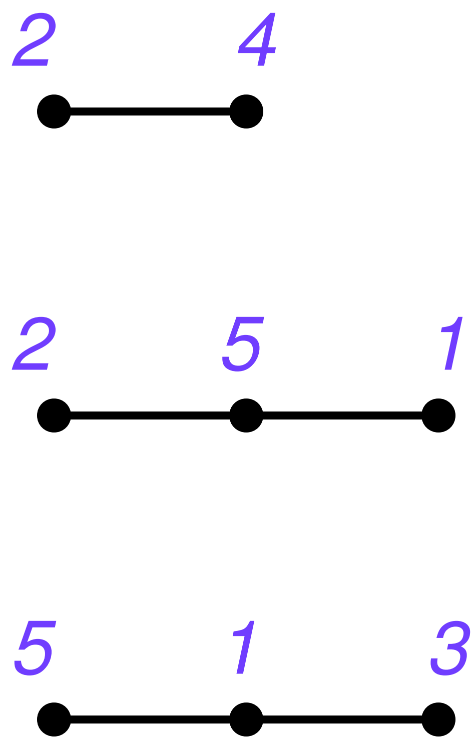
vertex 

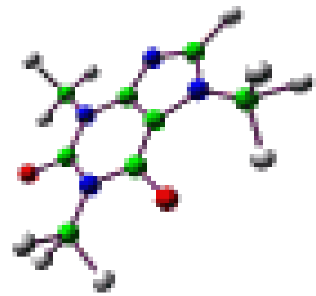
$$\longleftrightarrow \quad \int_{e^{i\epsilon}\mathbb{R}} du\; 2(\sinh \pi u)^2\, e^{\frac{\pi i a u^2}{2}}$$

edge

$$\longleftrightarrow \quad \frac{\sin \pi u v}{\sqrt{8i}\,\sinh \pi u \,\sinh \pi v}$$

Exercise: compute the integrals for

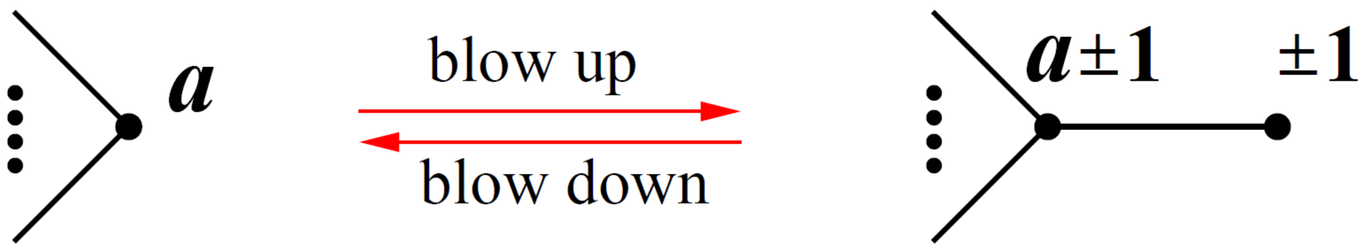


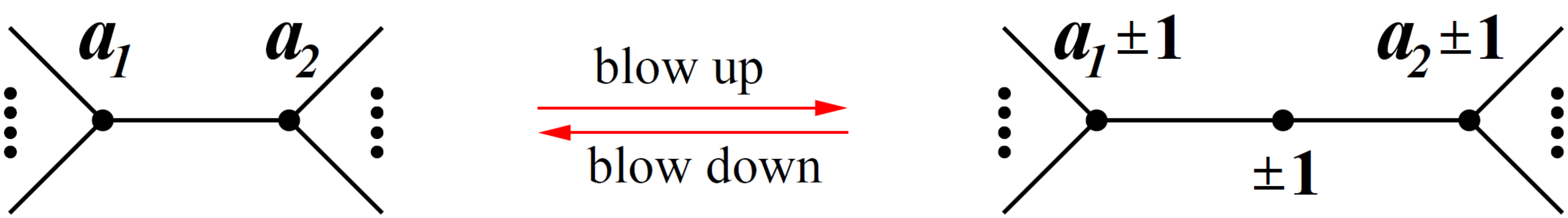


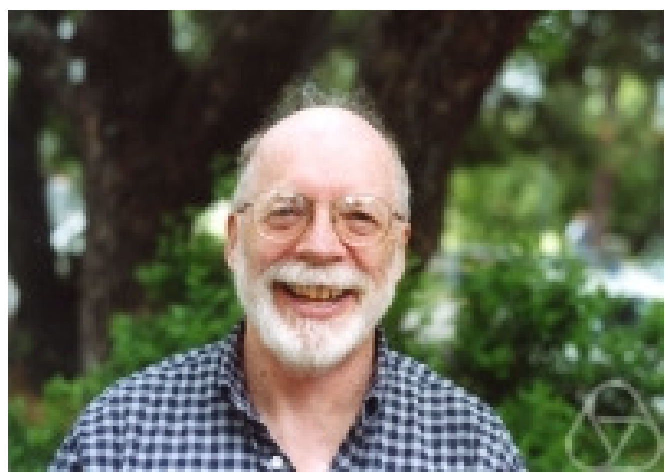
$$\int \prod_{i \in \text{Vertices}} \frac{dx_i}{2\pi i x_i} \frac{1}{1 - \prod_j x_j^{-Q^{ji}}} (1 - x_i)^{\deg(i)-2} x_i^{h_i}$$

$$h \in \operatorname{coker}(Q)$$

Kirby Calculus







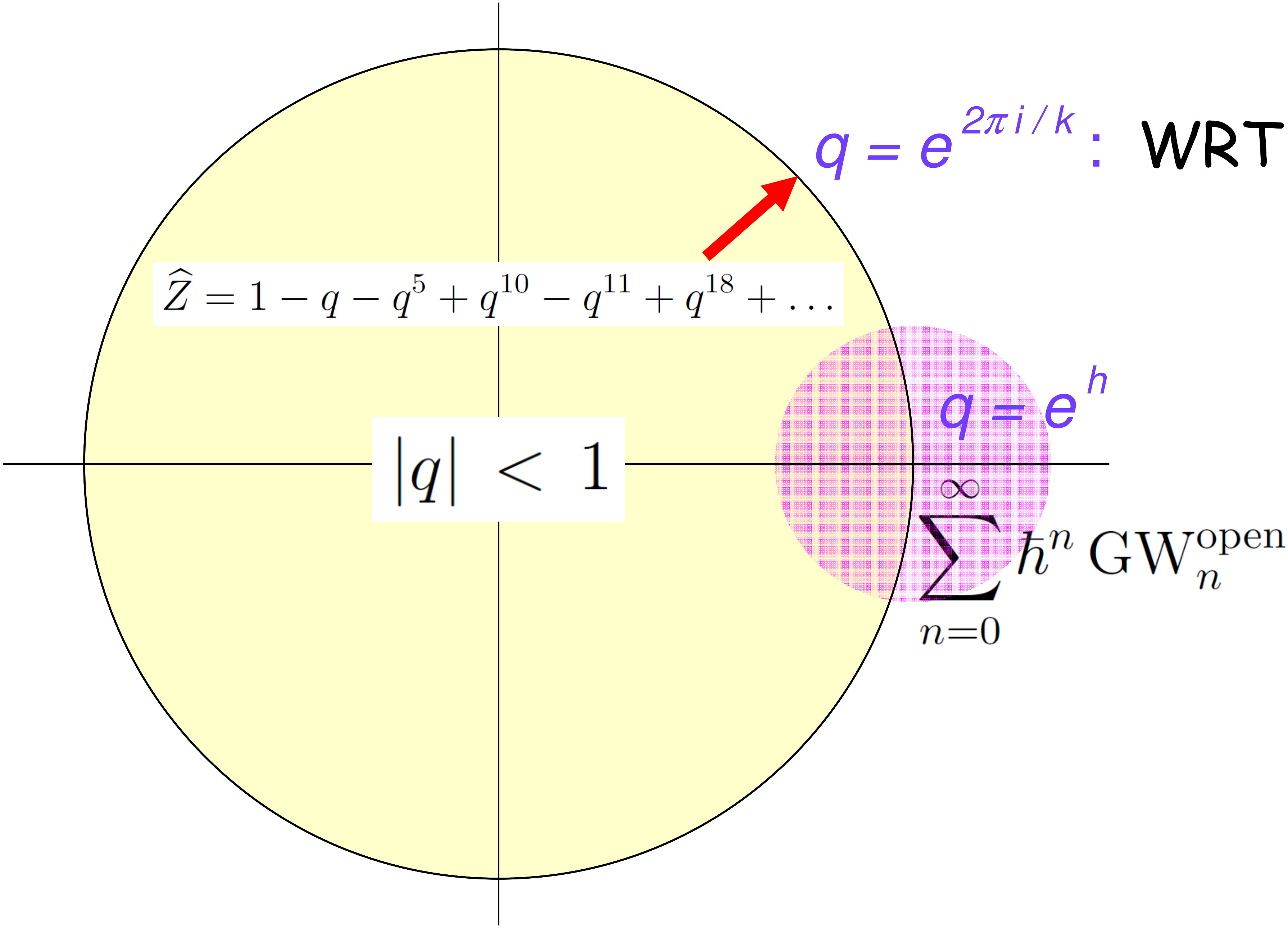
vertex 

$$\longleftrightarrow \quad q^{-\frac{a+3}{4}} \left(x - \frac{1}{x} \right)^2$$

edge

$$\longleftrightarrow \quad \frac{1}{\left(x_1 - \frac{1}{x_1} \right) \left(x_2 - \frac{1}{x_2} \right)}$$

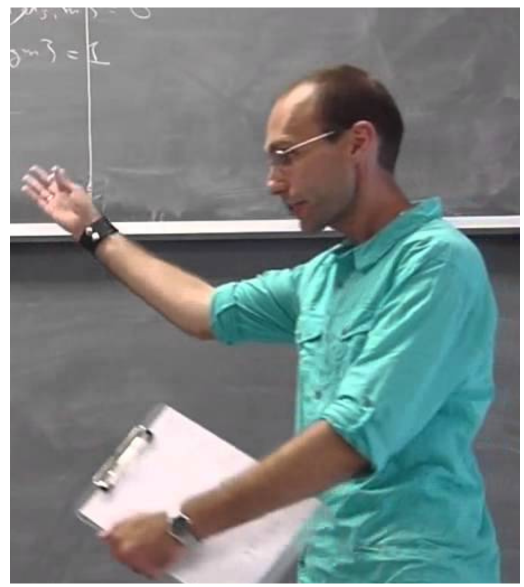
$$\widehat{Z}_a(q) = \text{v.p.} \int_{|x_j|=1} \prod_{j \in \text{Vertices}} \frac{dx_j}{2\pi i x_j} \ldots \prod_{(i,j) \in \text{Edges}} \ldots \Theta_a^Q(\vec{x})$$



$q = e^{2\pi i/k}$: WRT
$\widehat{Z} = 1 - q - q^5 + q^{10} - q^{11} + q^{18} + \ldots$
$q = e^h$
$|q| < 1$
$\sum_{n=0}^{\infty} \hbar^n \, \mathrm{GW}_n^{\mathrm{open}}$

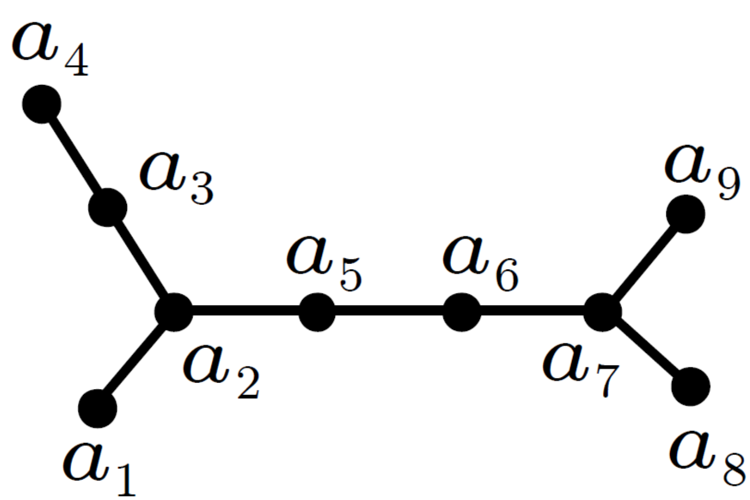
This correspondence should be viewed as a 3-dimensional analog of the AGT correspondence [4] that, in a similar way, associates to a Riemann surface C (possibly with punctures) a four-dimensional $\mathcal{N} = 2$ "effective" gauge theory:

What is 3d-3d correspondence?

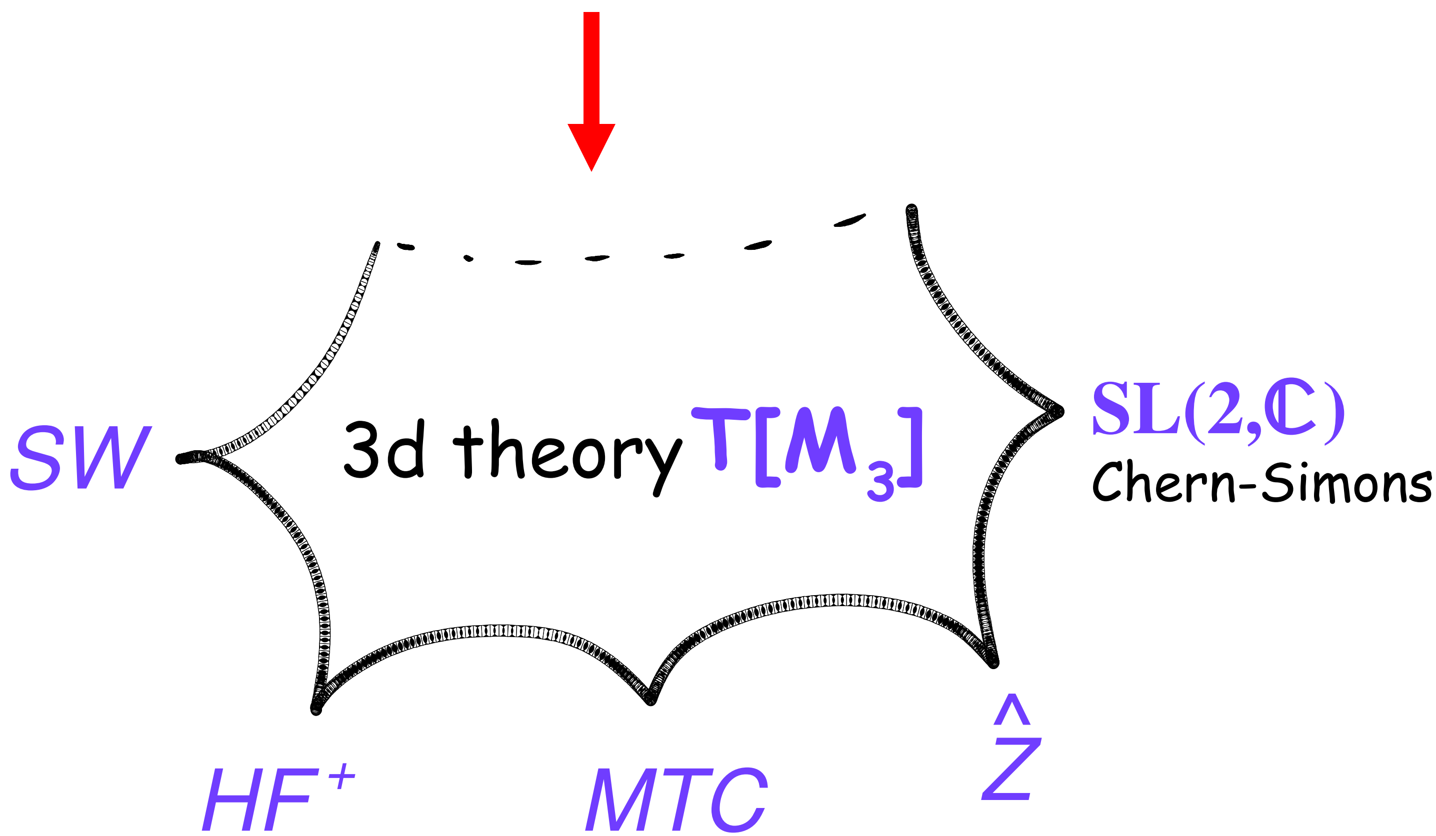


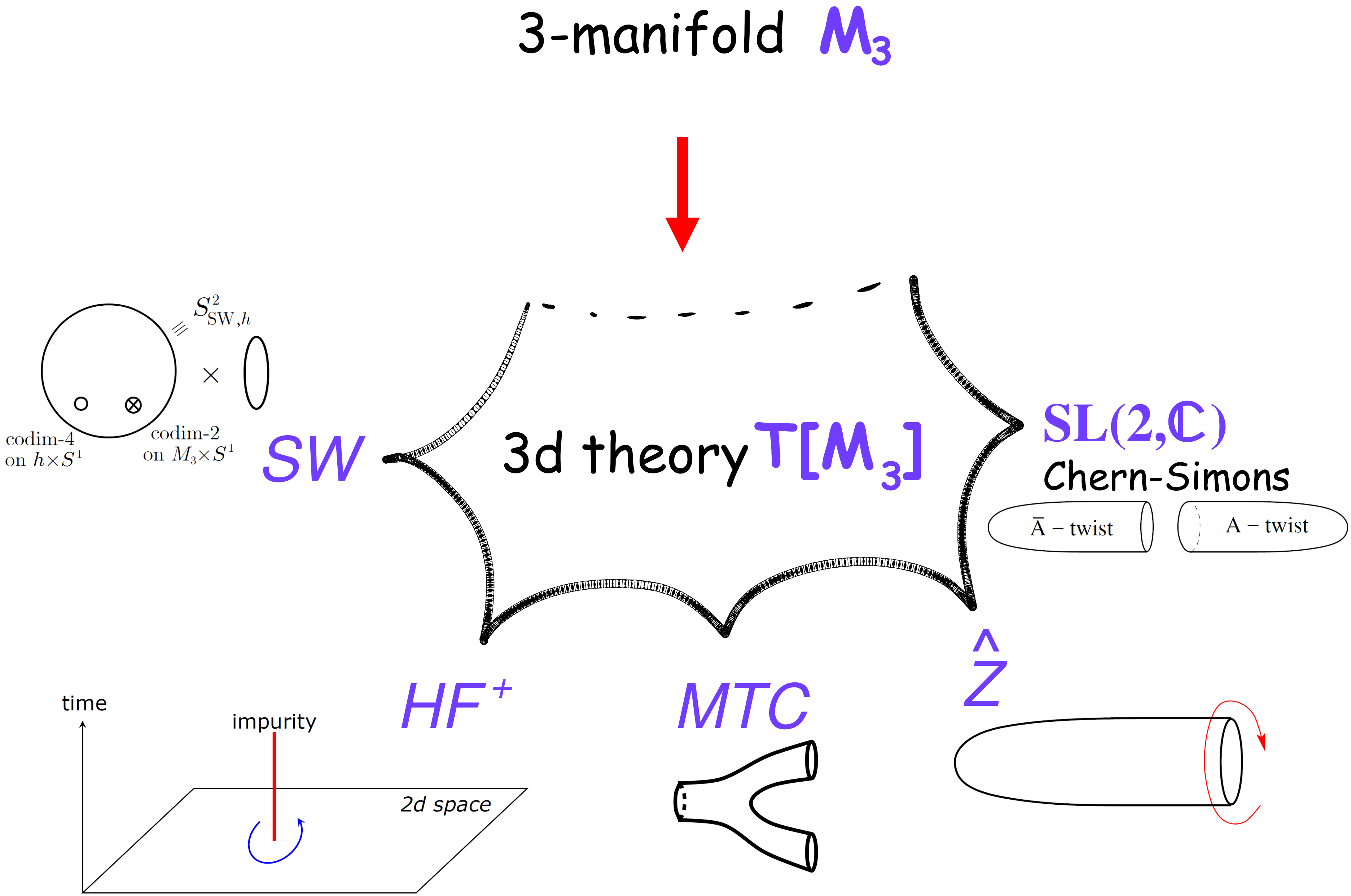
3-manifold M_3

3d theory $T[M_3]$



3-manifold M_3

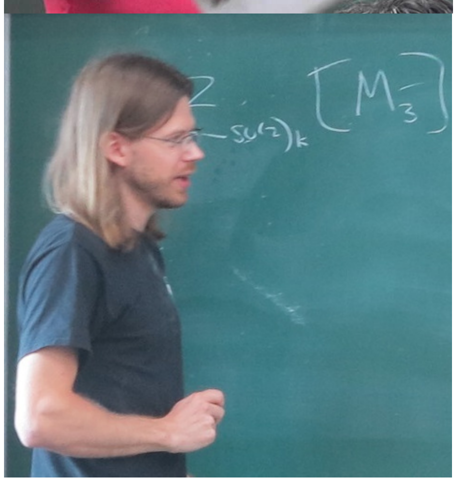


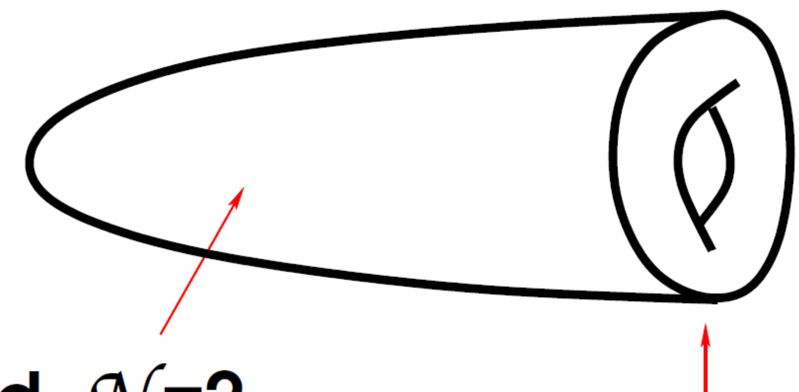


3-manifold M3
3d theory T[M3]
SW
SL(2,ℂ)
Chern-Simons
Ā − twist
A − twist
HF+
MTC
Ẑ
S²SW,h
codim-4 on h×S¹
codim-2 on M3×S¹
time
impurity
2d space

3d-2d half-Index

A.Gadde, S.G., P.Putrov '13
"Walls, Lines, … "

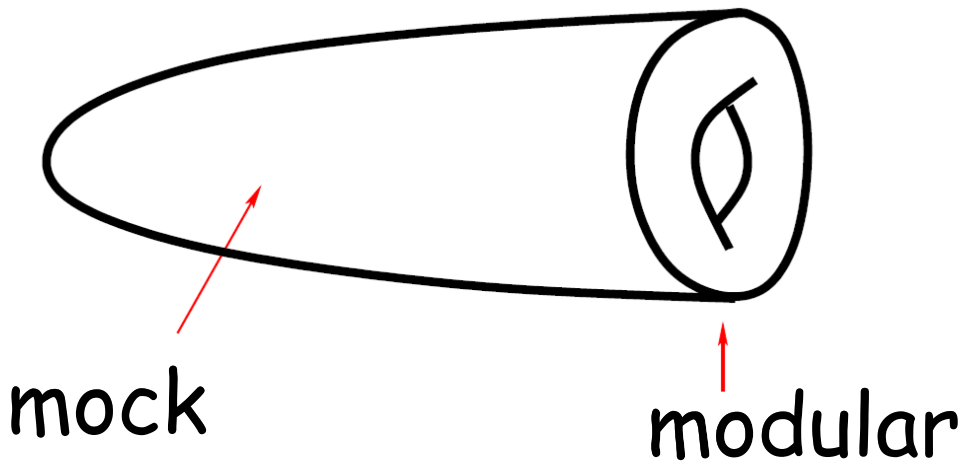




3d $\mathcal{N}$=2 theory

2d $\mathcal{N}$=(0,2) boundary condition

$$\hat{Z}_a(q) = Z(D^2 \times_q S^1; \mathcal{B}_a)$$

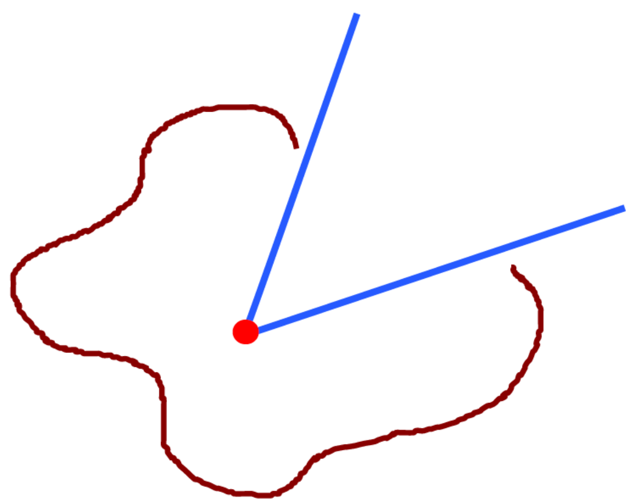


Chapter Two

4-manifolds

$$\begin{cases} F_A^+ = \sum_{i=1}^{N_f} (\Psi_i \overline{\Psi}_i)^+ \\ \not{D}\Psi_i = 0 \qquad i = 1, \ldots, N_f \end{cases}$$

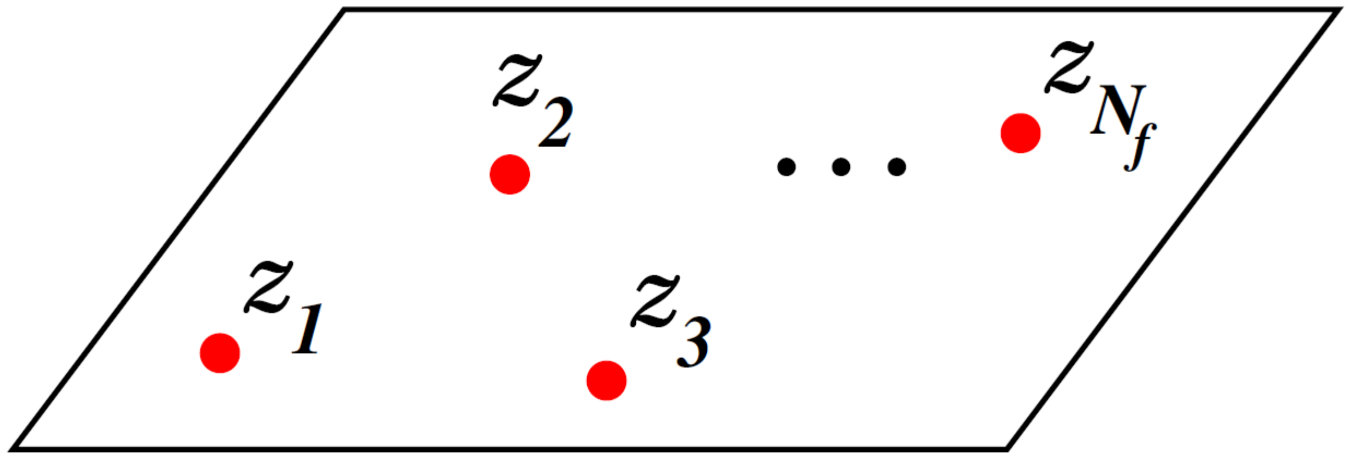
⋮
[J.Bryan, R.Wentworth]
[A.Haydys]
[A.Haydys, T. Walpuski]
⋮



$$\begin{cases} F_A^+ = \sum_{i=1}^{N_f} (\Psi_i \overline{\Psi}_i)^+ \\ \not{D}\Psi_i = 0 \qquad i = 1, \ldots, N_f \end{cases}$$

$$\int_{\mathcal{M}_{N_f}} c_1(\mathcal{L})^{d/2}$$

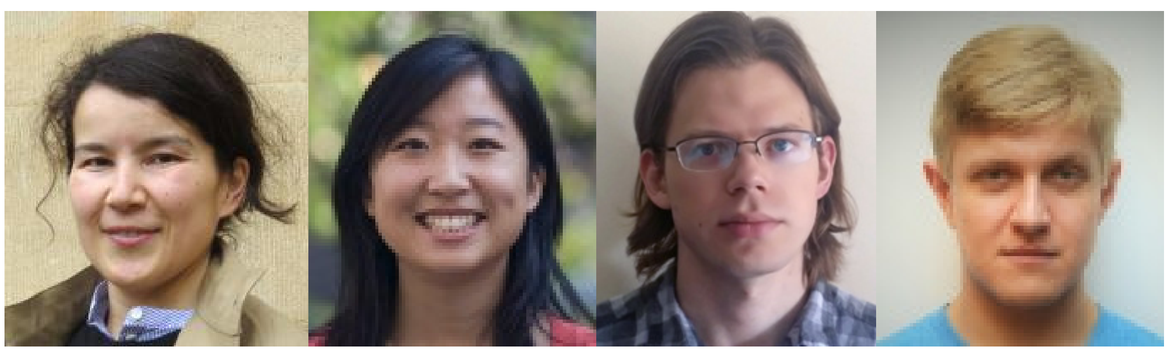
$$SU(N_f) \circlearrowright \mathcal{M}_{N_f}(M_4; \lambda)$$



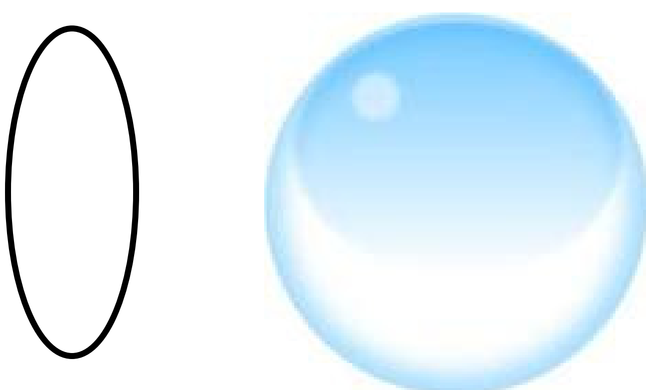
$$\int_{\mathcal{M}_{N_f}} c_1(\mathcal{L})^{d/2} = \langle 0| \, \mathcal{S}(z_1) \, \ldots \, \mathcal{S}(z_{N_f}) \, |\lambda\rangle_{T[M_4]}$$

2d $\mathcal{N} = (0,2)$ theory $T[M_4]$

Field		Chirality	Description
X_R^i	$i = 1, \ldots, b_2^+$	right-moving	compact real bosons
σ^i	$i = 1, \ldots, b_2^+$	non-chiral	non-compact real bosons
ψ_+^i	$i = 1, \ldots, b_2^+$	right-moving	complex Weyl fermions
$\phi_0, \overline{\phi}_0$		non-chiral	non-compact complex boson
χ_+		right-moving	complex Weyl fermion
X_L^j	$j = 1, \ldots, b_2^-$	left-moving	compact real bosons
A^k	$k = 1, \ldots, b_1$	non-chiral	vector fields
γ_-^k	$k = 1, \ldots, b_1$	left-moving	complex Weyl fermions

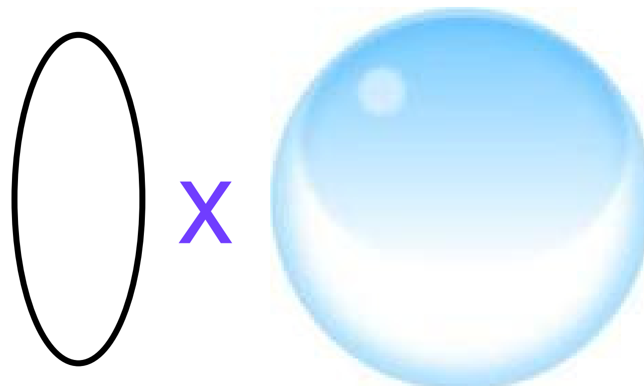


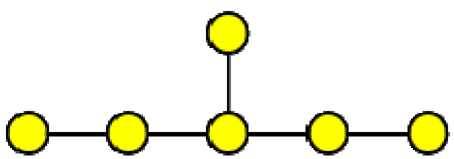
Topological $S^1 \times S^3$ index = Coulomb branch index



[M.Dedushenko, S.G., P.Putrov]

$$M_4 = S^1 \times S^3$$





E_6 SCFT

$$Z(M_4) = \frac{1}{1 - t^3}$$

$$M_4 = S^1 \times \text{Lens space}$$

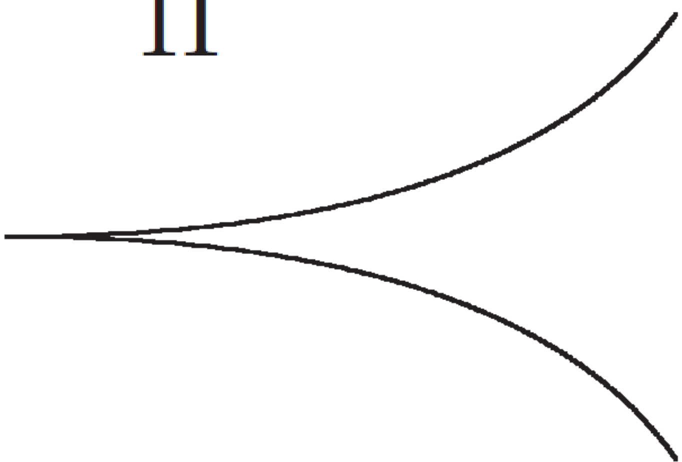
$$Z(M_4) = \frac{1}{(1-\mathfrak{t}^{\frac{2}{5}})(1-\mathfrak{t}^{\frac{3}{5}})} + \frac{\mathfrak{t}^{\frac{k}{5}}}{(1-\mathfrak{t}^{\frac{6}{5}})(1-\mathfrak{t}^{-\frac{1}{5}})}$$

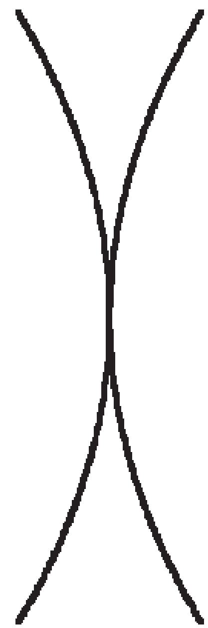
(A_1, A_2) Argyres-Douglas theory

[L.Fredrickson, D.Pei, W.Yan, K.Ye]

4d theory	a	c	$\Delta(u)$	$\dim_{\mathbb{H}}(\text{Higgs})$	Kodaira type
(A_1, A_2)	$\frac{43}{120}$	$\frac{11}{30}$	$\frac{6}{5}$	0	II
(A_1, A_3)	$\frac{11}{24}$	$\frac{1}{2}$	$\frac{4}{3}$	1	III

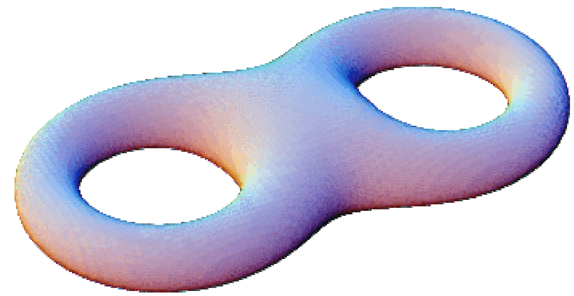
II





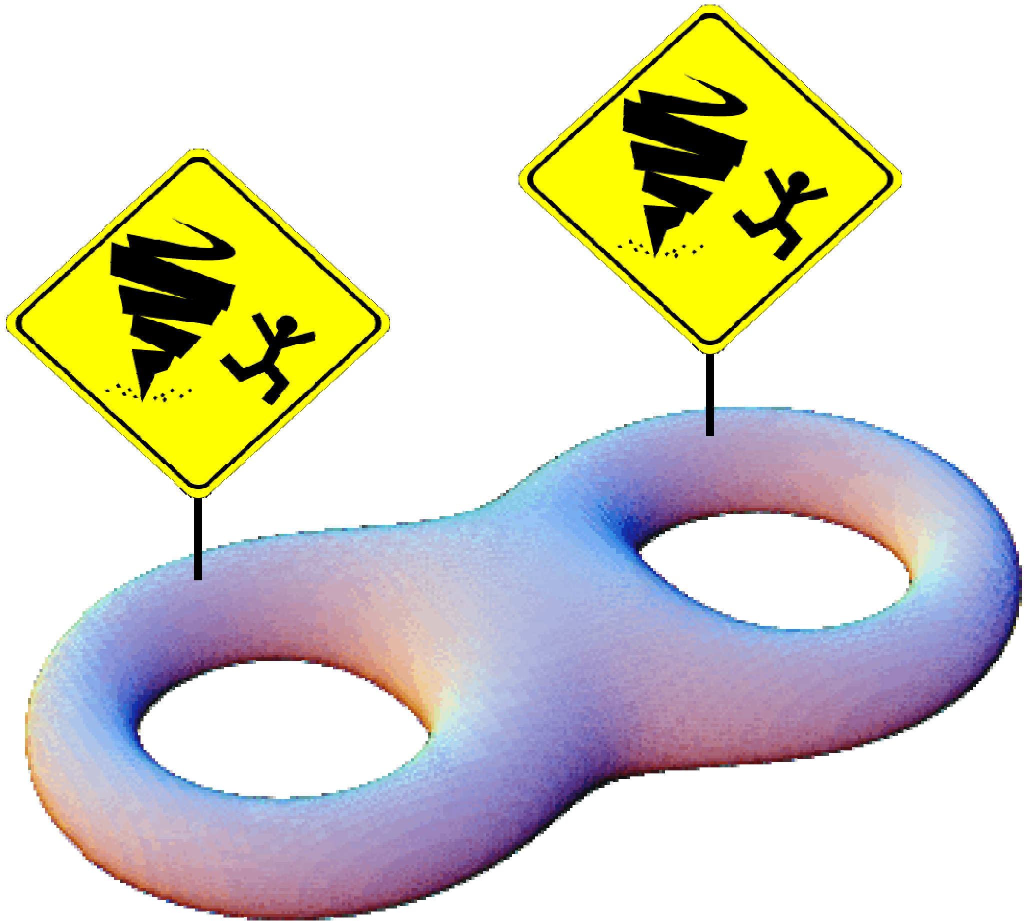
III

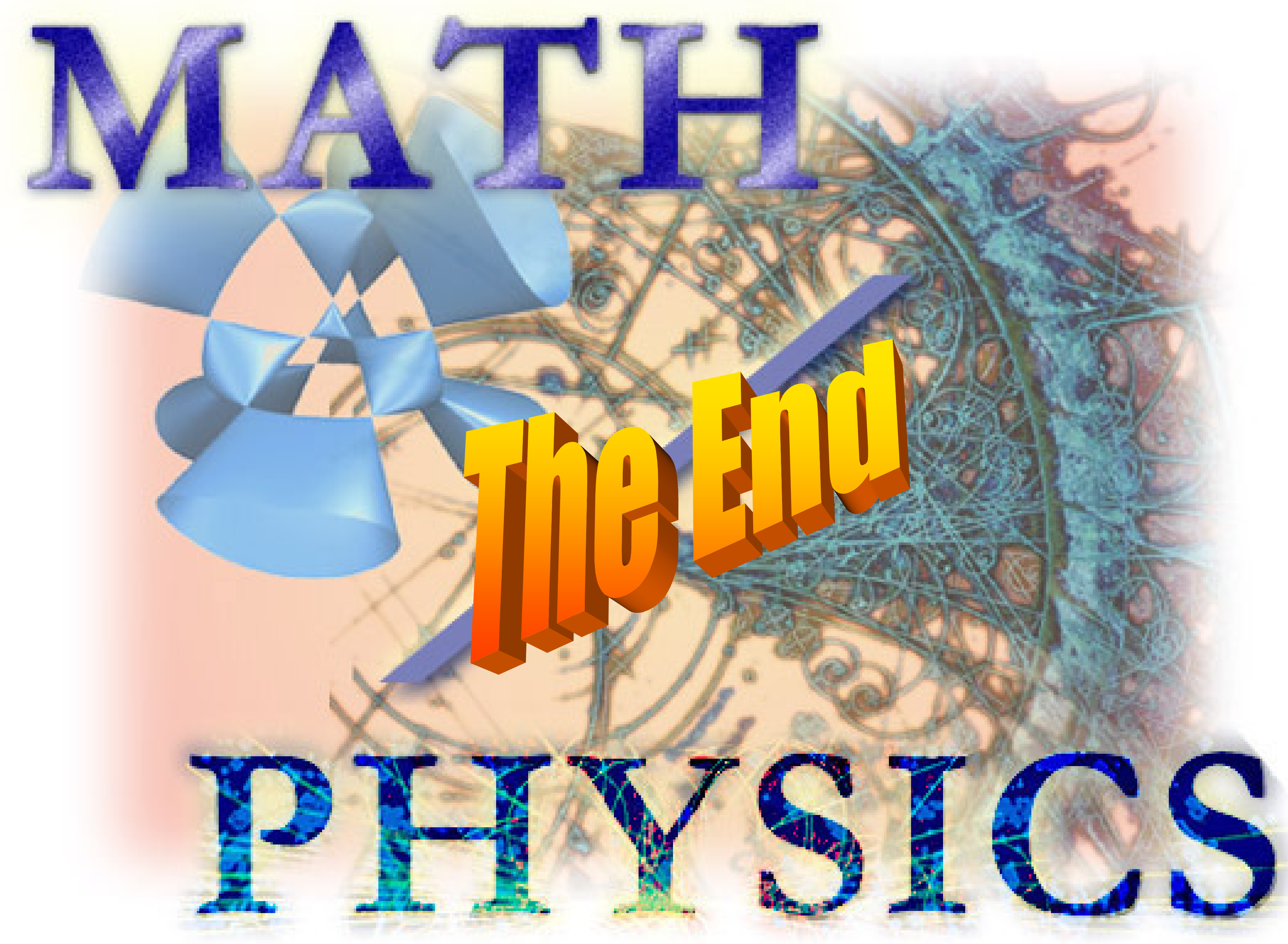
$$M_4 = T^2 \times F_g$$



$$Z = \begin{cases} -\frac{2}{(1-\mathfrak{t}^{1/3})(1-\mathfrak{t}^{4/3})}\,, & g=0 \\ 2\,, & g=1 \\ 8(1+\mathfrak{t}^{1/3}+\mathfrak{t}^{2/3}+\mathfrak{t})\,, & g=2 \\ 8(1+6\,\mathfrak{t}^{1/3}+\mathfrak{t}^{2/3})(1+\mathfrak{t}^{1/3}+\mathfrak{t}^{2/3}+\mathfrak{t})^2\,, & g=3 \\ \quad\vdots & \end{cases}$$

(A_1,A_3) Argyres-Douglas theory




MATH
The End
PHYSICS

Kirby Calculus