Classical and Quantum Gravitational Scattering, and the General Relativistic Two-Body Problem (lecture 2)

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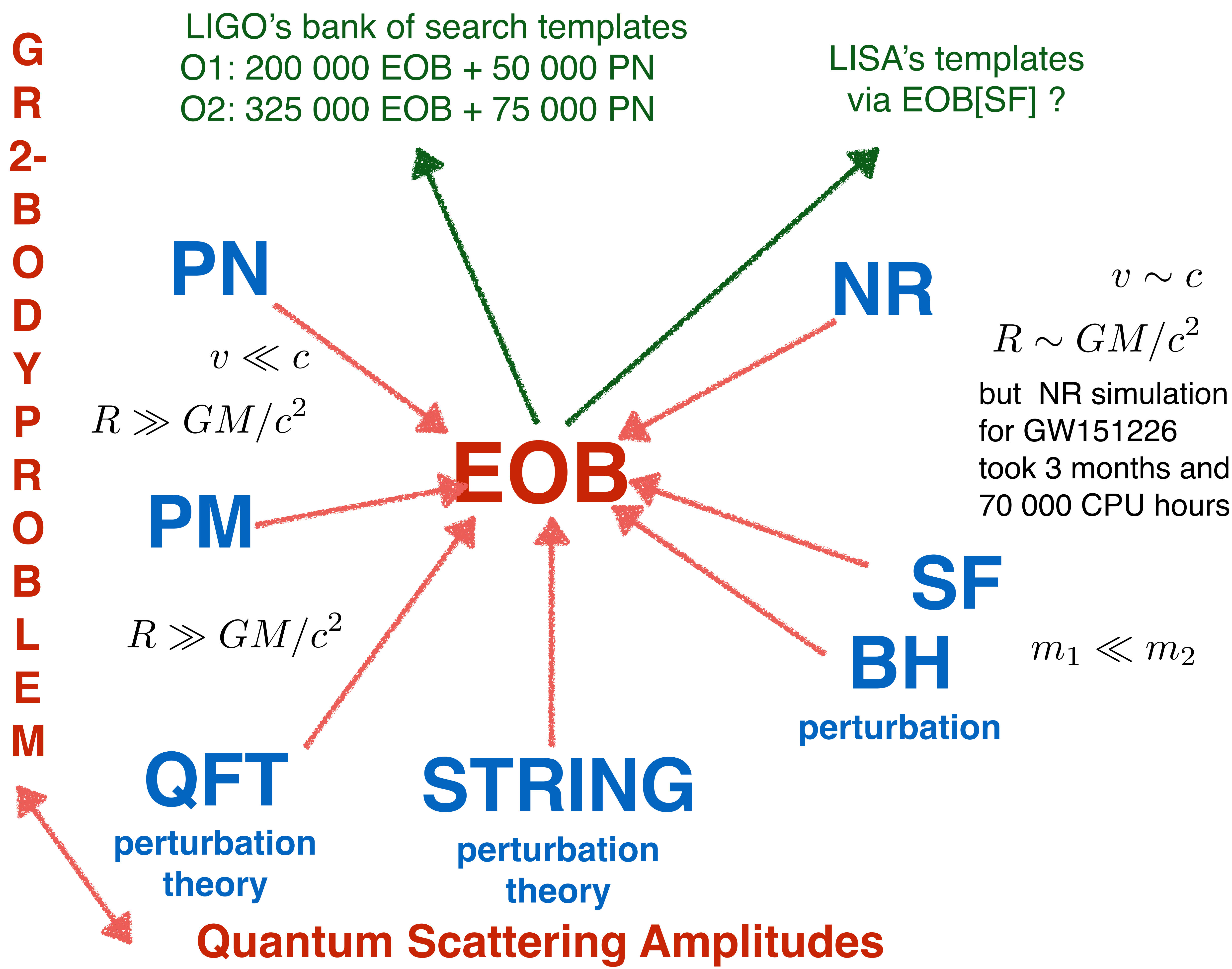

G R 2- B O D Y P R O B L E M
LIGO's bank of search templates
O1: 200 000 EOB + 50 000 PN
O2: 325 000 EOB + 75 000 PN
LISA's templates
via EOB[SF] ?
PN
$v \ll c$
$R \gg GM/c^2$
NR
$v \sim c$
$R \sim GM/c^2$
but NR simulation
for GW151226
took 3 months and
70 000 CPU hours
EOB
PM
$R \gg GM/c^2$
SF
BH
perturbation
$m_1 \ll m_2$
QFT
perturbation
theory
STRING
perturbation
theory
Quantum Scattering Amplitudes

TWO-BODY/EOB "CORRESPONDENCE":

THINK QUANTUM-MECHANICALLY (J.A. WHEELER)

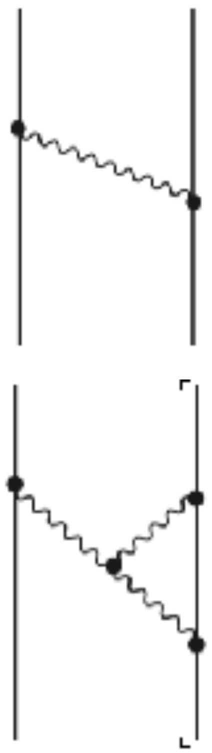
Real 2-body system
(in the c.o.m. frame)
(m_1, m_2)

1:1 map

An effective particle
in some effective metric

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$g_{\mu\nu}^{\text{eff}}$$



+ PN expansion in v^2/c^2 and G/c^2

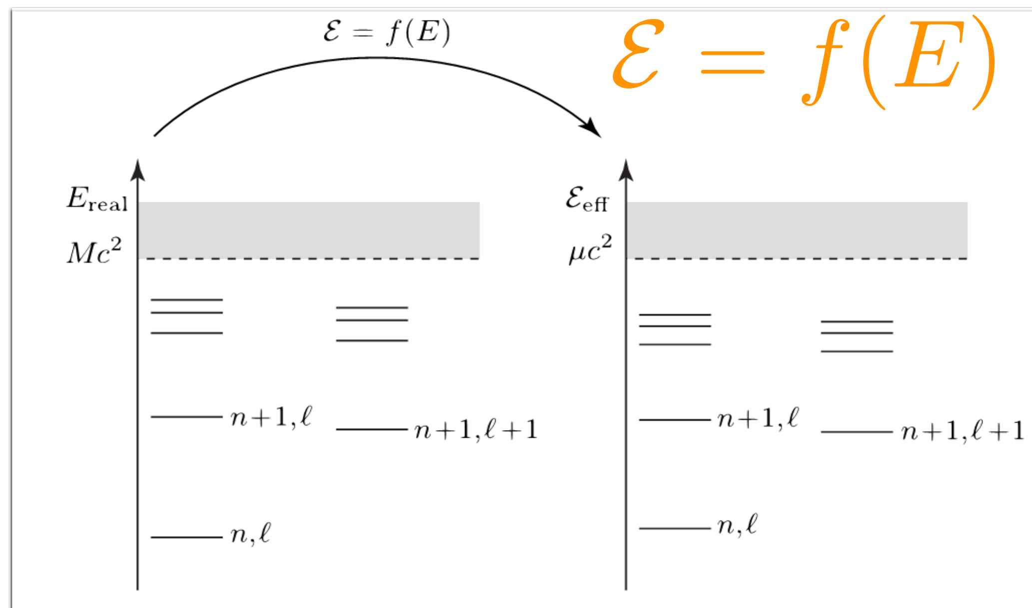


Figure 1: Sketch of the correspondence between the quantized energy levels of the real and effective conservative dynamics. n denotes the 'principal quantum

$$\mathcal{E} = f(E)$$

$$\mu^2 + g_{\text{eff}}^{\mu\nu} \frac{\partial S_{\text{eff}}}{\partial x^\mu} \frac{\partial S_{\text{eff}}}{\partial x^\nu} + \mathcal{O}(p^4) = 0$$

Bohr-Sommerfeld's
Quantization Conditions
(action-angle variables &
Delaunay Hamiltonian)

$$J = \ell \hbar = \frac{1}{2\pi} \oint p_\varphi d\varphi$$

$$N = n\hbar = I_r + J$$

$$I_r = \frac{1}{2\pi} \oint p_r dr$$

$$H^{\text{classical}}(q,p) \longrightarrow H^{\text{classical}}(I_a) \longrightarrow E^{\text{quantum}}(I_a = n_a h) = f^{-1}[\mathcal{E}_{\text{eff}}^{\text{quantum}}(I_a^{\text{eff}} = n_a h)]$$

3PN EOB

PN-expanded energy map

E_real=\cal E_real-Mc^2

$$\frac{\mathcal{E}_{\rm eff}}{m_0 c^2} = 1 + \frac{E_{\rm real}}{\mu c^2}\left(1 + \alpha_1 \frac{E_{\rm real}}{\mu c^2} + \alpha_2 \left(\frac{E_{\rm real}}{\mu c^2}\right)^2 + \dots\right)$$

post-geodesic effective action

$$0 = m_0^2 + g_{\rm eff}^{\alpha\beta}(x)\, p_\alpha p_\beta + A^{\alpha\beta\gamma\delta}(x)\, p_\alpha p_\beta p_\gamma p_\delta + \cdots$$

$$ds^2_{\rm eff} = -A(r;\nu)\, dt^2 + B(r;\nu)\, dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$$\frac{\mathcal{E}_{\rm eff}}{\mu c^2} = \frac{(\mathcal{E}_{\rm real}^{\rm tot})^2 - m_1^2 c^4 - m_2^2 c^4}{2\, m_1\, m_2\, c^4} \quad \text{(at the 1PN, 2PN, 3PN, and 4PN levels).}$$

$$\hat{H}^{\rm R}_{\rm eff}(\mathbf{q}', \mathbf{p}') = \sqrt{A(q')\left[1 + \mathbf{p}'^2 + \left(\frac{A(q')}{D(q')} - 1\right)(\mathbf{n}'\cdot\mathbf{p}')^2 + \frac{1}{q'^2}\left(z_1(\mathbf{p}'^2)^2 + z_2\,\mathbf{p}'^2(\mathbf{n}'\cdot\mathbf{p}')^2 + z_3(\mathbf{n}'\cdot\mathbf{p}')^4\right)\right]},$$

u=GM/(c^2r)

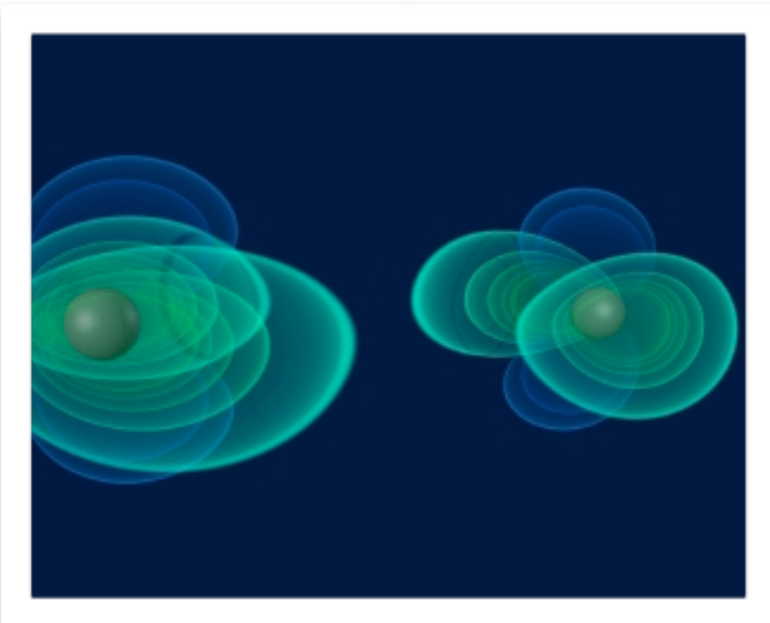
$$A^{\rm 3PN}(u) = 1 - 2u + 2\nu\, u^3 + \left(\frac{94}{3} - \frac{41}{32}\pi^2\right)\nu u^4,$$

AB=D=1/\bar D

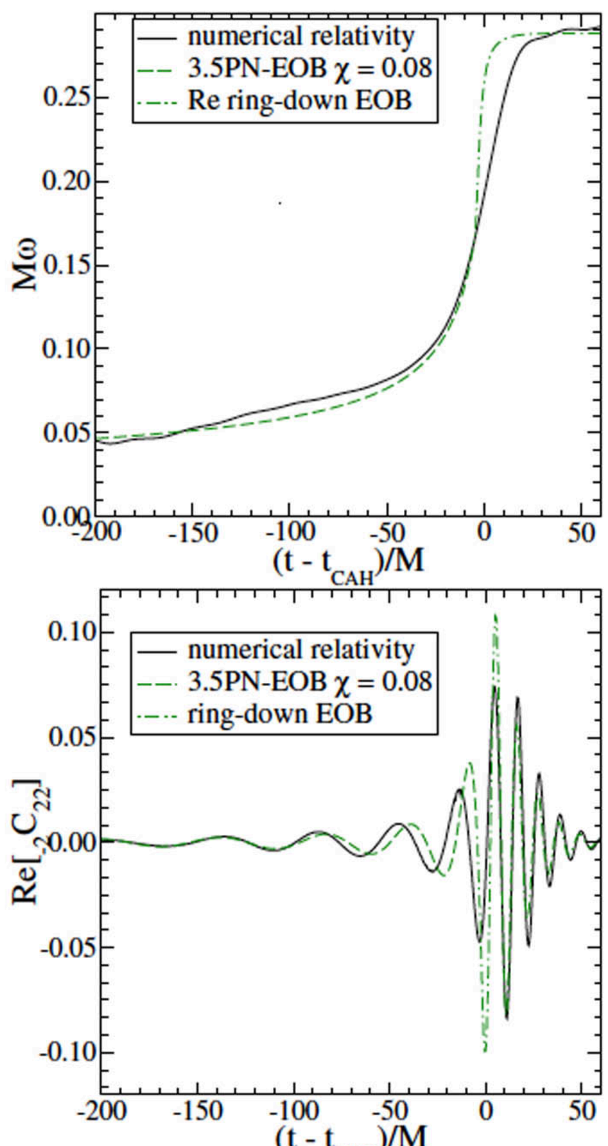
$$\overline{D}^{\rm 3PN}(u) = 1 + 6\nu\, u^2 + (52\nu - 6\nu^2)\, u^3,$$

$$\widehat{Q}^{\rm 3PN} \equiv \frac{Q}{\mu^2 c^2} = (8\nu - 6\nu^2)\, u^2\, \frac{p_r^4}{c^4}.$$

NR, EOB[NR] AND EOB MAIN RADIAL POTENTIAL A(R)



Buonanno-Cook-Pretorius 2007



Mathematical foundations : Darmois 27, Lichnerowicz 43,Choquet-Bruhat 52-

Breakthrough:

Pretorius 2005: generalized harmonic coordinates (Friedrich,Garfinkle); constraint damping (Brodbeck et al., Gundlach et al., Pretorius, Lindblom et al.); excision;

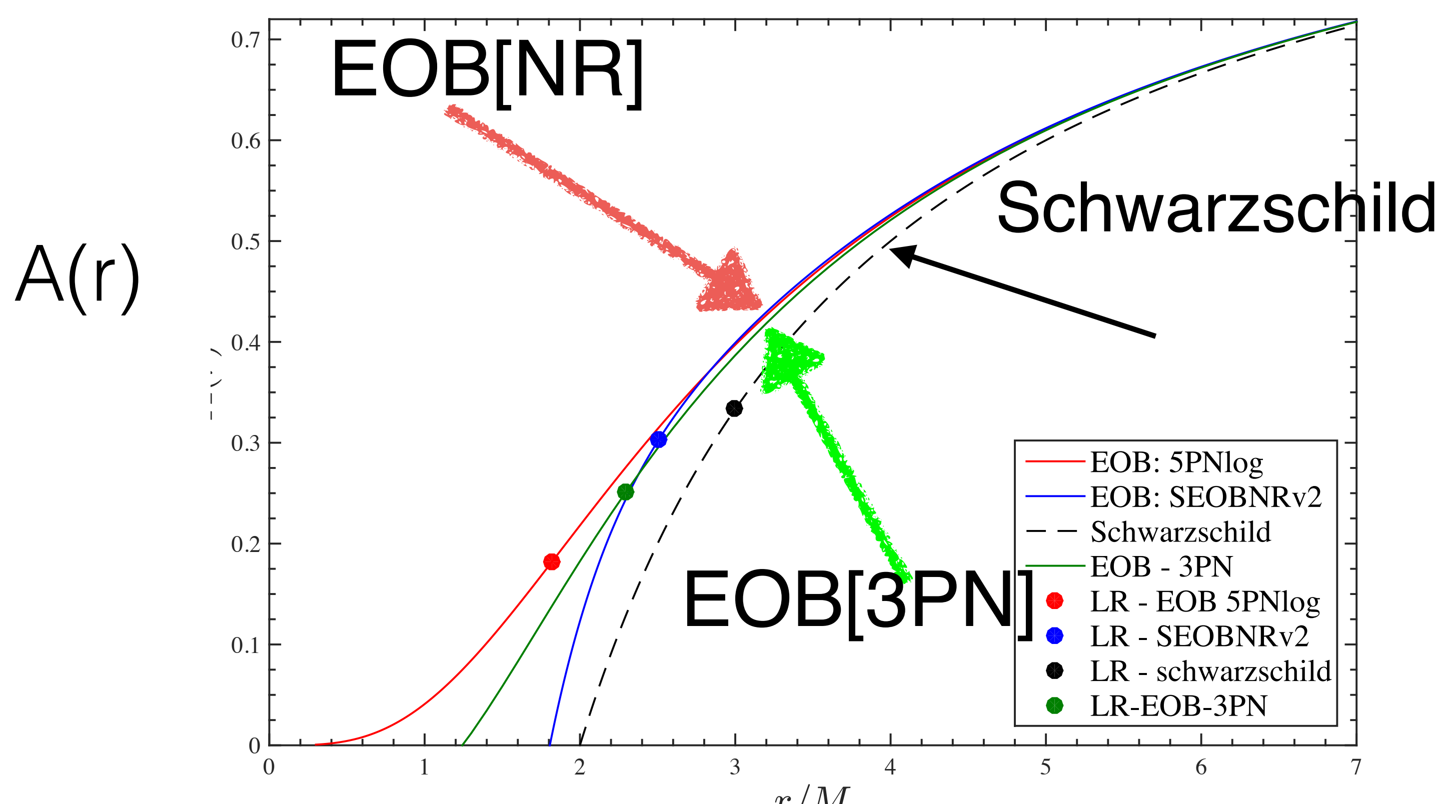
Moving punctures: Campanelli-Lousto-Maronetti-Zlochover 2006

Baker-Centrella-Choi-Koppitz-van Meter 2006

$$A(u;\nu,a_6^c) = P_5^1\Big[1 - 2u + 2\nu u^3 + \nu\left(\frac{94}{3} - \frac{41}{32}\pi^2\right)u^4 + \nu\left[-\frac{4237}{60} + \frac{2275}{512}\pi^2 + \left(-\frac{221}{6} + \frac{41}{32}\pi^2\right)\nu + \frac{64}{5}\ln(16e^{2\gamma}u)\right]u^5 + \nu\left[a_6^c(\nu) - \left(\frac{7004}{105} + \frac{144}{5}\nu\right)\ln u\right]u^6\Big]$$

$$a_6^{c\,\mathrm{NR\text{-}tuned}}(\nu) = 81.38 - 1330.6\,\nu + 3097.3\,\nu^2$$

m1=m2 case



Link radiative multipoles <-> source variables

(Blanchet-Damour '89'92, Damour-Iyer'91, Blanchet '95…)

$$H_{ij}^{\mathrm{TT}}(U,\mathbf{X}) = \frac{4G}{c^2R}\mathcal{P}_{ijab}(\boldsymbol{N})\sum_{\ell=2}^{+\infty}\frac{1}{c^\ell \ell!}\left\{N_{L-2}\mathrm{U}_{abL-2}(U) - \frac{2\ell}{c(\ell+1)}N_{cL-2}\epsilon_{cd(a}\mathrm{V}_{b)dL-2}(U)\right\} + \mathcal{O}\left(\frac{1}{R^2}\right).$$

$$\begin{aligned}\mathrm{U}_{ij}(U) = {} & \mathrm{M}_{ij}^{(2)}(U) + \frac{2GM}{c^3}\int_0^{+\infty} d\tau\,\mathrm{M}_{ij}^{(4)}(U-\tau)\left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{11}{12}\right] \\ & + \frac{G}{c^5}\left\{-\frac{2}{7}\int_0^{+\infty} d\tau\,\mathrm{M}_{a\langle i}^{(3)}(U-\tau)\mathrm{M}_{j\rangle a}^{(3)}(U-\tau)\right. \\ & \left. - \frac{2}{7}\mathrm{M}_{a\langle i}^{(3)}\mathrm{M}_{j\rangle a}^{(2)} - \frac{5}{7}\mathrm{M}_{a\langle i}^{(4)}\mathrm{M}_{j\rangle a}^{(1)} + \frac{1}{7}\mathrm{M}_{a\langle i}^{(5)}\mathrm{M}_{j\rangle a} + \frac{1}{3}\varepsilon_{ab\langle i}\mathrm{M}_{j\rangle a}^{(4)}\mathrm{S}_b\right\} \\ & + \frac{2G^2\mathrm{M}^2}{c^6}\int_0^{+\infty} d\tau\,\mathrm{M}_{ij}^{(5)}(U-\tau)\left[\ln^2\left(\frac{c\tau}{2r_0}\right) + \frac{57}{70}\ln\left(\frac{c\tau}{2r_0}\right) + \frac{124627}{44100}\right] \\ & + \mathcal{O}\left(\frac{1}{c^7}\right).\end{aligned}$$

tail

memory

instant.

tail-of-tail

$$\mathrm{M}_{ij} = \mathrm{I}_{ij} - \frac{4G}{c^5}\left[\mathrm{W}^{(2)}\mathrm{I}_{ij} - \mathrm{W}^{(1)}\mathrm{I}_{ij}^{(1)}\right] + \mathcal{O}\left(\frac{1}{c^7}\right)$$

$$\begin{aligned}\mathrm{I}_L(u) = \mathcal{FP}\int d^3\mathbf{x}\int_{-1}^{1} dz\Big\{ & \delta_l\hat{x}_L\Sigma - \frac{4(2l+1)}{c^2(l+1)(2l+3)}\delta_{l+1}\hat{x}_{iL}\Sigma_i^{(1)} \\ & + \frac{2(2l+1)}{c^4(l+1)(l+2)(2l+5)}\delta_{l+2}\hat{x}_{ijL}\Sigma_{ij}^{(2)}\Big\}(\mathbf{x}, u+z|\mathbf{x}|/c),\end{aligned} \quad (85)$$

$$\mathrm{J}_L(u) = \mathcal{FP}\int d^3\mathbf{x}\int_{-1}^{1} dz\,\varepsilon_{ab\langle i_l}\Big\{\delta_l\hat{x}_{L-1\rangle a}\Sigma_b - \frac{2l+1}{c^2(l+2)(2l+3)}\delta_{l+1}\hat{x}_{L-1\rangle ac}\Sigma_{bc}^{(1)}\Big\}(\mathbf{x}, u+z|\mathbf{x}|/c).$$

$$\Sigma = \frac{\overline{\tau}^{00}+\overline{\tau}^{ii}}{c^2},$$

$$\Sigma_i = \frac{\overline{\tau}^{0i}}{c},$$

$$\Sigma_{ij} = \overline{\tau}^{ij}$$

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Hereditary effects in gravitational radiation

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$$\int_{-\infty}^{U} dV \ln\left(\frac{U-V}{2P^{\mathrm{rad}}}\right) M_L^{(\ell+2)}(V) = \frac{1}{2P^{\mathrm{rad}}} M_L^{(\ell)}(U-2P^{\mathrm{rad}}) + \int_{U-2P^{\mathrm{rad}}}^{U} dV \ln\left(\frac{U-V}{2P^{\mathrm{rad}}}\right) M_L^{(\ell+2)}(V) - \int_{-\infty}^{U-2P^{\mathrm{rad}}} \frac{dV}{(U-V)^2} M_L^{(\ell)}(V) \,. \quad (2.44)$$

The new form (2.44) shows that the influence of the remote-past activity of the source enters radiative moments via a quadratically decreasing kernel $\mathcal{K}_M^{\mathrm{quad}}(U-V) \propto (U-V)^{-2}$. Therefore, in a scattering situation, where $M_L^{(\ell)}(V)$ is expected to have a finite, nonzero, limit as $V \to -\infty$ [see the expressions below relating $M_L(V)$ to the matter distribution] the remote past history of the system gives a "tail" contribution to the radiative moments which falls off only as the inverse of the time span between now and the considered period in the past.

EOB

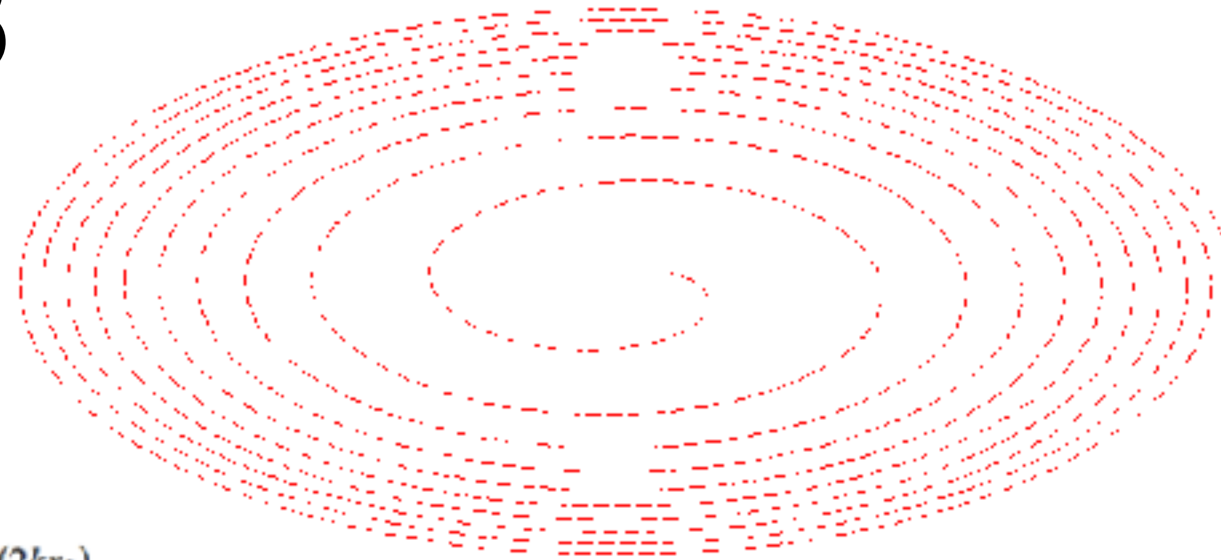
$$\frac{dr}{dt} = \left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{\rm EOB}}{\partial p_{r_*}},$$

$$\frac{dp_{r_*}}{dt} = -\left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{\rm EOB}}{\partial r},$$

$$\Omega \equiv \frac{d\varphi}{dt} = \frac{\partial \hat{H}_{\rm EOB}}{\partial p_\varphi}$$

$$\frac{dp_\varphi}{dt} = \hat{\mathcal{F}}_\varphi .$$

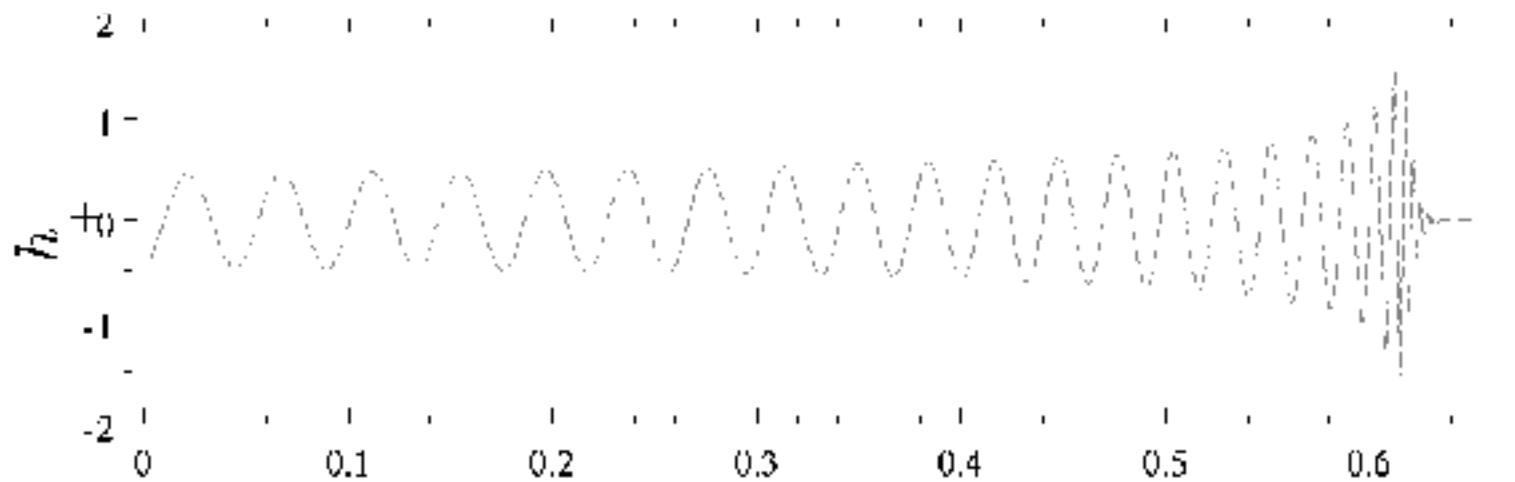
$$T_{\ell m} = \frac{\Gamma(\ell + 1 - 2i\hat{\hat{k}})}{\Gamma(\ell+1)} e^{\pi \hat{k}} e^{2i\hat{k}\log(2kr_0)},$$



$$h_{\ell m} \equiv h_{\ell m}^{(N,\epsilon)} \hat{h}_{\ell m}^{(\epsilon)} \hat{h}_{\ell m}^{\rm NQC}$$

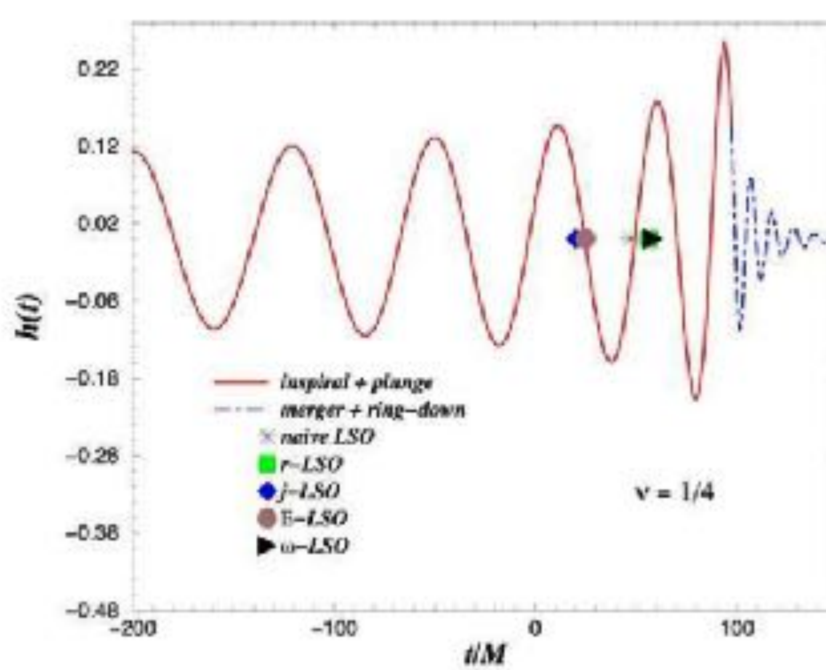
$$\hat{h}_{\ell m}^{(\epsilon)} = \hat{S}_{\rm eff}^{(\epsilon)} T_{\ell m} e^{i\delta_{\ell m}} \rho_{\ell m}^{\ell}$$

$$\mathcal{F}_\varphi \equiv -\frac{1}{8\pi\Omega} \sum_{\ell=2}^{\ell_{\max}} \sum_{m=1}^{\ell} (m\Omega)^2 |R h_{\ell m}^{(\epsilon)}|^2$$



$$h_{\ell m}^{\rm EOB} = \theta(t_m - t) h_{\ell m}^{\rm insplunge}(t) + \theta(t - t_m) h_{\ell m}^{\rm ringdow}$$

$$h_{\ell m}^{\rm ringdown}(t) = \sum_N C_N^+ e^{-\sigma_N^+ (t - t_m)}$$



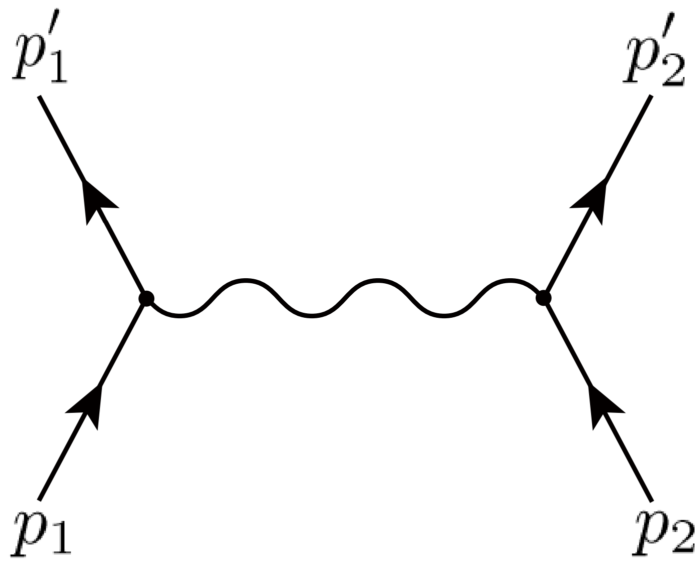
First complete waveforms for BBH coalescences: analytical EOB (Buonanno-Damour'00, Buonanno-Chen-Damour'05)

Gravitational Scattering and the GR 2-body problem

Beyond the PN approximation: (Possibly) High-energy Classical Scattering:

Post-Minkowskian (PM) approximation: expansion in G^n keeping all orders in v/c:

1PM O(G^1) classical scattering in Fourier space



$$\frac{dp_{1\mu}}{d\sigma_1} = \frac{1}{2}\partial_\mu g_{\alpha\beta}(x_1) p_1^\alpha p_1^\beta. \qquad \Delta p_{1\mu} = \int_{-\infty}^{+\infty} d\sigma_1 \frac{1}{2} p_1^\alpha p_1^\beta \partial_\mu h_{\alpha\beta}(x_1),$$

$$\Box h_{\alpha\beta} = -16\pi G \left(T_{\alpha\beta} - \frac{1}{2} T \eta_{\alpha\beta} \right).$$

$$T_2^{\alpha\beta}(x) = \int_{-\infty}^{+\infty} d\sigma_2 p_2^\alpha p_2^\beta \delta^4(x - x_2(\sigma_2)),$$

$$\Delta p_{1\mu} = 8\pi G \int \frac{d^4k}{(2\pi)^4} i k_\mu p_1^\alpha p_1^\beta \frac{P_{\alpha\beta;\alpha'\beta'}}{k^2} p_2^{\alpha'} p_2^{\beta'} \times \int d\sigma_1 \int d\sigma_2 e^{ik.(x_1(\sigma_1) - x_2(\sigma_2))}.$$

$$T_2^{\alpha'\beta'}(k) = \int d^4x e^{-ik.x} T_2^{\alpha'\beta'}(x) = \int_{-\infty}^{+\infty} d\sigma_2 e^{-ik.x_2(\sigma_2)} p_2^{\alpha'} p_2^{\beta'}.$$

$$P_{\alpha\beta;\alpha'\beta'} \equiv \eta_{\alpha\alpha'}\eta_{\beta\beta'} - \frac{1}{2}\eta_{\alpha\beta}\eta_{\alpha'\beta'}$$

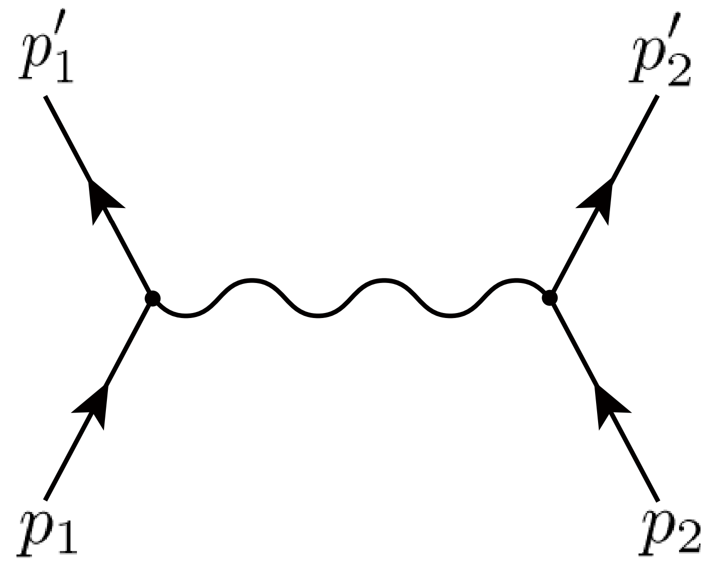
$$(2\pi)^2 e^{ik.(x_1(0) - x_2(0))} \delta(k.p_1)\delta(k.p_2).$$

$$x_1^\mu(\sigma_1) = x_1^\mu(0) + p_1^\mu \sigma_1;$$
$$x_2^\mu(\sigma_2) = x_2^\mu(0) + p_2^\mu \sigma_2.$$

[Equivalent to old, x-space PM results of Bel-Martin ’75-’81, Portilla ’79, Westpfahl-Goller ’79, Portilla ’80, Bel-Damour-Deruelle-Ibanez-Martin’81, Westpfahl ’85]

1PM Classical Gravitational Scattering: scattering angle chi in c.m. frame

G^1



$$\frac{1}{2}\chi_{1PM}^{\text{real}} = 2\,\frac{G}{b\,p_{\text{c.m.}}}\,\frac{p_1^\alpha p_1^\beta P_{\alpha\beta;\alpha'\beta'}\,p_2^{\alpha'}p_2^{\beta'}}{\mathcal{D}}$$

$$= 2\,\frac{G}{J}\,\frac{p_1^\alpha p_1^\beta P_{\alpha\beta;\alpha'\beta'}\,p_2^{\alpha'}p_2^{\beta'}}{\mathcal{D}}.$$

$$P_{\alpha\beta;\alpha'\beta'} \equiv \eta_{\alpha\alpha'}\eta_{\beta\beta'} - \tfrac{1}{2}\eta_{\alpha\beta}\eta_{\alpha'\beta'}$$

$$\mathcal{D}^2 = |p_1 \wedge p_2|^2 = -\frac{1}{2}(p_1^\mu p_2^\nu - p_1^\nu p_2^\mu)(p_{1\mu}p_{2\nu} - p_{1\nu}p_{2\mu})$$

$$= (p_1.p_2)^2 - p_1^2 p_2^2. \qquad (52)$$

classical scattering angle

$$\frac{1}{2}\chi_{1PM}^{\text{real}} = \frac{G}{J}\,\frac{2(p_1.p_2)^2 - p_1^2p_2^2}{\sqrt{(p_1.p_2)^2 - p_1^2p_2^2}}.$$

quantum scattering amplitude

$$\mathcal{M}^{(\frac{G}{\hbar})}(s,t) = 16\pi\,\frac{G}{\hbar}\,\frac{2(p_1\cdot p_2)^2 - p_1^2p_2^2}{-t}.$$

same numerator

Gravitational Scattering and the EOB description of the GR 2-body dynamics

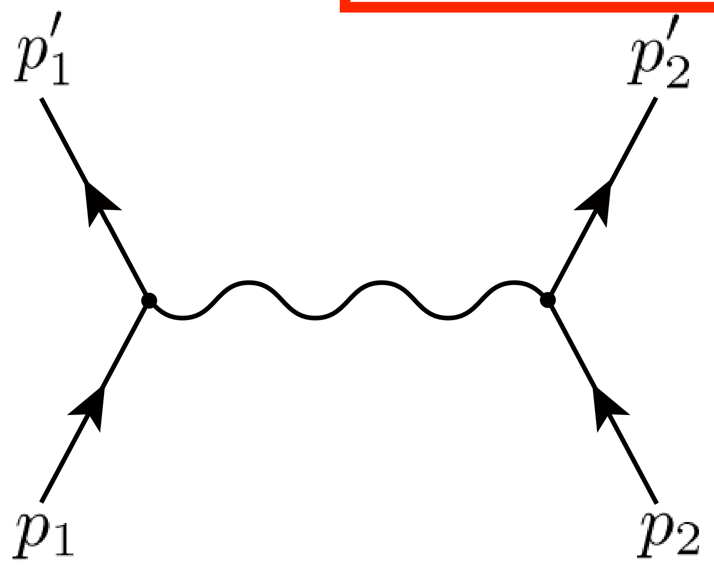
Original EOB dictionary based on bound states.

$$f(E_{\rm real}(I_a)) = E_{\rm eff}(I_a)$$

New (equivalent) dictionary for scattering states:
applicable to the PM approximation (no restriction on v/c).
[Damour2016]

$$\chi_{\rm eff}(\mathcal{E}_{\rm eff}, J) = \chi_{\rm real}(\mathcal{E}_{\rm real}, J);$$

G^1



$$\frac{1}{2}\chi^{\rm real}_{1PM} = \frac{G}{J}\frac{2(p_1.p_2)^2 - p_1^2 p_2^2}{\sqrt{(p_1.p_2)^2 - p_1^2 p_2^2}}.$$

$$\frac{1}{2}\chi_{\rm class}(E,J) = \frac{1}{j}\chi_1(\hat{E}_{\rm eff},\nu) + \frac{1}{j^2}\chi_2(\hat{E}_{\rm eff},\nu) + O(G^3)$$

Computing chi_eff: scattering of m_0 =\mu in some effective metric

$$g^{\text{eff}}_{\mu\nu}(M_0,\beta_1)dx^\mu dx^\nu = -\left(1-\frac{R_g}{R}\right)dt^2 + \left(1+\beta_1\frac{R_g}{R}\right)dR^2 + R^2(d\theta^2+\sin^2\theta d\varphi^2), \quad (62)$$

R_g= 2 GM_0

Hamilton-Jacobi eq

$$g^{\mu\nu}_{\text{eff}}\partial_\mu S_{\text{eff}}\partial_\nu S_{\text{eff}} = -m_0^2,$$

$$S_{\text{eff}} = -\mathcal{E}_0 t + J_0\varphi + S_R^{\text{eff}}(R).$$

Computing $P_R = dS_R^{\text{eff}}/dR$

$$P_R(R;\mathcal{E}_0,J_0) = \pm\left(1+\frac{1}{2}\beta_1\frac{R_g}{R}\right) \times\sqrt{\mathcal{E}_0^2\left(1+\frac{R_g}{R}\right)-\left(m_0^2+\frac{J_0^2}{R^2}\right)},$$

$$\pi+\chi_{\text{eff}} = -\int_{-\infty}^{+\infty} dR\,\frac{\partial P_R(R;\mathcal{E}_0,J_0)}{\partial J_0}.$$

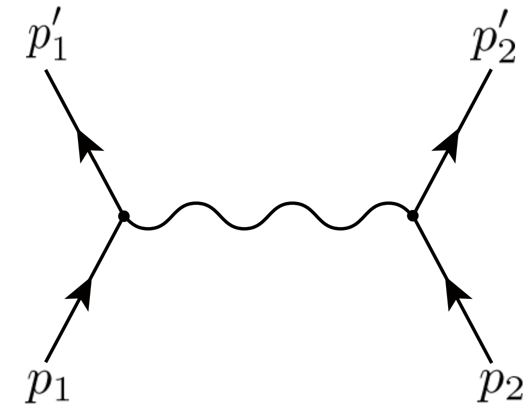
$$\frac{1}{2}\chi^{\text{eff}}_{1PM}(\mathcal{E}_0,J_0) = \frac{GM_0m_0}{J_0}\frac{(1+\beta_1)(\mathcal{E}_0/m_0)^2-\beta_1}{\sqrt{(\mathcal{E}_0/m_0)^2-1}}.$$

$$\frac{1}{2}\chi^{\text{real}}_{1PM} = \frac{G}{J}\frac{2(p_1.p_2)^2-p_1^2p_2^2}{\sqrt{(p_1.p_2)^2-p_1^2p_2^2}}.$$

New results already at the 1PM order (linear in G)

Derivation of EOB energy map to all orders in v/c:

$$\mathcal{E}_{\rm eff} = \frac{(\mathcal{E}_{\rm real})^2 - m_1^2 - m_2^2}{2(m_1+m_2)} \,.$$



to order G^1, the relativistic dynamics of a two-body system (of masses m_1 , m_2) is equivalent to the relativistic dynamics of an effective test particle of mass $\mu = m_1 m_2/(m_1 + m_2)$ moving in a Schwarzschild metric of mass $M = m_1 + m_2$, i.e. the rather complicated 1PM Hamiltonian of Ledvinka-Schaefer-Bicak2010: with

$$\begin{aligned}
H_{\rm lin} &= \sum_a \overline{m}_a + \frac{1}{4}G\sum_{a,b\neq a}\frac{1}{r_{ab}}\left(7\,\mathbf{p}_a\!\cdot\!\mathbf{p}_b + (\mathbf{p}_a\!\cdot\!\mathbf{n}_{ab})(\mathbf{p}_b\!\cdot\!\mathbf{n}_{ab})\right) - \frac{1}{2}G\sum_{a,b\neq a}\frac{\overline{m}_a\overline{m}_b}{r_{ab}} \\
&\times\left(1+\frac{p_a^2}{\overline{m}_a^2}+\frac{p_b^2}{\overline{m}_b^2}\right) - \frac{1}{4}G\sum_{a,b\neq a}\frac{1}{r_{ab}}\frac{(\overline{m}_a\overline{m}_b)^{-1}}{(y_{ba}+1)^2 y_{ba}}\Bigg[2\Big(2(\mathbf{p}_a\!\cdot\!\mathbf{p}_b)^2(\mathbf{p}_b\!\cdot\!\mathbf{n}_{ba})^2 \qquad (6)\\
&-2(\mathbf{p}_a\!\cdot\!\mathbf{n}_{ba})(\mathbf{p}_b\!\cdot\!\mathbf{n}_{ba})(\mathbf{p}_a\!\cdot\!\mathbf{p}_b)\mathbf{p}_b^2 + (\mathbf{p}_a\!\cdot\!\mathbf{n}_{ba})^2\mathbf{p}_b^4 - (\mathbf{p}_a\!\cdot\!\mathbf{p}_b)^2\mathbf{p}_b^2\Big)\frac{1}{\overline{m}_b^2} + 2\Big[-\mathbf{p}_a^2(\mathbf{p}_b\!\cdot\!\mathbf{n}_{ba})^2 \\
&+(\mathbf{p}_a\!\cdot\!\mathbf{n}_{ba})^2(\mathbf{p}_b\!\cdot\!\mathbf{n}_{ba})^2 + 2(\mathbf{p}_a\!\cdot\!\mathbf{n}_{ba})(\mathbf{p}_b\!\cdot\!\mathbf{n}_{ba})(\mathbf{p}_a\!\cdot\!\mathbf{p}_b) + (\mathbf{p}_a\!\cdot\!\mathbf{p}_b)^2 - (\mathbf{p}_a\!\cdot\!\mathbf{n}_{ba})^2\mathbf{p}_b^2\Big] \\
&+\Big[-3\mathbf{p}_a^2(\mathbf{p}_b\!\cdot\!\mathbf{n}_{ba})^2 + (\mathbf{p}_a\!\cdot\!\mathbf{n}_{ba})^2(\mathbf{p}_b\!\cdot\!\mathbf{n}_{ba})^2 + 8(\mathbf{p}_a\!\cdot\!\mathbf{n}_{ba})(\mathbf{p}_b\!\cdot\!\mathbf{n}_{ba})(\mathbf{p}_a\!\cdot\!\mathbf{p}_b) \\
&+\mathbf{p}_a^2\mathbf{p}_b^2 - 3(\mathbf{p}_a\!\cdot\!\mathbf{n}_{ba})^2\mathbf{p}_b^2\Big]y_{ba}\Bigg]\,, \qquad y_{ba} = \frac{1}{\overline{m}_b}\sqrt{m_b^2+(\mathbf{n}_{ba}\!\cdot\!\mathbf{p}_b)^2}\,.
\end{aligned}$$

$$\overline{m}_a = (m_a^2 + \mathbf{p}_a^2)^{\frac{1}{2}}$$

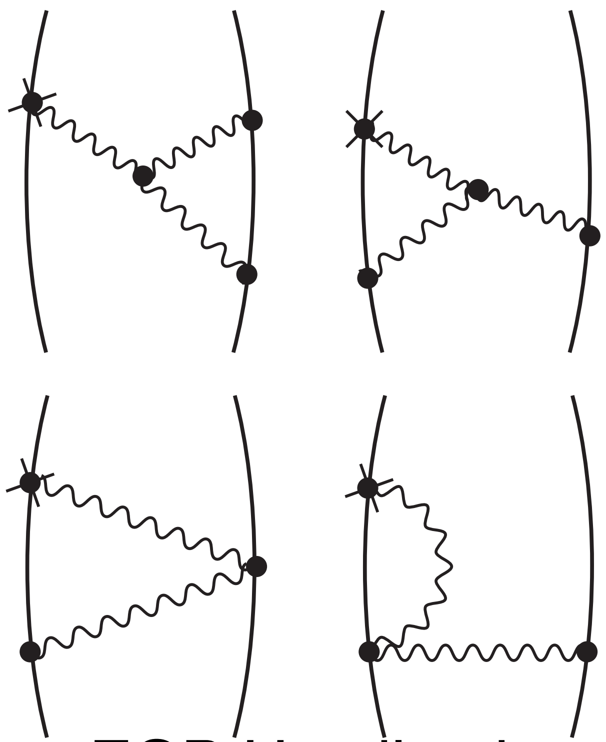
is fully described by the EOB energy map applied to

$$ds^2_{\rm lin} = -(1-2\frac{GM}{r})dt^2 + (1+2\frac{GM}{r})dr^2 + r^2 d\Omega^2$$

Classical Gravitational Scattering at the 2PM level (one-loop)

Damour'18, using Westpfahl-Goller '79, Bel-Damour-Deruelle-Ibanez-Martin'81, Westpfahl '85

G^2



+ Ladder

$$\frac{1}{2}\chi_{\rm class}(E,J) = \frac{1}{j}\chi_1(\hat{E}_{\rm eff},\nu) + \frac{1}{j^2}\chi_2(\hat{E}_{\rm eff},\nu) + O(G^3)$$

$$\chi_1(\hat{\mathcal{E}}_{\rm eff},\nu) = \frac{2\hat{\mathcal{E}}_{\rm eff}^2 - 1}{\sqrt{\hat{\mathcal{E}}_{\rm eff}^2 - 1}}, \qquad \hat{\mathcal{E}}_{\rm eff} \equiv \frac{\mathcal{E}_{\rm eff}}{\mu} \equiv \frac{(E_{\rm real})^2 - m_1^2 - m_2^2}{2m_1 m_2} = \frac{s - m_1^2 - m_2^2}{2m_1 m_2}.$$

$$\chi_2(\hat{\mathcal{E}}_{\rm eff},\nu) = \frac{3\pi}{8}\frac{5\hat{\mathcal{E}}_{\rm eff}^2 - 1}{\sqrt{1 + 2\nu(\hat{\mathcal{E}}_{\rm eff} - 1)}}.$$

Effective EOB Hamiltonian transcription of chi2PM as a post-Schwarzschild Hamiltonian

$$0 = g^{\mu\nu}_{\rm Schwarz} P_\mu P_\nu + Q(\mathbf{R},\mathbf{P}) \longrightarrow \frac{1}{2}(\chi(\mathcal{E}_{\rm eff},J) - \chi^{\rm Schw}(\mathcal{E}_{\rm eff},J)) = \frac{1}{4}\frac{\partial}{\partial J}\int d\sigma_{(0)} Q + O(G^4).$$

gauge-freedom $Q'(\mathbf{R},\mathbf{P}) = Q(\mathbf{R},\mathbf{P}) + \frac{d}{d\sigma_{(0)}} G(\mathbf{R},\mathbf{P})$ use an « energy gauge »

$$\widehat{H}^2_{\rm eff}(p_r, r, p_\varphi; \nu) = \widehat{H}^2_{\rm Schw} + \frac{3}{2}(1-2u)u^2\left(5\,\widehat{H}^2_{\rm Schw} - 1\right)\left(1 - \frac{1}{\sqrt{1 + 2\nu(\widehat{H}_{\rm Schw} - 1)}}\right)$$

$$\hat{H}^2_{\rm Schw}(p_r, r, p_\varphi) \equiv (1-2u)[1 + (1-2u)p_r^2 + p_\varphi^2 u^2],$$

Predicted High-Energy Regge-like Behavior

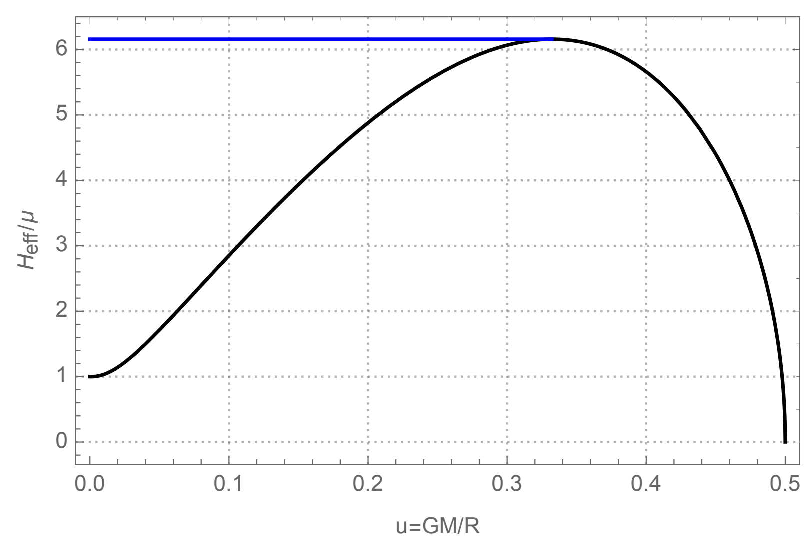
$$u \equiv \frac{GM}{R}$$

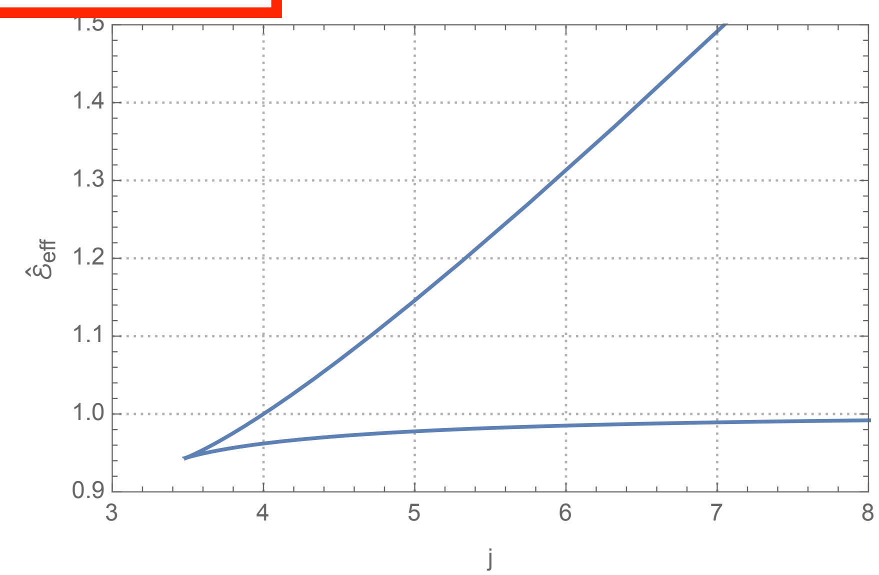
$$\widehat{H}^2_{\rm eff}(p_r, r, p_\varphi; \nu) = \widehat{H}^2_{\rm Schw} + \frac{3}{2}(1-2u)u^2\left(5\,\widehat{H}^2_{\rm Schw} - 1\right)\left(1 - \frac{1}{\sqrt{1+2\nu(\widehat{H}_{\rm Schw}-1)}}\right)$$

$$H^2_{\rm Schw} = \left(1 - \frac{2GM}{R}\right)\left[\mu^2 + (1-\frac{2GM}{R})P_R^2 + \frac{J^2}{R^2}\right]$$

High J —> H^2_eff ~ B(u) J^2
but

$$\mathcal{E}_{\rm eff} = \frac{(\mathcal{E}_{\rm real})^2 - m_1^2 - m_2^2}{2(m_1+m_2)}.$$





HE unstable circular bound states:
asymptotic constant Regge slope

string-like

$$s = E^2_{\rm real} \propto J$$

$$\frac{ds}{dJ} = \frac{2}{G}\frac{d\hat{\mathcal{E}}_{\rm eff}}{dj} \overset{\rm HE}{\approx} \frac{0.719\,964}{G} \qquad E^2_{\rm real} \overset{\rm HE}{=} C\,\frac{J}{G},$$

Self-Force Expansion , Light-Ring Behavior

Small mass-ratio expansion: \nu ->0

$$\widehat{H}^2_{\rm eff}(p_r, r, p_\varphi; \nu) = \widehat{H}^2_{\rm Schw} + \frac{3}{2}(1-2u)u^2\left(5\,\widehat{H}^2_{\rm Schw} - 1\right)\left(1 - \frac{1}{\sqrt{1+2\nu(\widehat{H}_{\rm Schw}-1)}}\right)$$

$$1 - \frac{1}{\sqrt{1+2\nu(\hat{H}_{\rm Schw}-1)}} = \nu(\hat{H}_{\rm Schw}-1) - \frac{3}{2}\nu^2(\hat{H}_{\rm Schw}-1)^2 + \frac{5}{2}\nu^3(\hat{H}_{\rm Schw}-1)^3 + \cdots$$

$$\hat{H}^2_{\rm eff} = \hat{H}^2_{\rm Schw} + \frac{3}{2}\nu(1-2u)u^2(5\hat{H}_{\rm Schw}{}^2 - 1)(\hat{H}_{\rm Schw}-1) \times \left[1 - \frac{3}{2}\nu(\hat{H}_{\rm Schw}-1) + \frac{5}{2}\nu^2(\hat{H}_{\rm Schw}-1)^2 + \cdots\right]. \quad (8.7$$

Singular Light-Ring Behavior of Self-Force expansion in DJS gauge (Akcay-Barack-Damour-Sago'12)

$$\bar{A}^{\rm SF}(\bar{u};\nu) = 1 - 2\bar{u} + \nu a_{1\rm SF}(\bar{u}) + \nu^2 a_{2\rm SF}(\bar{u}) + O(\nu^3). \quad a_{1\rm SF}(\bar{u}) \underset{\bar{u}\to\frac{1}{3}}{\sim} \frac{1}{4}\zeta(1-3\bar{u})^{-1/2}, \quad \text{with} \quad \zeta \approx 1.$$

1PM and 2PM-accurate spin-orbit couplings

1PM: Bini-Damour'17; (see also Vines'17); 2PM: Bini-Damour '18

New concepts: scattering holonomy and spin rotation

$$
\begin{aligned}
H_{\rm eff} &= \sqrt{A\left(\mu^2 + \mathbf{P}^2 + \left(\frac{1}{B} - 1\right) P_R^2 + Q\right)} \\
&+ \frac{G}{R^3}\left(g_S\, \mathbf{L}\cdot\mathbf{S} + g_{S_*}\, \mathbf{L}\cdot\mathbf{S}_*\right) ,
\end{aligned}
$$

$$
\begin{aligned}
g_S &= g_S^{\rm 1PM}(H_{\rm eff}) + g_S^{\rm 2PM}(H_{\rm eff})\, u + O(u^2) , \\
g_{S*} &= g_{S*}^{\rm 1PM}(H_{\rm eff}) + g_{S*}^{\rm 2PM}(H_{\rm eff})\, u + O(u^2) ,
\end{aligned}
$$

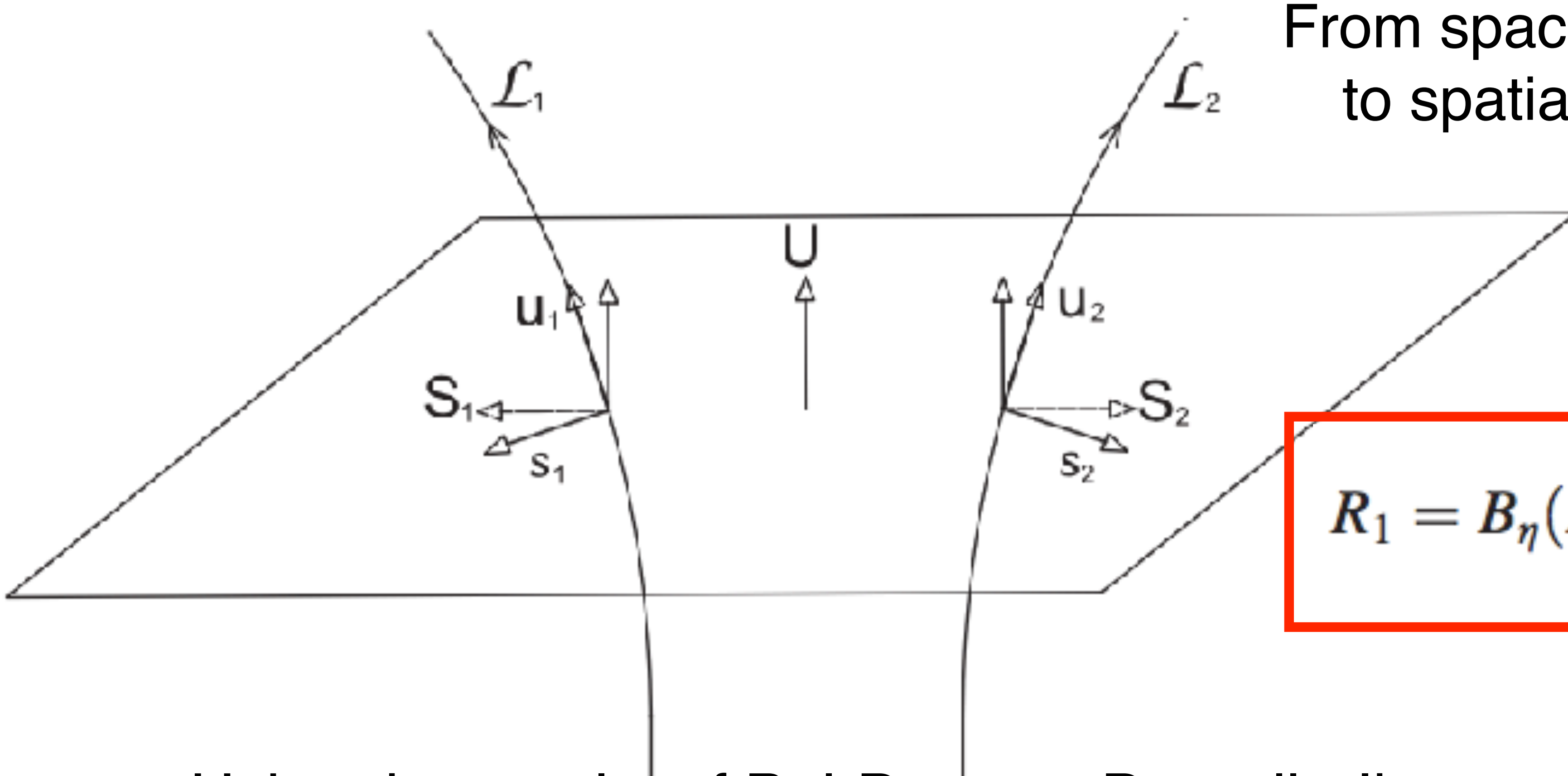
$$D_g u_1 = 0 = D_g s_1, \qquad D_g = d + \omega_1.$$

$$\omega^\mu{}_\nu = \Gamma^\mu{}_{\nu\lambda} dx^\lambda, \qquad \Gamma^\mu{}_{\nu\lambda} = \frac{1}{2} g^{\mu\sigma}(\partial_\nu g_{\lambda\sigma} + \partial_\lambda g_{\nu\sigma} - \partial_\sigma g_{\nu\lambda}).$$

$$u_1^+ = \Lambda_1 u_1^-; \qquad s_1^+ = \Lambda_1 s_1^-.$$

$$
\begin{aligned}
\Lambda_1 &= T_{\mathcal{L}_1}[e^{-\int \omega_1}] \\
&= 1 - \int_{-\infty}^{+\infty} \omega_1 + \frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} T[\omega_1 \omega_1'] + \cdots
\end{aligned}
$$

1PM and 2PM-accurate spin-rotation

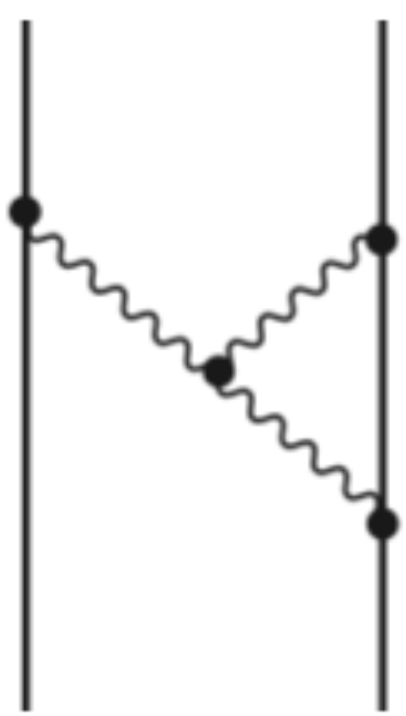


From spacetime spin four-vectors s_1,s_2 to spatial spin three-vectors S_1, S_2

Bini-Damour ’17,‘18

$$R_1 = B_\eta(\Lambda_1 u_1^- \to U^{\rm as})\Lambda_1[B_\eta(u_1^- \to U^{\rm as})]^{-1}.$$

Using the results of Bel-Damour-Deruelle-Ibanez-Martin’81 for the 2PM metric, one gets the 2PM-accurate value of the integrated spin rotation (spin holonomy)



$$\begin{aligned}\theta_1 \;=\; & -\frac{2}{hj\sqrt{\gamma^2-1}}\left[\gamma X_2 + (2\gamma^2-1)(X_1-h)\right] \\ & +\frac{\pi}{4h^2j^2}\left[-3(5\gamma^2-1)(X_1-h) - 6\gamma X_2 \right. \\ & \left. +\gamma(5\gamma^2-3)X_1X_2\right] . \qquad (5\end{aligned}$$

Spinning EOB effective Hamiltonian

Damour'01, Damour-Jaranowski-Schaefer'08, Barausse-Buonanno'10, Damour-Nagar'14, ……

$$H_{\rm eff} = H_{\rm orb} + H_{\rm so} \quad \to H_{\rm EOB} = Mc^2\sqrt{1+2\nu\left(\frac{H_{\rm eff}}{\mu c^2}-1\right)}$$

$$\hat{H}^{\rm eff}_{\rm orb} = \sqrt{A\left(1 + B_p \boldsymbol{p}^2 + B_{np}(\boldsymbol{n}\cdot\boldsymbol{p})^2 - \frac{1}{1+\frac{(\boldsymbol{n}\cdot\boldsymbol{\chi}_0)^2}{r^2}}\frac{(r^2+2r+(\boldsymbol{n}\cdot\boldsymbol{\chi}_0)^2)}{\mathcal{R}^4+\Delta(\boldsymbol{n}\cdot\boldsymbol{\chi}_0)^2}((\boldsymbol{n}\times\boldsymbol{p})\cdot\boldsymbol{\chi}_0)^2 + Q_4\right)}.$$

$$H_{\rm so} = G_S \boldsymbol{L}\cdot\boldsymbol{S} + G_{S^*}\boldsymbol{L}\cdot\boldsymbol{S}^*,$$

$$\mathbf{S} = \mathbf{S_1} + \mathbf{S_2}\,;\ \mathbf{S}_* = \frac{m_2}{m_1}\mathbf{S_1} + \frac{m_1}{m_2}\mathbf{S_2}\,,$$

Gyrogravitomagnetic ratios
(when neglecting spin^2 effects)

$$g_S = R^3 G_S\,;\ g_{S*} = R^3 G_{S*}$$

$$r^3 G_S^{\rm PN} = 2 - \frac{5}{8}\nu u - \frac{27}{8}\nu p_r^2 + \nu\left(-\frac{51}{4}u^2 - \frac{21}{2}u p_r^2 + \frac{5}{8}p_r^4\right) + \nu^2\left(-\frac{1}{8}u^2 + \frac{23}{8}u p_r^2 + \frac{35}{8}p_r^4\right)$$

$$r^3 G_{S_*}^{\rm PN} = \frac{3}{2} - \frac{9}{8}u - \frac{15}{8}p_r^2 + \nu\left(-\frac{3}{4}u - \frac{9}{4}p_r^2\right) - \frac{27}{16}u^2 + \frac{69}{16}u p_r^2 + \frac{35}{16}p_r^4 + \nu\left(-\frac{39}{4}u^2 - \frac{9}{4}u p_r^2 + \frac{5}{2}p_r^4\right) + \nu^2\left(-\frac{3}{16}u^2 + \frac{57}{16}u p_r^2 + \frac{45}{16}p_r^4\right)$$

EOB transcription of the 2PM-accurate spin-rotation

energy spin-gauge
instead of DJS gauge

$$
\begin{aligned}
g_S &= g_S^{1\mathrm{PM}}(H_{\mathrm{eff}}) + g_S^{2\mathrm{PM}}(H_{\mathrm{eff}})\, u + O(u^2)\,,\\
g_{S*} &= g_{S*}^{1\mathrm{PM}}(H_{\mathrm{eff}}) + g_{S*}^{2\mathrm{PM}}(H_{\mathrm{eff}})\, u + O(u^2)\,,
\end{aligned}
$$

$$
\begin{aligned}
g_S^{1\mathrm{PM}}(\gamma,\nu) &= \frac{(2\gamma+1)(2\gamma+h)-1}{h(h+1)\gamma(\gamma+1)}\\
&= \frac{1}{h(h+1)}\left[4+\frac{h-1}{\gamma+1}+\frac{h-1}{\gamma}\right]\\
g_{S*}^{1\mathrm{PM}}(\gamma,\nu) &= \frac{2\gamma+1}{h\gamma(\gamma+1)}\\
&= \frac{1}{h}\left[\frac{1}{\gamma+1}+\frac{1}{\gamma}\right].
\end{aligned}
$$

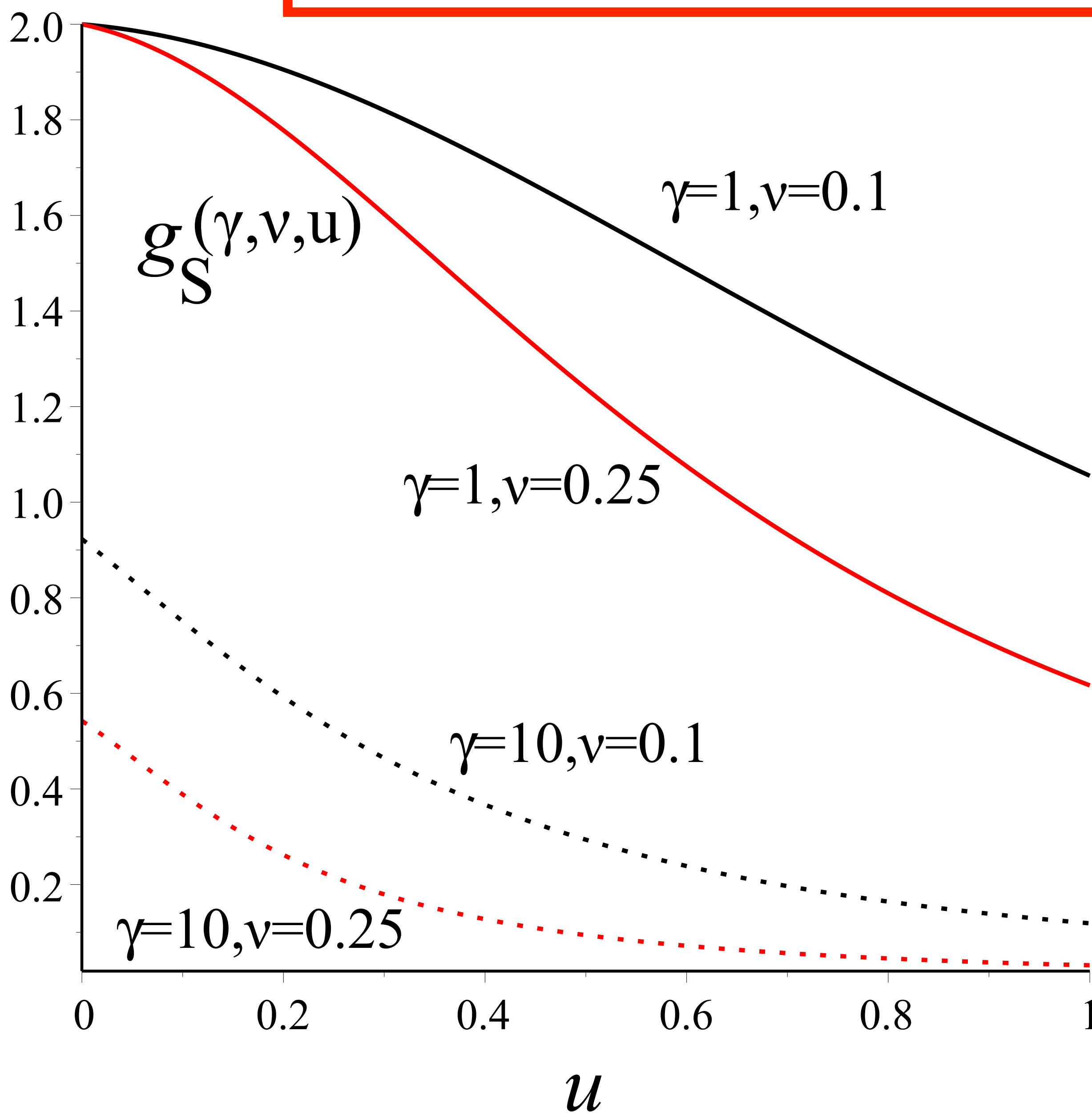
$$
\theta_1^{\mathrm{EOB}} = G\int \frac{\mathbf{L}}{R^3}\left(g_S + \frac{m_2}{m_1} g_{S*}\right)\frac{B}{A} E_{\mathrm{eff}} \frac{dR}{P_R}\,,
$$

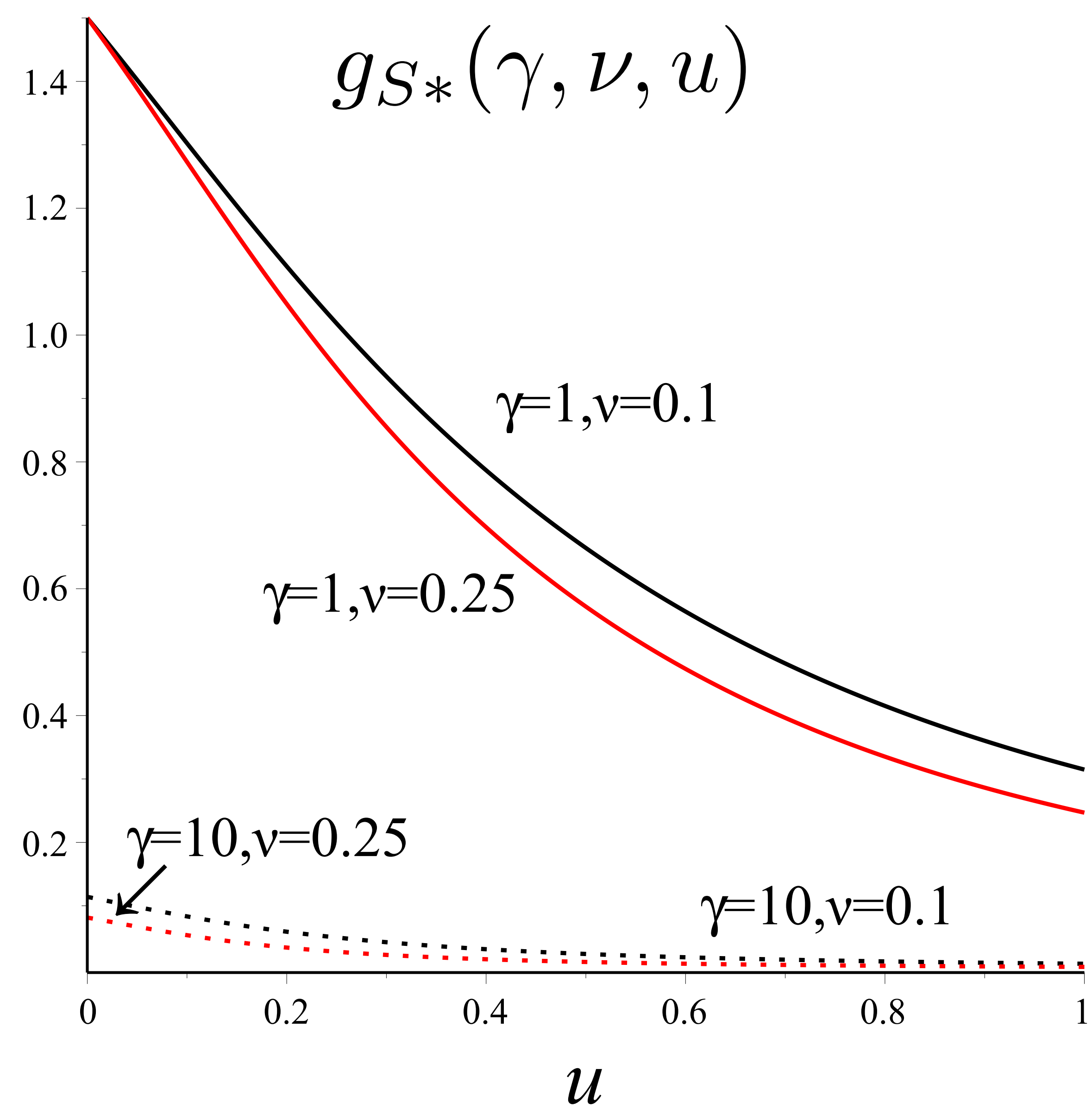
$$
\gamma = \hat{H}_{\mathrm{eff}} \qquad h = \sqrt{1+2\nu(\gamma-1)}
$$

$$
\begin{aligned}
g_S^{2\mathrm{PM}}(\gamma,\nu) &= -\frac{\nu}{\gamma(\gamma+1)^2h^2(h+1)^2}[2(2\gamma+1)(5\gamma^2-3)h+(\gamma+1)(35\gamma^3-15\gamma^2-15\gamma+3)]\\
&= \frac{\nu}{h^2(h+1)^2}\left[-5(7\gamma+4h-10)+\frac{8(3h-4)}{\gamma+1}-\frac{4h}{(\gamma+1)^2}+\frac{3(2h-1)}{\gamma}\right]\\
g_{S*}^{2\mathrm{PM}}(\gamma,\nu) &= -\frac{1}{2\gamma(\gamma+1)^2h^2(h+1)}\left[(5\gamma^2+6\gamma+3)(h+1)+4\nu(1+2\gamma)(5\gamma^2-3)\right]\\
&= \frac{1}{h^2(h+1)}\left[-20\nu+\frac{24\nu-h-1}{\gamma+1}+\frac{h+1-4\nu}{(\gamma+1)^2}-\frac{3}{2}\frac{h+1-4\nu}{\gamma}\right]\\
&= \frac{1}{h^2(h+1)}\left[-\frac{20\gamma\nu}{\gamma+1}+(h+1-4\nu)\left(\frac{1}{(\gamma+1)^2}-\frac{1}{\gamma+1}-\frac{3}{2}\frac{1}{\gamma}\right)\right].
\end{aligned}
$$

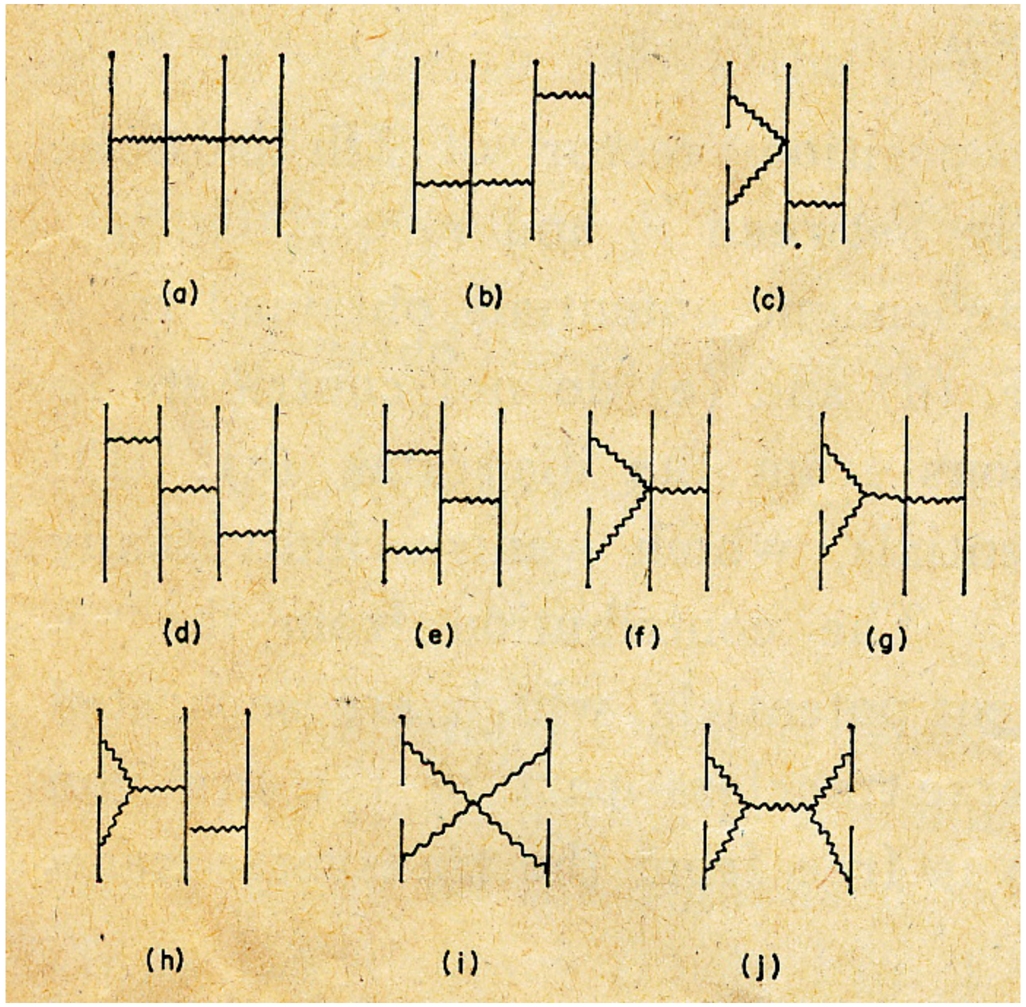
High-energy behavior, strong-field behavior and resummation of gS, gS*

$$g_{S,S*}(\gamma,\nu,u) = \frac{g^{1\mathrm{PM}}_{S,S*}(\gamma,\nu)}{1 + u\,\tilde{c}^1_{S,S*}(\gamma,\nu) + u^2\,\tilde{c}^2_{S,S*}(\gamma,\nu)}$$





Quantum Scattering Amplitudes and 2-body Dynamics



• Quantum Scattering Amplitudes —> Potential
one-graviton exchange : Corinaldesi '56 '71, Barker-Gupta-Haracz 66, Barker-O'Connell 70, Hiida-Okamura72

Nonlinear: Iwasaki 71 [1PN], Okamura-Ohta-Kimura-Hiida 73[2 PN]

New technique: use EOB as a scattering-> Hamiltonian translation device

Progress in gravity amplitudes (Bern, Carrasco et al., Cachazo et al., Bjerrum-Bohr et al., Cachazo-Guevara,…) can be used (Damour '17) to improve the classical 2-body dynamics: need a quantum/classical dictionary.

Amati-Ciafaloni-Veneziano 1987-2008

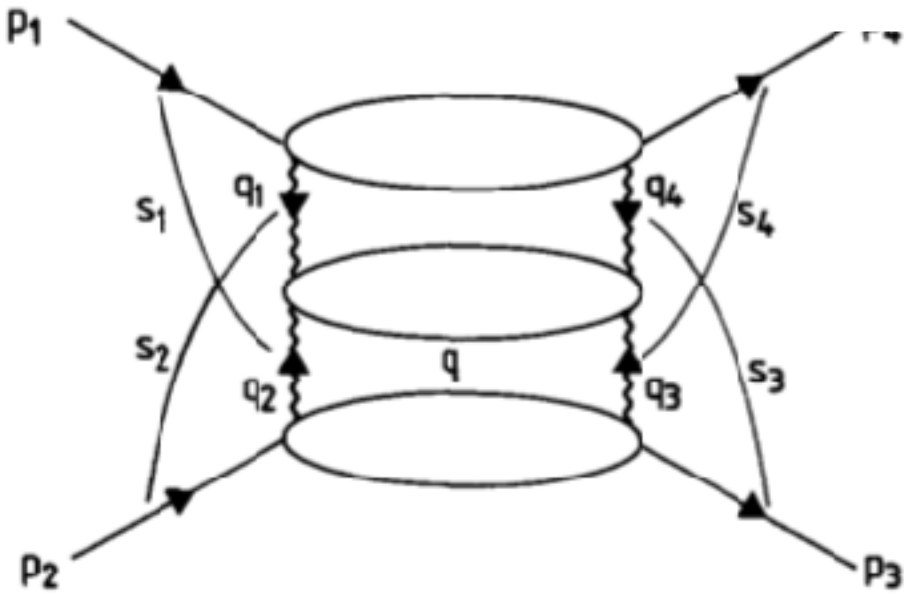
Ultra-High-Energy (s >> M_Planck^2)

Four-graviton Scattering at 2 loops

Impact-parameter (b) representation

$$\frac{1}{s} A(s,q) = (\varepsilon_a \varepsilon_d)(\varepsilon_b \varepsilon_c) 4 \int d^{D-2} b \exp(i\boldsymbol{q}\cdot\boldsymbol{b})\, a(s,b)$$

$$a(s,b) = \sum_{h=0}^{\infty} a^{(h)}(s,b) = \langle 0 | (1/2i) \{\exp[2i\delta(s,b;\hat{X},\hat{X}')] - 1\} | 0 \rangle$$



g. 3. The “H” diagram that provides the leading correction to the eikonal.

Eikonal phase \delta in D=4
with one- and two-loop corrections
using the Regge-Gribov approach

$$\delta = \frac{Gs}{\hbar}\left(\log\left(\frac{L_{IR}}{b}\right) + \frac{6\ell_s^2}{\pi b^2} + \frac{2G^2 s}{b^2}\left(1 + \frac{2i}{\pi}\log(\cdots)\right)\right)$$

High-energy limit of 2-body scattering and 2-body dynamics

Using the (eikonal) ultra-high-energy results of Amati-Ciafaloni-Veneziano: get HE information up to G^4

$$\frac{1}{2}\chi^{ACV} = \frac{2GE_{\rm real}}{b} + \frac{7}{6}\left(\frac{2GE_{\rm real}}{b}\right)^3 + O\left(\left(\frac{2GE_{\rm real}}{b}\right)^5\right)$$

In HE limit the EOB energy map is such that

$$\alpha = \frac{GME_{\rm eff}}{J} = \frac{G}{2}\frac{(E_{\rm real})^2 - m_1^2 - m_2^2}{J} \approx_{\rm HE} \frac{GE_{\rm real}}{b}$$

HE (J-> infinity, E_eff-> infinity) scattering of test particle in effective metric

$$ds^2_{\rm eff} = -A(R)dt^2 + B(R)dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$\pi + \chi \stackrel{\rm HE}{=} \int J\frac{dR}{C}\frac{\sqrt{AB}}{\pm\sqrt{\mathcal{E}_{\rm eff}^2 - J^2\frac{A}{C}}} = \int \frac{dR}{C}\frac{\sqrt{AB}}{\pm\sqrt{\frac{\mathcal{E}_{\rm eff}^2}{J^2} - \frac{A}{C}}}$$

Conformally invariant

Effective 4PM-accurate metric equivalent to ACV HE scattering

The masses disappear and the HE scattering is equivalent to a null geodesic in the « effective HE metric »

$$ds^2 = -A_{\rm HE}(u)dT^2 + \frac{dR^2}{1-2u} + R^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

$$A_{\rm HE}(u) = (1-2u)\left(1 + \frac{15}{2}u^2 - 3u^3 + \frac{1749}{16}u^4 + O(u^5)\right)$$

2PM 3PM 4PM

OK with above — NEW derived from 2-loop ACV result

Translating quantum scattering amplitudes into classical dynamical information

How to translate a scattering amplitude into a classical Hamiltonian ?

$$\mathcal{M}(s,t) = \mathcal{M}^{(\frac{G}{\hbar})}(s,t) + \mathcal{M}^{(\frac{G^2}{\hbar^2})}(s,t) + \cdots$$

$$\mathcal{M}^{(\frac{G}{\hbar})}(s,t) = 16\pi \frac{G}{\hbar} \frac{2(p_1 \cdot p_2)^2 - p_1^2 p_2^2}{-t}.$$

Problem: The domain of validity of the Born expansion
is GE_1 E_2/(hbar v) << 1, while
the domain of validity of the classical scattering
is GE_1 E_2/(hbar v) >> 1!

It is an accident that the Born approximation of
a 1/r potential yields the exact cross section.

A way out: quantize the classical EOB Hamiltonian dynamics.

$$\mathbf{p}^2 = p_\infty^2 + \bar{W}(\bar{u}) = p_\infty^2 + w_1\bar{u} + w_2\bar{u}^2 + O(\bar{u}^3),$$

$$p_\infty^2 = \hat{\mathcal{E}}_{\rm eff}^2 - 1,$$

$$w_1 = 2(2\hat{\mathcal{E}}_{\rm eff}^2 - 1),$$

$$w_2 = \frac{3}{2}\frac{5\hat{\mathcal{E}}_{\rm eff}^2 - 1}{h(\hat{\mathcal{E}}_{\rm eff})}.$$

Quantized version:

$$-\hat{\hbar}^2\Delta_{\mathbf{x}}\psi(\mathbf{x}) = \left[p_\infty^2 + \frac{w_1}{\bar{r}} + \frac{w_2}{\bar{r}^2} + O\left(\frac{1}{\bar{r}^3}\right)\right]\psi(\mathbf{x}).$$

Scattering amplitude for this potential scattering at the second Born approximation

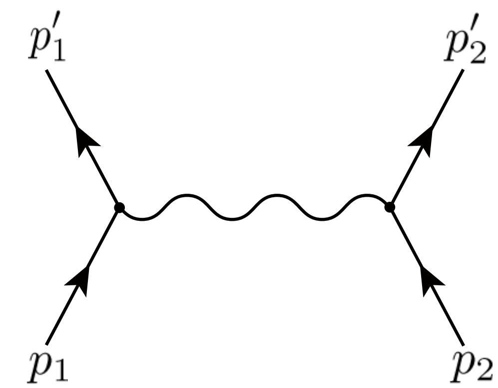
$$f_{\mathbf{k}_a}^{+\mathrm{B}1}(\mathbf{k}_b) = \frac{1}{\hat{\hbar}^2}\left[e^{\delta_C}\frac{w_1}{q^2} + \frac{\pi}{2}\frac{w_2}{q}\right],$$

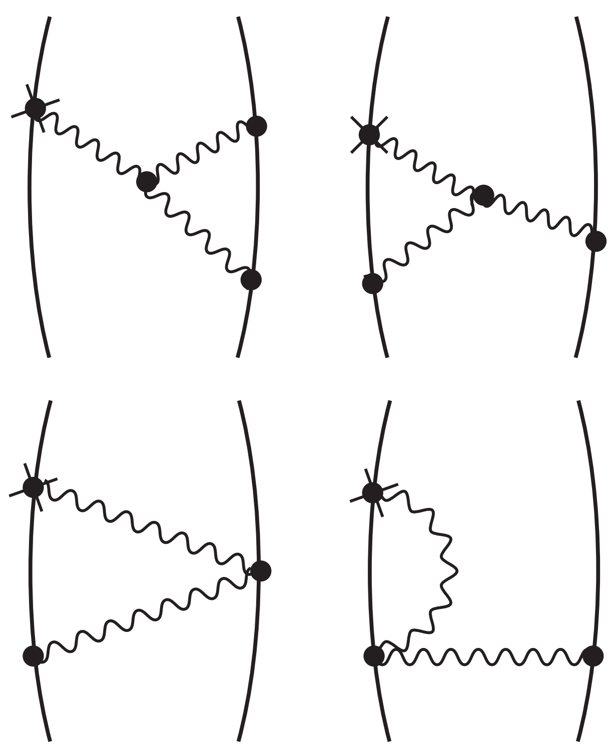
$$\delta_C = i\frac{w_1}{2k\hat{\hbar}^2}\ln\left(\sin^2\frac{\theta}{2}\right) + 2i\arg\Gamma\left(1 - i\frac{w_1}{2k\hat{\hbar}^2}\right).$$

Classical/quantum dictionary: prediction for one-loop result

$\mathcal{M}^{G^2}/\mathcal{M}^{G^1}$ with

$$\mathcal{M}^{(\frac{G}{\hbar})}(s,t) = 16\pi \frac{G}{\hbar} \frac{2(p_1\cdot p_2)^2 - p_1^2 p_2^2}{-t}.$$



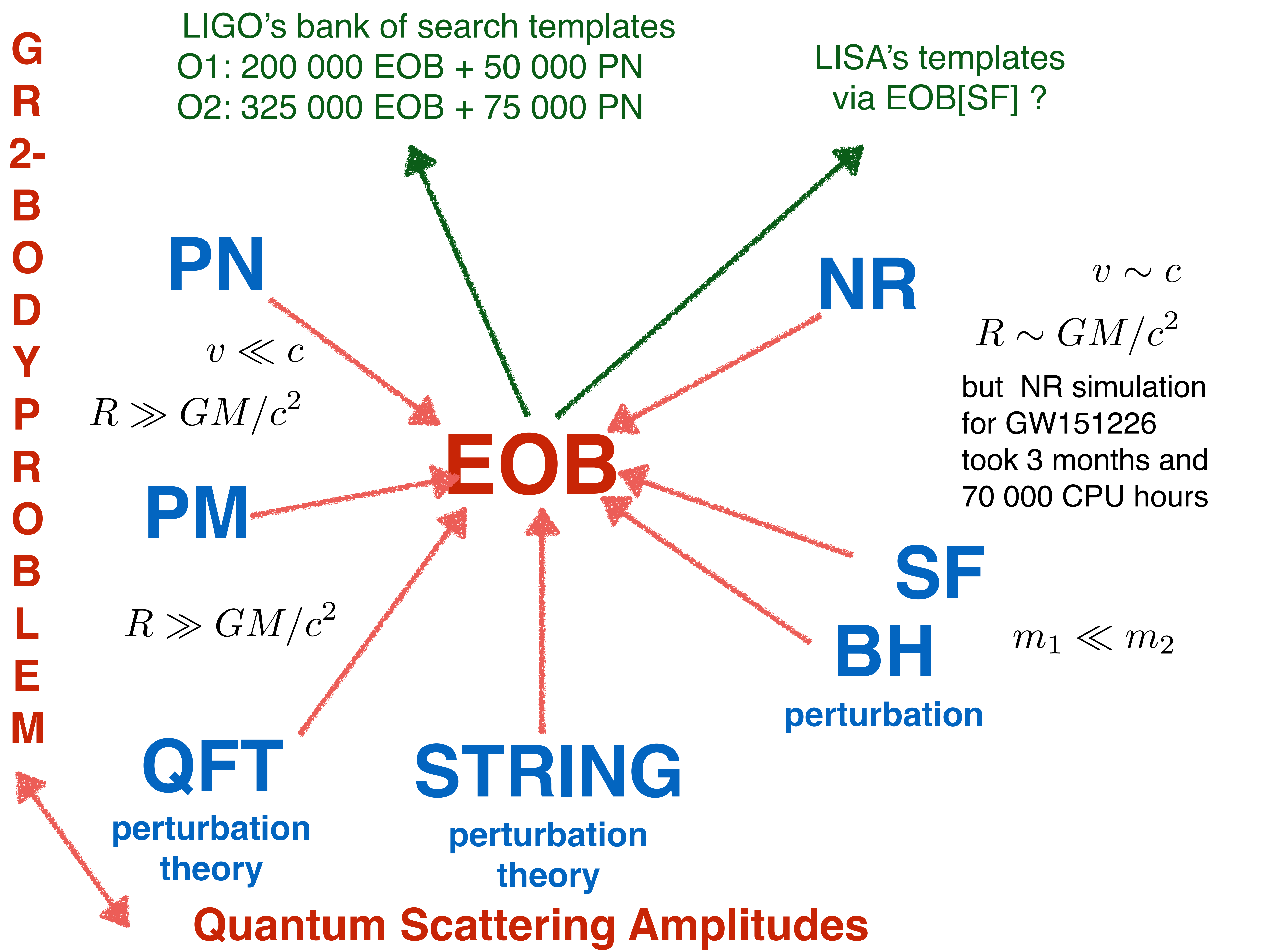


+ Ladder

$$\frac{f^+_{(1/q)}}{f^+_{(1/q^2)}} = \frac{3\pi}{8} \frac{5\hat{\mathcal{E}}^2_{\text{eff}} - 1}{2\hat{\mathcal{E}}^2_{\text{eff}} - 1} \frac{G(m_1+m_2)\sqrt{-t}}{\hbar} + O(G^2).$$

OK with one-loop result of Guevara 1706.02314;

2-loop amplitude ?? would give 3PM O(G^3) EOB Hamiltonian



LIGO's bank of search templates
O1: 200 000 EOB + 50 000 PN
O2: 325 000 EOB + 75 000 PN
LISA's templates
via EOB[SF] ?
G R 2- B O D Y P R O B L E M
PN
$v \ll c$
$R \gg GM/c^2$
NR
$v \sim c$
$R \sim GM/c^2$
but NR simulation
for GW151226
took 3 months and
70 000 CPU hours
EOB
PM
$R \gg GM/c^2$
SF
BH
perturbation
$m_1 \ll m_2$
QFT
perturbation
theory
STRING
perturbation
theory
Quantum Scattering Amplitudes