Classical and Quantum Gravitational Scattering, and the General Relativistic Two-Body Problem (lecture 1)

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Quantum Gravity, Strings and Fields
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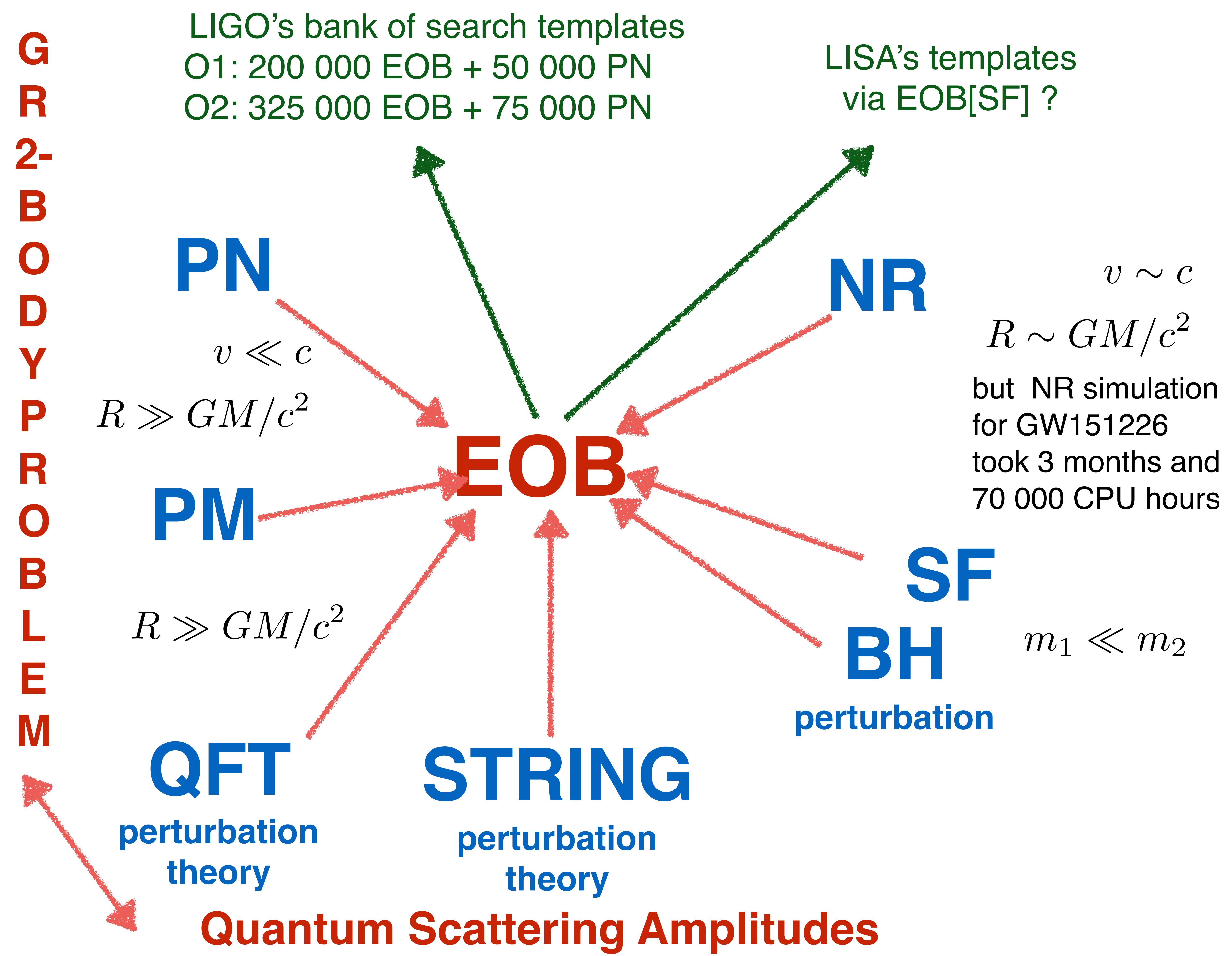
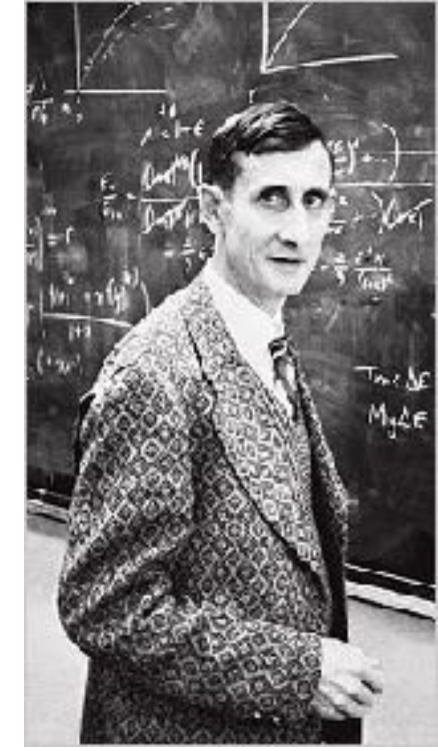
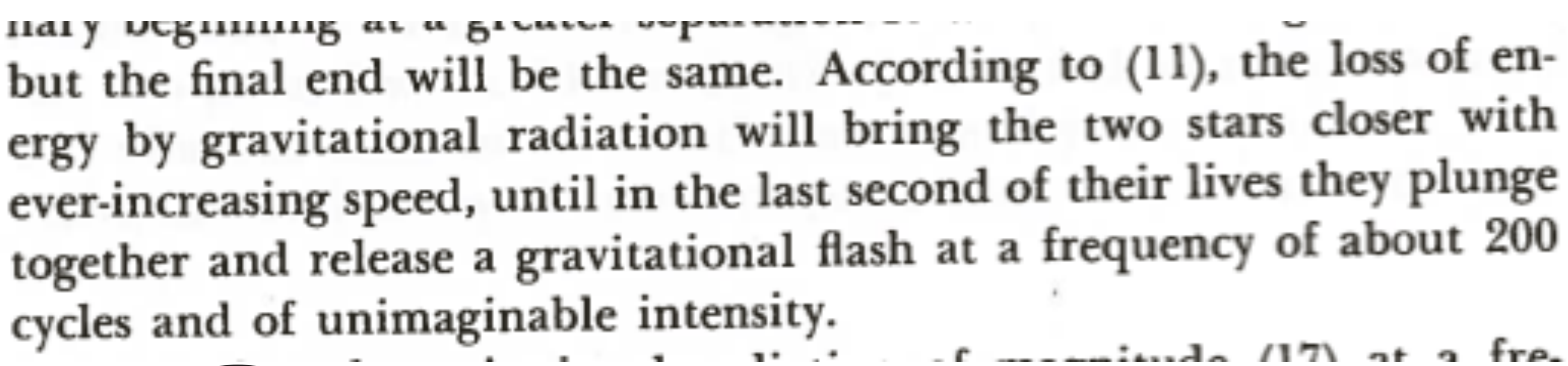


LIGO's bank of search templates
O1: 200 000 EOB + 50 000 PN
O2: 325 000 EOB + 75 000 PN
LISA's templates
via EOB[SF] ?
G R 2- B O D Y P R O B L E M
PN
$v \ll c$
$R \gg GM/c^2$
NR
$v \sim c$
$R \sim GM/c^2$
but NR simulation for GW151226 took 3 months and 70 000 CPU hours
EOB
PM
$R \gg GM/c^2$
SF
BH
perturbation
$m_1 \ll m_2$
QFT
perturbation theory
STRING
perturbation theory
Quantum Scattering Amplitudes

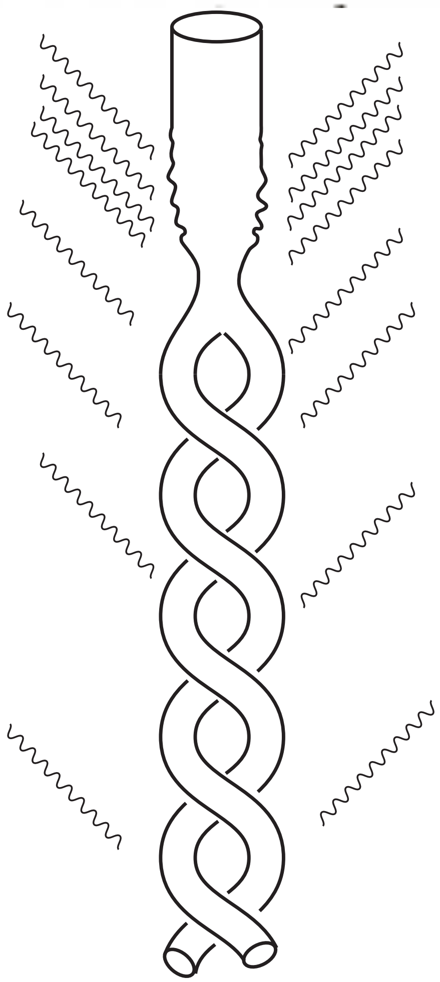
Pioneering the GWs from coalescing compact binaries



Freeman Dyson 1963, using Einstein 1918 + Landau-Lifshitz 1951 (+ Peters '64)
first vision of an intense GW flash from coalescing binary NS



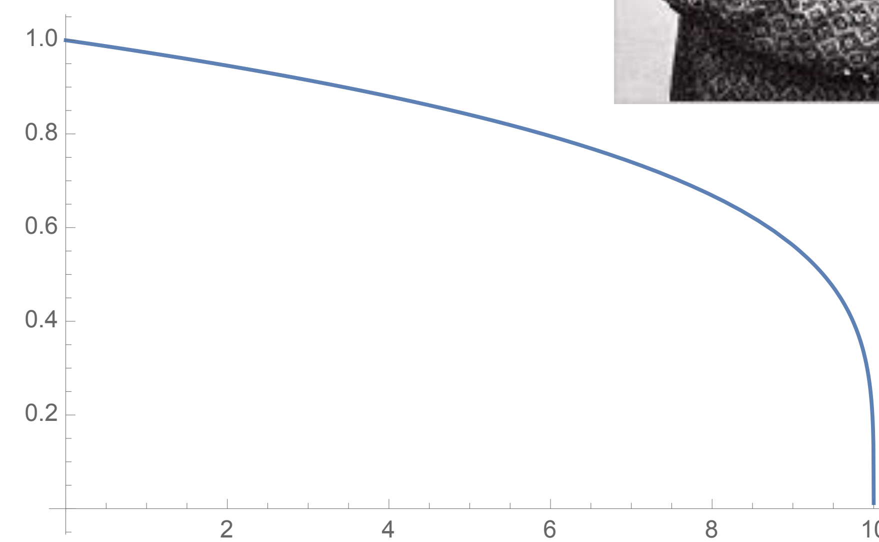
but the final end will be the same. According to (11), the loss of energy by gravitational radiation will bring the two stars closer with ever-increasing speed, until in the last second of their lives they plunge together and release a gravitational flash at a frequency of about 200 cycles and of unimaginable intensity.

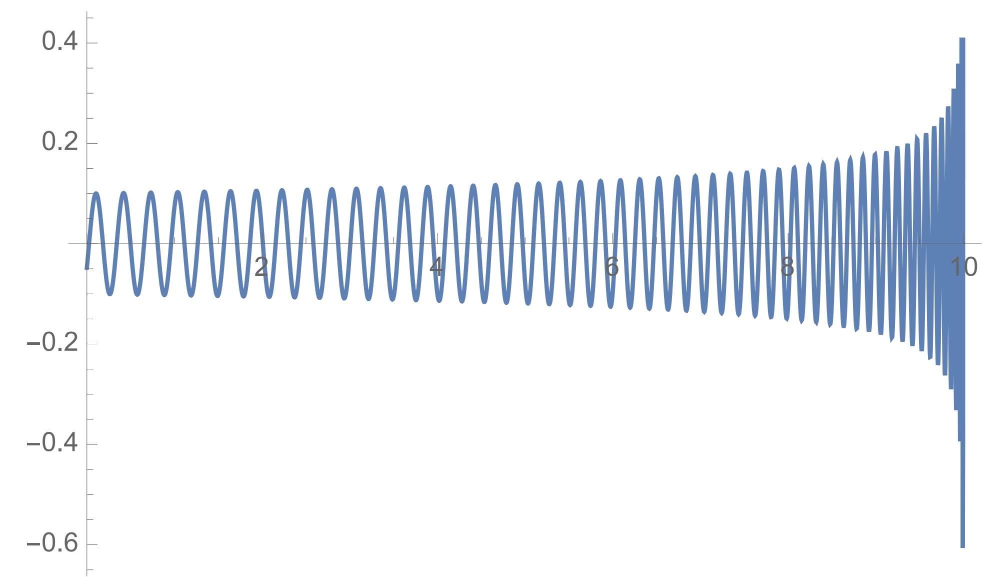


$$E = -\frac{G\, m_1\, m_2}{2r}$$

$$\frac{d}{dt} E = -F$$

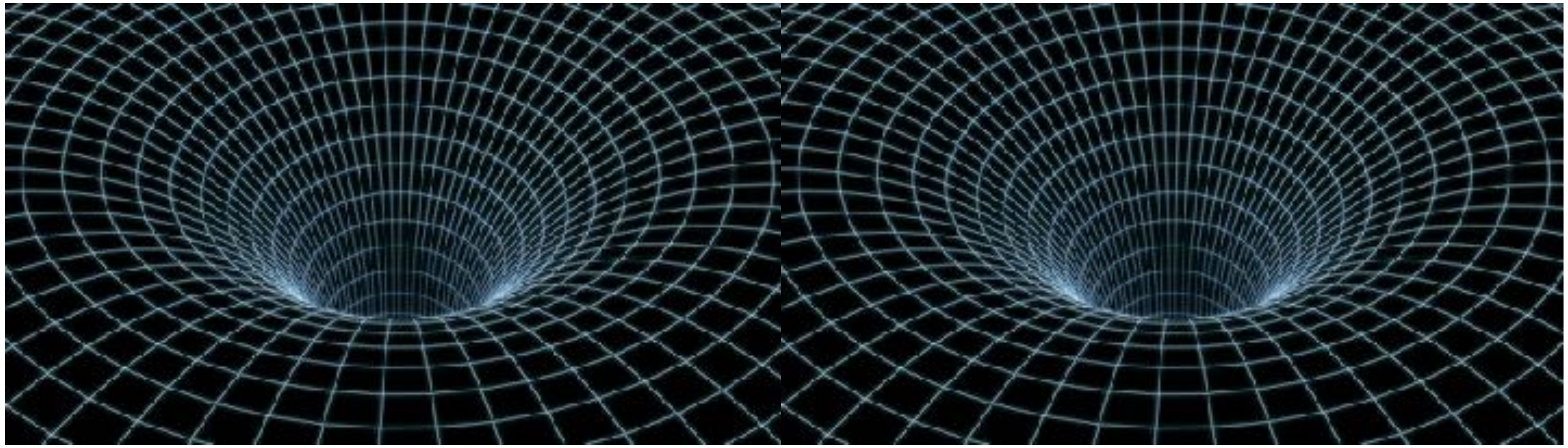
$$F = \frac{32\, G^4}{5\, c^5} \frac{m_1^2\, m_2^2 (m_1 + m_2)}{r^5}$$

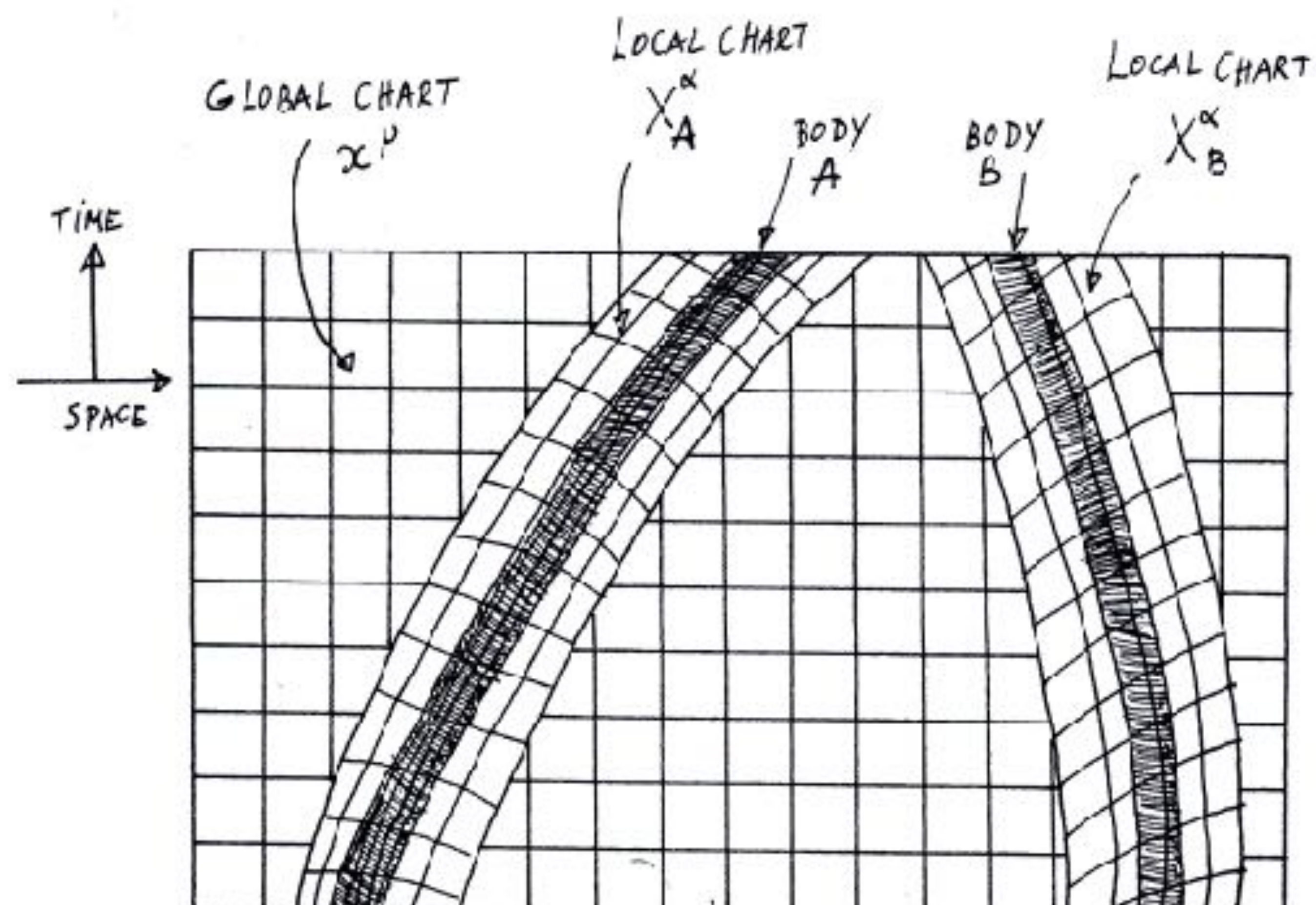




Challenge: describe the intense flash of GWs emitted by the last orbits and the merger of a binary BH, when v~c and r~GM/c^2

Challenge: Motion of Strongly Self-gravitating Bodies (NS, BH)





Multi-chart approach to motion of strong-self-gravity bodies, and matched asymptotic expansions [EIH '38], Manasse '63, Demianski-Grishchuk '74, D'Eath'75, Kates '80, Damour '82

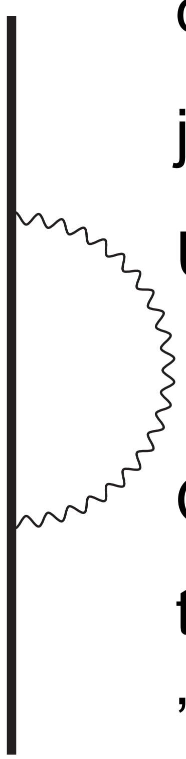
Useful even for weakly self-gravitating bodies, i.e."relativistic celestial mechanics", Brumberg-Kopeikin '89, Damour-Soffel-Xu '91-94

Combine two expansions in two charts:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + G h^{(1)}_{\mu\nu}(x) + G^2 h^{(2)}_{\mu\nu}(x) + \ldots \quad G_{\alpha\beta}(x) = G^{(0)}_{\alpha\beta}(x) + G^{(1)}_{\alpha\beta}(x) + \cdots$$

Practical Techniques for Computing the Motion of Compact Bodies (NS or BH)

Skeletonization : $T_{\mu\nu} \longrightarrow$ point-masses (Mathisson '31)



delta-functions in GR : Infeld '54, Infeld-Plebanski '60

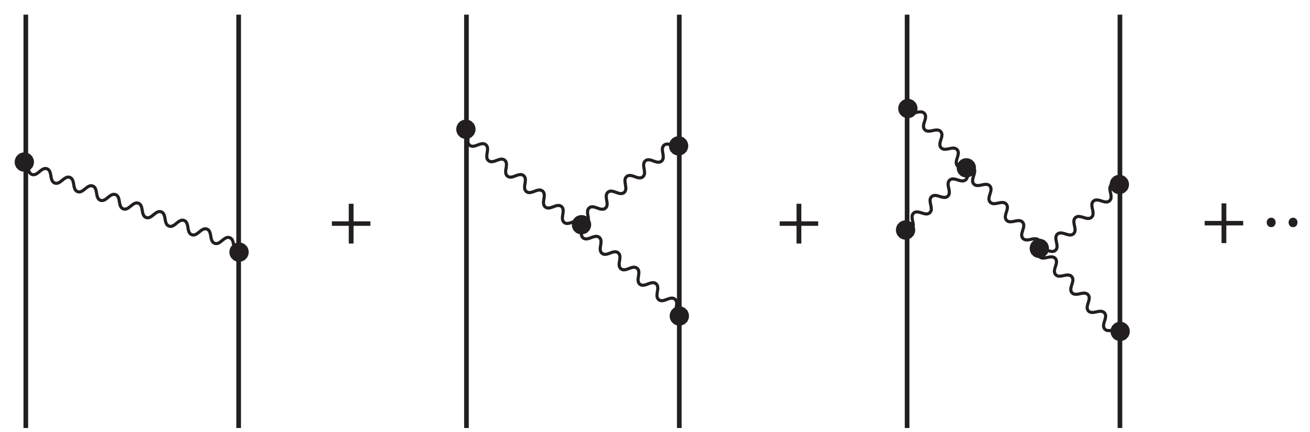
justified by Matched Asymptotic Expansions (« Effacing Principle » Damour '83)

UV divergences linked to self-field effects (loops on external lines) [Dirac, 1938]

QFT's analytic (Riesz '49) or dimensional regularization (Bollini-Giambiagi '72, t'Hooft-Veltman '72) imported in GR (Damour '80, Damour-Jaranowski-Schäfer '01, …)

Feynman-like diagrams and

« Effective Field Theory » techniques



Bertotti-Plebanski '60,

Damour-Esposito-Farèse '96,

Goldberger-Rothstein '06, Porto '06, Gilmore-Ross' 08, Levi '10,

Foffa-Sturani '11 '13, Levi-Steinhoff '14, '15

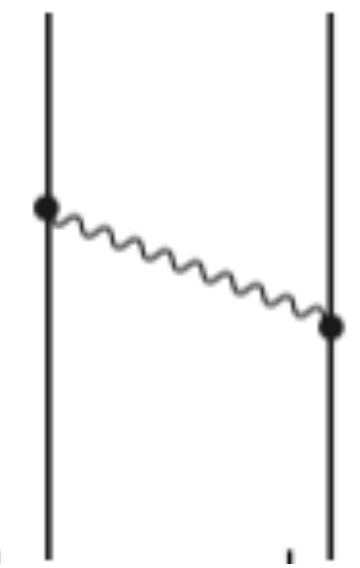
Fokker Action in Electrodynamics (1929)

$$S_{\rm tot}[x_a^\mu, A_\mu] = -\sum_a \int m_a ds_a + \sum_a \int e_a dx_a^\mu A_\mu(x_a) - \int d^D x \frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu} + S_{\rm gf}$$

« Integrate out » the field A_mu in the total (particle+field) action

$$S_{\rm eff}^{\rm class}[x_a(s_a)] = -\sum_a m_a \int \mathrm{d}s_a + \frac{1}{2}\sum_{a,b} e_a e_b \iint \mathrm{d}x_a^\mu \, \mathrm{d}x_{b\mu}\, \delta\left((x_a - x_b)^2\right).$$

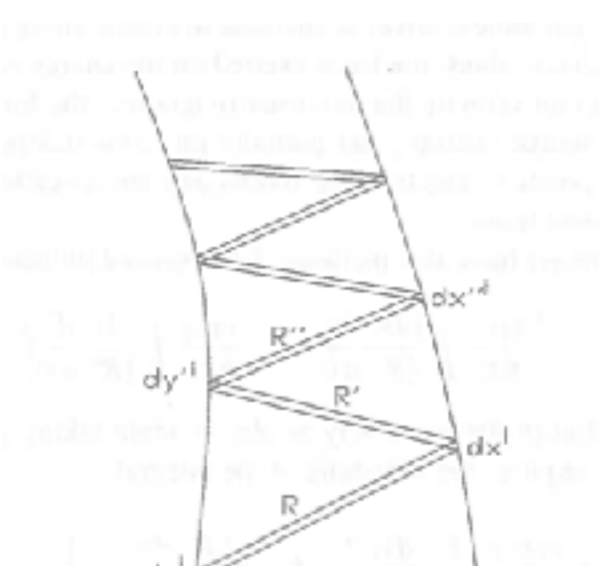
One-photon-exchange diagram

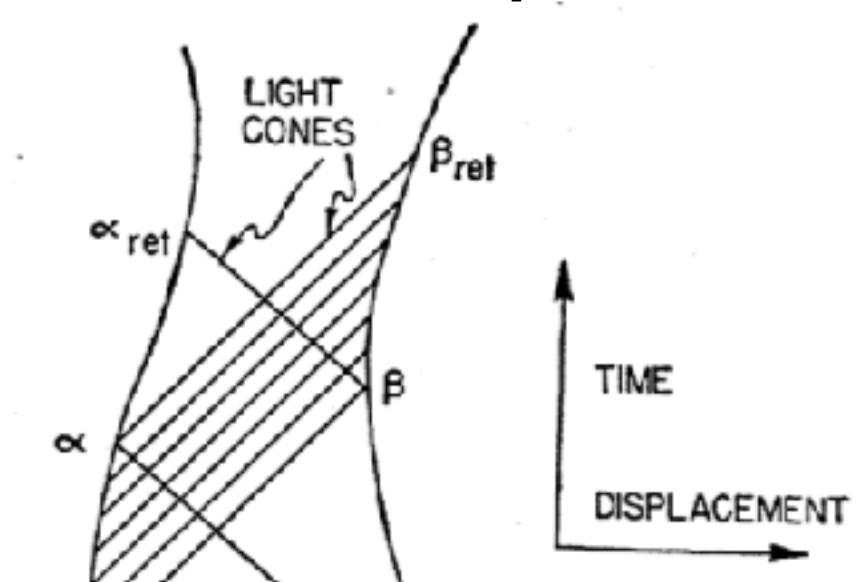


time-symmetric Green function G.

$$G(x) = \delta(-\eta_{\mu\nu}x^\mu x^\nu) = \frac{1}{2r}\left(\delta(t-r) + \delta(t+r)\right) \; ; \; \Box G(x) = -4\pi\delta^4(x)$$

The effective action S_eff(x_a) was heavily used in the (second) Wheeler-Feynman paper (1949) together with similar diagrams to those used by Fokker





Fokker-type Action in Gravity and its Diagrammatic Expansion

Damour-Esposito-Farese '96

$$S_{\rm eff}^{\rm class}[x_a(s_a)] = [S_{\rm pm} + S_{\rm EH} + S_{\rm gf}]_{g_{\mu\nu}(x)\to g^{\rm gf}_{\mu\nu}[x_a(s_a)]}$$

Needs gauge-fixed* action and time-symmetric Green function G.

*E.g. Arnowitt-Deser-Misner Hamiltonian formalism or harmonic coordinates.

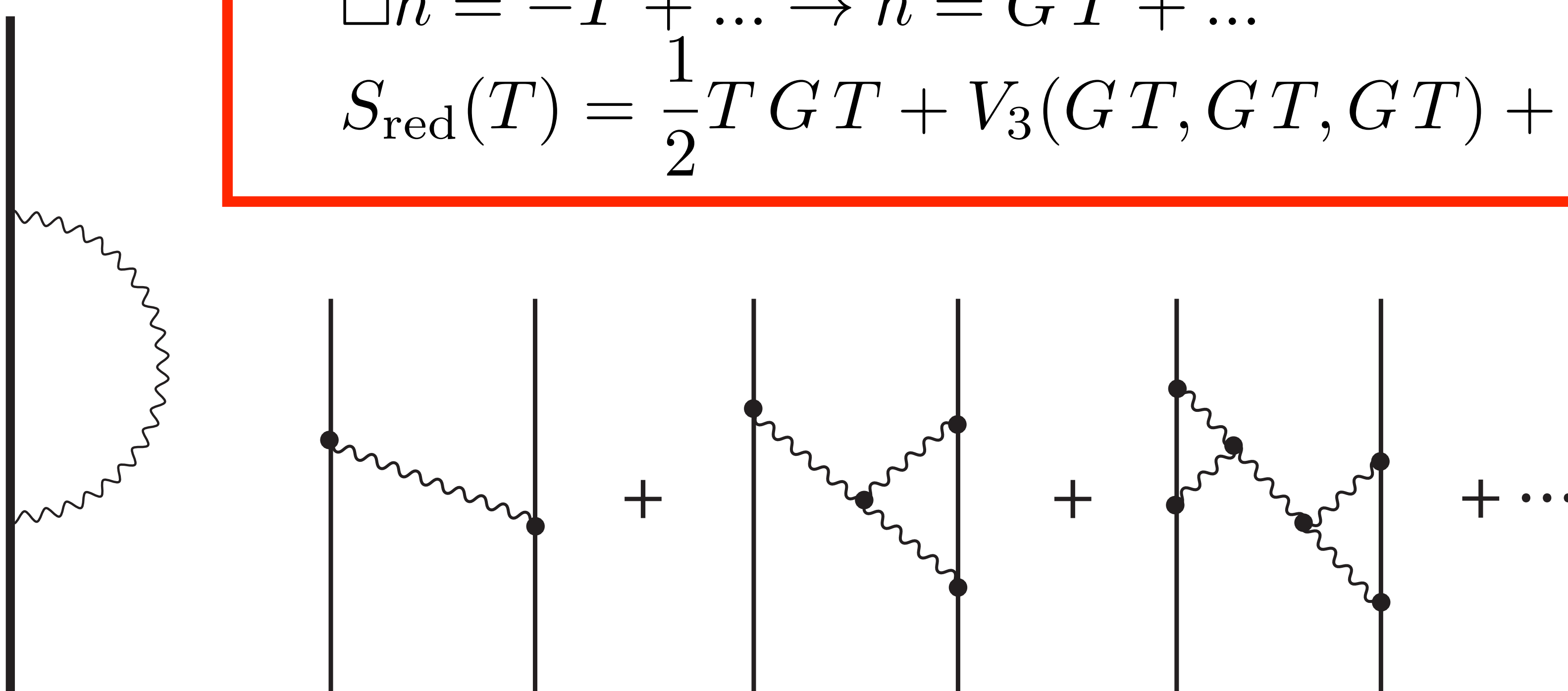
Perturbatively solving (in dimension D=4 - eps) Einstein's equations to get the equations of motion and the action for the conservative dynamics

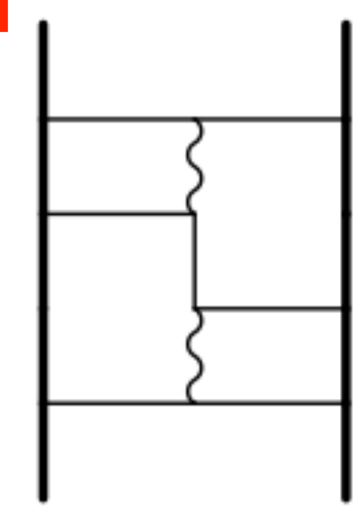
$$g = \eta + h$$

$$S(h,T) = \int \left(\frac{1}{2} h \Box h + \partial\partial hhh + \ldots + (h + hh + \ldots)T \right)$$

$$\Box h = -T + \ldots \to h = G\,T + \ldots$$

$$S_{\rm red}(T) = \frac{1}{2} T\,G\,T + V_3(G\,T, G\,T, G\,T) + \ldots$$





O(G)= Newtonian + (v/c)^n corrections

O(G^2)=1PN =1 loop

O(G^3)=2PN =2 loop

O(G^5)=4PN =4 loop

Arnowitt-Deser-Misner (ADM) Hamiltonian approach (Jaranowski-Schaefer'18)

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -(Nc\,\mathrm{d}t)^2 + \gamma_{ij}(\mathrm{d}x^i + N^i c\,\mathrm{d}t)(\mathrm{d}x^j + N^j c\,\mathrm{d}t),$$

$$\gamma_{ij} \equiv g_{ij}, \quad N \equiv (-g^{00})^{-1/2}, \quad N^i = \gamma^{ij}N_j \quad \text{with} \quad N_i \equiv g_{0i},$$

$$\pi_{ij} \equiv -\gamma^{1/2}(K_{ij} - K\gamma_{ij}), \tag{2.9}$$

with $K \equiv \gamma^{ij}K_{ij}$, where $K_{ij} = -N\Gamma^0_{ij}$ is the extrinsic curvature of $t = \text{const}$ slices, Γ^0_{ij} denote

$$S = \int \mathrm{d}t \left(\frac{c^3}{16\pi G} \int \mathrm{d}^3x\, \pi^{ij}\partial_t \gamma_{ij} + \sum_a p_{ai}\dot{x}^i_a - H_0 \right) \qquad H_0 = \int \mathrm{d}^3x\,(N\mathcal{H} - cN^i\mathcal{H}_i) + \frac{c^4}{16\pi G}\oint \mathrm{d}S_i\, \partial_j(\gamma_{ij} - \delta_{ij}\gamma_{kk})$$

constraints

$$\mathcal{H} = \frac{c^4}{16\pi G}\left[\frac{1}{\gamma^{1/2}}\left(\pi^i_j\pi^j_i - \frac{1}{2}\pi^2\right) - \gamma^{1/2}R\right] + \sum_a c\left(m_a^2c^2 + \gamma^{ij}_a p_{ai}p_{aj}\right)^{1/2}\delta_a,$$

$$\mathcal{H}_i = \frac{c^3}{8\pi G}\nabla_j \pi^j_i + \sum_a p_{ai}\delta_a,$$

Elliptic eqs for phi and V^i and
Hyperbolic eqs for hTT,piTT

ADMTT
coordinate
gauge

$$g_{ij} = A(\phi)\delta_{ij} + h^{\mathrm{TT}}_{ij},$$

$$\pi^{ij} = \tilde{\pi}^{ij}(V^k) + \pi^{ij}_{\mathrm{TT}},$$

where

$$A(\phi) \equiv \left(1 + \frac{d-2}{4(d-1)}\phi\right)^{4/(d-2)},$$

$$\tilde{\pi}^{ij}(V^k) \equiv \partial_i V^j + \partial_j V^i - \frac{2}{d}\delta^{ij}\partial_k V^k,$$

$$\Delta\phi_{(2)} = -\sum m_a\delta_a,$$

$$\partial_j\tilde{\pi}^{ij}_{(3)}(V) \equiv \Delta V^i_{(3)} + \left(1 - \frac{2}{d}\right)\partial_{ij}V^j_{(3)} = -\frac{1}{2}\sum_a p_{ai}\delta_a.$$

$$\Delta h^{\mathrm{TT}}_{(4)ij} = \left(-\sum_a \frac{p_{ai}p_{aj}}{m_a}\delta_a - \frac{d-2}{2(d-1)}\phi_{(2),i}\phi_{(2),j}\right)^{\mathrm{TT}},$$

$$\pi^{ij}_{(5)\mathrm{TT}} = \frac{1}{2}\dot{h}^{\mathrm{TT}}_{(4)ij} + \frac{d-2}{d-1}\left(\phi_{(2)}\tilde{\pi}^{ij}_{(3)}\right)^{\mathrm{TT}}.$$

Alternative Computation of Effective Action

Instead of classically « integrating out » the field dofs

$$S_{\rm eff}^{\rm class}[x_a(s_a)] = [S_{\rm pm} + S_{\rm EH} + S_{\rm gf}]_{g_{\mu\nu}(x)\to g_{\mu\nu}^{\rm gf}[x_a(s_a)]}$$

Formal functional integral over the field (QED: Feynman '50; …;
GR: « Effective Field Theory » approach
(Goldberger-Rothstein '06, Porto '06, Gilmore-Ross' 08, Levi '10,Foffa-Sturani '11 '13, Levi-Steinhoff '14, '15; Foffa-Mastrolia-Sturani-Sturm'16, Damour-Jaranowski '17)

$$e^{\frac{i}{\hbar}S_{\rm eff}^{\rm quant}} = \int Dg_{\mu\nu}\, e^{\frac{i}{\hbar}(S_{\rm pm}+S_{\rm EH}+S_{\rm gf})}.$$

Saddle-point estimation: $S_{\rm eff}^{\rm quant}[x_a(s_a)] = S_{\rm eff}^{\rm class}[x_a(s_a)] + O(\hbar).$

However, the explicit computations are done differently: by means of Wick's theorem, and p-space integrations

$$e^{\frac{i}{\hbar}S_{\rm eff}} = \int D\varphi\, e^{\frac{i}{\hbar}(\int[\frac{1}{2}\varphi\mathcal{K}\varphi+\varphi s+g\varphi^3+\cdots]}.$$

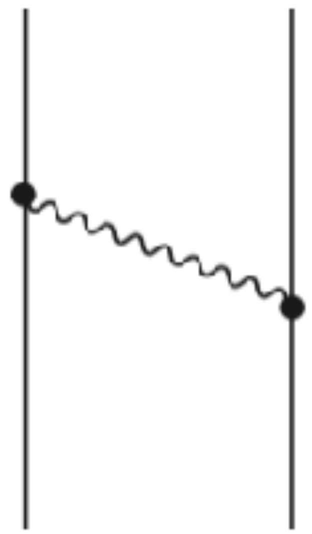
$$\int D\varphi\, e^{\frac{i}{\hbar}\int[\frac{1}{2}\varphi\mathcal{K}\varphi]}\sum_n \frac{(i/\hbar)^n}{n!}\left(\int(\varphi s+g\varphi^3+\cdots)\right)^n$$

$$\langle\varphi(x)\varphi(y)\rangle = \int D\varphi\, e^{\frac{i}{\hbar}\int[\frac{1}{2}\varphi\mathcal{K}\varphi]}\varphi(x)\varphi(y) = i\hbar\mathcal{K}^{-1}_{x,y},$$

PM computation of the Fokker-type Gravity Action

Free-motion action $$-\sum_A m_A c \int \sqrt{-\eta_{\mu\nu} dx_A^\mu dx_A^\nu} = \int dt L^{(1)}$$

Free-motion Lagrangian $$L^{(1)} = \sum_{A=1}^{N} -m_A c^2 \sqrt{1 - v_A^2/c^2},$$



One-graviton exchange $$-(8\pi G_N/c^4) T^{\mu\nu} \Box^{-1} (T_{\mu\nu} - \frac{1}{2} T \eta_{\mu\nu})$$

$$S_{AB} = (-1)^s g_s \iint \mathrm{d}\tau_A \, \mathrm{d}\tau_B \, m_A m_B f_s(u_A, u_B) G(z_A(\tau_A) - z_B(\tau_B))$$

Time-symmetric Green's function

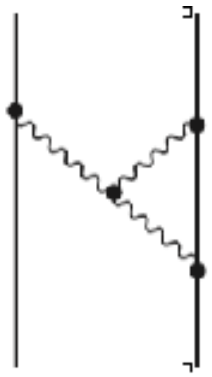
$$G(x-y) = \delta\left(-\eta_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu)\right)$$
$$= \frac{1}{2|\mathbf{x}-\mathbf{y}|} \left(\delta(x^0 - y^0 - |\mathbf{x}-\mathbf{y}|) + \delta(x^0 - y^0 + |\mathbf{x}-\mathbf{y}|)\right)$$

$$f_0(u_A, u_B) = 1$$

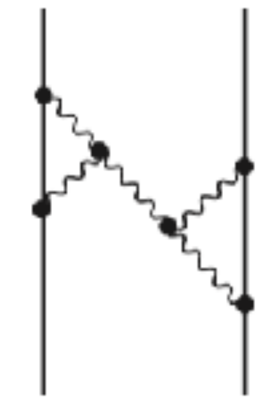
$$f_1(u_A, u_B) = -u_A^\mu f_{\mu\nu} u_B^\nu \equiv -(u_A \cdot u_B)$$

$$f_2(u_A, u_B) = u_A^\mu u_A^\nu (2 f_{\mu\lambda} f_{\nu\rho} - f_{\mu\nu} f_{\lambda\rho}) u_B^\lambda u_B^\rho \equiv 2(u_A \cdot u_B)^2 - 1.$$

PN computation of the Fokker-type Gravity Action



2PM (one-loop) has been explicitly computed
(Westpfahl et al. ’79,’85; Bel-Damour-Deruelle-Ibanez-Martin’81)
but, no classical PM calculations beyond one-loop

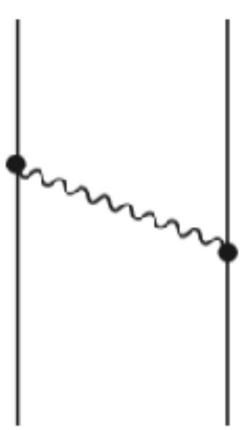


Use slow-motion-weak-field PN expansion: in powers of 1/c^2:
1PN= (v/c)^2; 2PN= (v/c)^4, etc nPN=(v/c)^(2n)

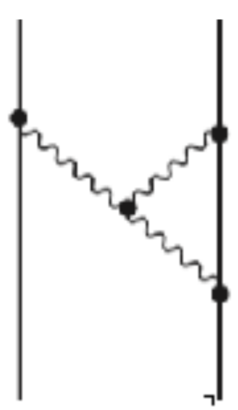
$$\Box^{-1} = (\Delta - \frac{1}{c^2}\partial_t^2)^{-1} = \Delta^{-1} + \frac{1}{c^2}\partial_t^2\Delta^{-2} + \ldots$$

1PN= G[(v/c)^2+ Gm/(r c^2)]

$$L^{(1)} = \sum_A -m_A c^2 \sqrt{1 - \frac{v_A^2}{c^2}} = \sum_A \left(-m_A c^2 + \frac{1}{2} m_A v_A^2 + \frac{1}{8c^2} m_A v_A^4 + \cdots \right)$$



$$L^{(2)} = \frac{1}{2} \sum_{A \neq B} \frac{G_N\, m_A\, m_B}{r_{AB}} \left[1 + \frac{3}{2c^2}(\boldsymbol{v}_A^2 + \boldsymbol{v}_B^2) - \frac{7}{2c^2}(\boldsymbol{v}_A \cdot \boldsymbol{v}_B) - \frac{1}{2c^2}(\boldsymbol{n}_{AB} \cdot \boldsymbol{v}_A)(\boldsymbol{n}_{AB} \cdot \boldsymbol{v}_B) + O\left(\frac{1}{c^4}\right) \right].$$

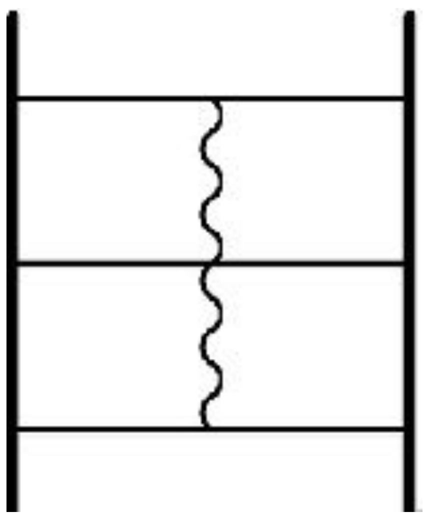


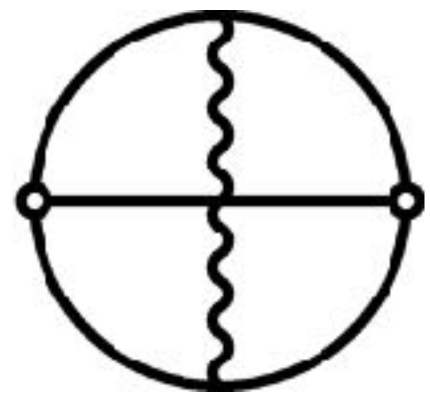
$$L^{(3)} = -\frac{1}{2} \sum_{B \neq A \neq C} \frac{G_N^2\, m_A\, m_B\, m_C}{r_{AB}\, r_{AC}\, c^2} + O\left(\frac{1}{c^4}\right)$$

PN computation of the Fokker-type Gravity Action

Slow Motion (PN) expansion: in powers of 1/c^2: 1PN= (v/c)^2; 2PN= (v/c)^4, etc nPN=(v/c)^(2n)

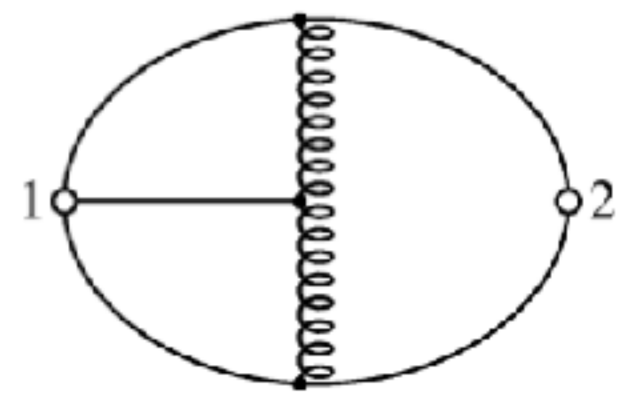
$$\Box^{-1} = (\Delta - \frac{1}{c^2}\partial_t^2)^{-1} = \Delta^{-1} + \frac{1}{c^2}\partial_t^2\Delta^{-2} + \dots$$



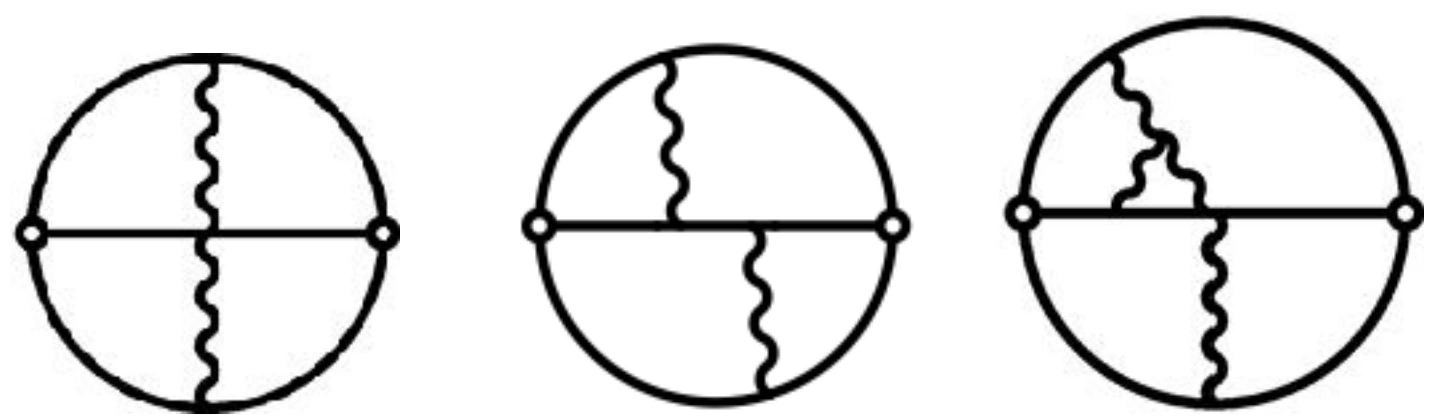


Use PN-expanded Green function for explicit computations.
This transforms spacetime diagrams (between two worldlines) into (massless) two-point space diagrams (in three dimensions)

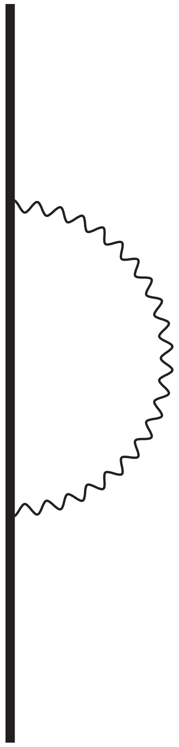
E.g. at 3PN, a 3-loop space diagram
~ G^4 m_1^3 m_2^2
(Damour-Jaranowski-Schaefer 2001)



E.g. at 4PN, some 4-loop space diagrams
~ G^5 m_1^3 m_2^3
among 515 4PN-level diagrams,
(Damour-Jaranowski-Schaefer '14,
Bernard et al '16, Foffa et al '17)



Analytic Continuation and Dimensional Regularization



Self-energy of classical point-particles

in dimension d=3

$$\Delta\phi = -4\pi G m \delta^{(3)}\,,\ \phi = \frac{Gm}{r}$$

$$E_{self} \sim \int d^3x (\nabla\phi)^2 \sim Gm^2 \int r^2 dr \frac{1}{r^4} = Gm^2 \int dr r^{-2} = \infty$$

$$E_{self} \sim m\phi(0) = Gm^2 [\frac{1}{r}]_{r=0} = \infty$$

in dimension d=3+ eps

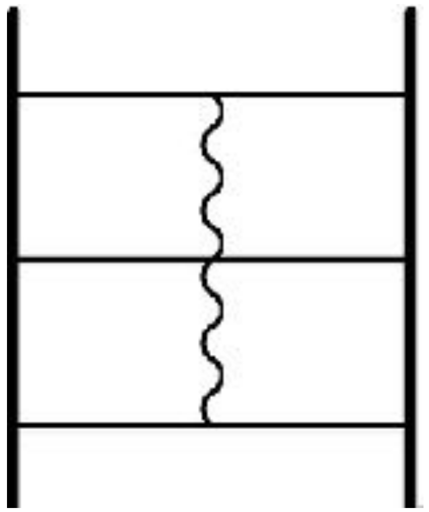
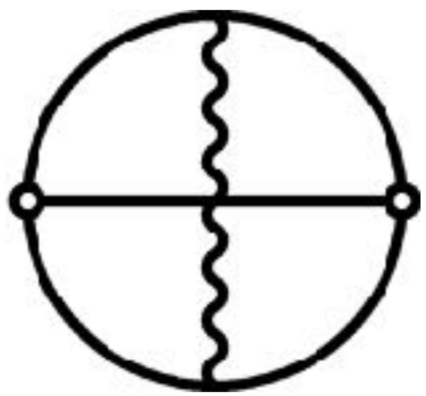
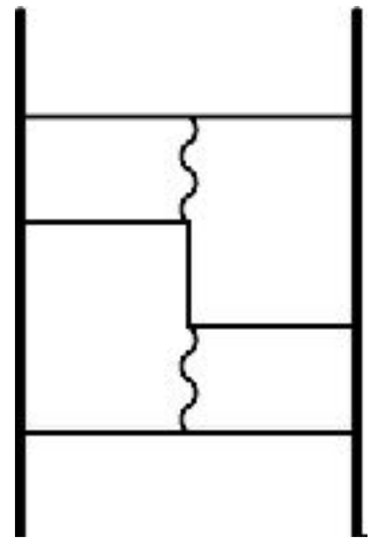
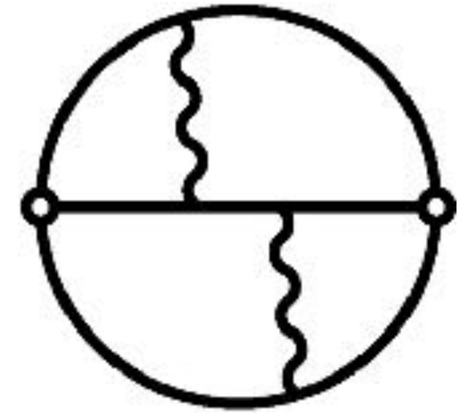
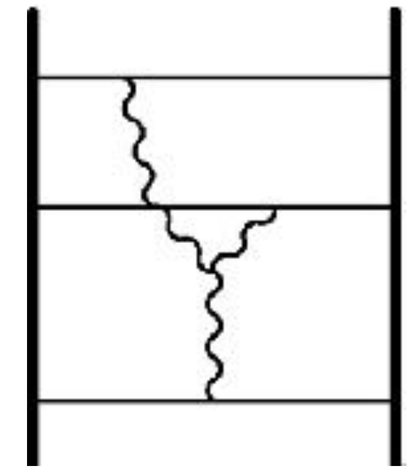
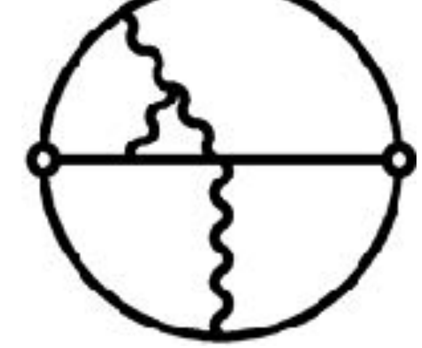
$$\Delta\phi = -4\pi G m k_d \delta^{(d)}\,,\ \phi = \frac{Gm}{r^{d-2}}$$

$$E_{self} \sim \int d^dx (\nabla\phi)^2 \sim Gm^2 \int r^{d-1} dr \frac{1}{r^{2(d-1)}} \sim Gm^2 \int dr r^{-(d-1)} = 0$$

$$E_{self} \sim m\phi(0) = Gm^2 \left[\frac{1}{r^{d-2}}\right]_{r=0} = 0$$

Justified by Matched Asymptotic Expansions, at least when being able to use Marcel Riesz kernels (Damour '82)

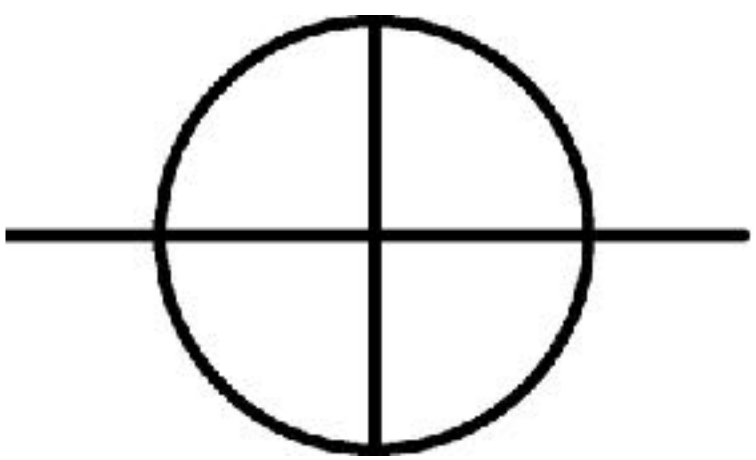
Four-loop static contribution to the gravitational interaction potential of two point masses (Foffa-Mastrolia-Sturani-Sturm’16, Damour-Jaranowski ’17)

$$L_{33} = (32 - 2\pi^2)\frac{G^5 m_1^3 m_2^3}{c^8 r_{12}^5}.$$

$$L_{49} = (64 - 6\pi^2)\frac{G^5 m_1^3 m_2^3}{c^8 r_{12}^5}.$$

$$L_{50} = \left(-\frac{248}{3} + 8\pi^2\right)\frac{G^5 m_1^3 m_2^3}{c^8 r_{12}^5}.$$

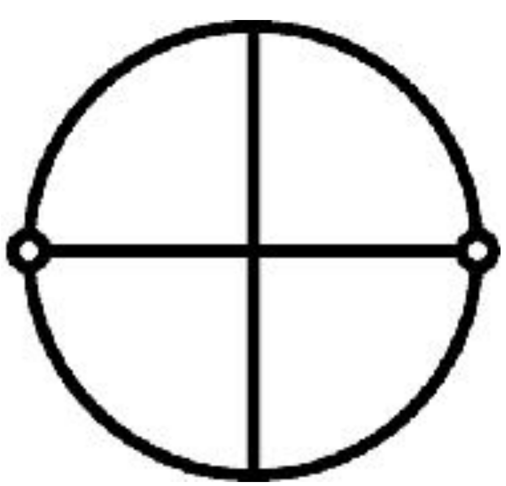


$$\mathcal{M}_{3,6}(\boldsymbol{p}) = \widehat{\mathcal{M}}_{3,6}|\boldsymbol{p}|^{4\varepsilon-4},$$

(num.) Lee-Mingulov’15, Damour-Jaranowski ‘17

where (with γ denoting Euler’s constant)

$$\widehat{\mathcal{M}}_{3,6} = (4\pi)^{-4-2\varepsilon}\frac{e^{2\gamma\varepsilon}}{2}\left[\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon} + \frac{\pi^2}{12} - 8 + O(\varepsilon)\right].$$



All integrals computable by the x-space generalized Riesz integral (Jaranowski-Schaefer ’00)

$$\int d^3x\, r_1^a r_2^b (r_1 + r_2 + r_{12})^c = 2\pi R(a,b,c) r_{12}^{a+b+c+3},$$

Post-Newtonian Equations of Motion [2-body, wo spins]

- 1PN (including v^2/c^2) [Lorentz-Droste '17], Einstein-Infeld-Hoffmann '38
- 2PN (inc. v^4/c^4) Ohta-Okamura-Kimura-Hiida '74, Damour-Deruelle '81
  Damour '82, Schäfer '85, Kopeikin '85
- 2.5 PN (inc. v^5/c^5) Damour-Deruelle '81, Damour '82, Schäfer '85,
  Kopeikin '85
- 3 PN (inc. v^6/c^6) Jaranowski-Schäfer '98, Blanchet-Faye '00,
  Damour-Jaranowski-Schäfer '01, Itoh-Futamase '03,
  Blanchet-Damour-Esposito-Farèse' 04, Foffa-Sturani '11
- 3.5 PN (inc. v^7/c^7) Iyer-Will '93, Jaranowski-Schäfer '97, Pati-Will '02,
  Königsdörffer-Faye-Schäfer '03, Nissanke-Blanchet '05, Itoh '09
- 4PN (inc. v^8/c^8) Jaranowski-Schäfer '13, Foffa-Sturani '13,'16
  Bini-Damour '13, Damour-Jaranowski-Schäfer '14, Bernard et al'16

New feature : non-locality in time (linked to IR divergences of formal PN-expansion)

Inclusion of spin-dependent effects: Barker-O' Connell'75, Faye-Blanchet-Buonanno'06, Damour-Jaranowski-Schaefer'08, Porto-Rothstein '06, Levi '10, Steinhoff-Hergt-Schaefer '10, Steinhoff'11, Levi-Steinhoff'15-18

2-body Taylor-expanded N + 1PN + 2PN Hamiltonian

$$H_{\mathrm{N}}(\mathbf{x}_a, \mathbf{p}_a) = \frac{\mathbf{p}_1^2}{2m_1} - \frac{1}{2}\frac{Gm_1m_2}{r_{12}} + (1 \leftrightarrow 2)$$

$$c^2 H_{1\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) = -\frac{1}{8}\frac{(\mathbf{p}_1^2)^2}{m_1^3} + \frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left(-12\frac{\mathbf{p}_1^2}{m_1^2} + 14\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2} + 2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right) + \frac{1}{4}\frac{Gm_1m_2}{r_{12}}\frac{G(m_1+m_2)}{r_{12}} + (1 \leftrightarrow 2),$$

$$c^4 H_{2\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) = \frac{1}{16}\frac{(\mathbf{p}_1^2)^3}{m_1^5} + \frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left(5\frac{(\mathbf{p}_1^2)^2}{m_1^4} - \frac{11}{2}\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2} - \frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2m_2^2} + 5\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2} - 6\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2} - \frac{3}{2}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right) + \frac{1}{4}\frac{G^2m_1m_2}{r_{12}^2}\left(m_2\left(10\frac{\mathbf{p}_1^2}{m_1^2} + 19\frac{\mathbf{p}_2^2}{m_2^2}\right) - \frac{1}{2}(m_1+m_2)\frac{27(\mathbf{p}_1\cdot\mathbf{p}_2) + 6(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right) - \frac{1}{8}\frac{Gm_1m_2}{r_{12}}\frac{G^2(m_1^2 + 5m_1m_2 + m_2^2)}{r_{12}^2} + (1 \leftrightarrow 2),$$

2-body Taylor-expanded 3PN Hamiltonian [DJS 01]

$$
\begin{aligned}
c^6 H_{3\mathrm{PN}}(\mathbf{x}_a,\mathbf{p}_a) = & -\frac{5}{128}\frac{(\mathbf{p}_1^2)^4}{m_1^7} + \frac{1}{32}\frac{G m_1 m_2}{r_{12}}\left(-14\frac{(\mathbf{p}_1^2)^3}{m_1^6} + 4\frac{((\mathbf{p}_1\cdot\mathbf{p}_2)^2+4\mathbf{p}_1^2\mathbf{p}_2^2)\mathbf{p}_1^2}{m_1^4 m_2^2} + 6\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^4 m_2^2}\right.\\
& - 10\frac{(\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2+\mathbf{p}_2^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2)\mathbf{p}_1^2}{m_1^4 m_2^2} + 24\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^4 m_2^2}\\
& + 2\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3 m_2^3} + \frac{(7\mathbf{p}_1^2\mathbf{p}_2^2-10(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3 m_2^3}\\
& + \frac{(\mathbf{p}_1^2\mathbf{p}_2^2-2(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3 m_2^3} + 15\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3 m_2^3}\\
& \left. - 18\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3 m_2^3} + 5\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3 m_2^3}\right) + \frac{G^2 m_1 m_2}{r_{12}^2}\left(\frac{1}{16}(m_1-27m_2)\frac{(\mathbf{p}_1^2)^2}{m_1^4}\right.\\
& - \frac{115}{16}m_1\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3 m_2} + \frac{1}{48}m_2\frac{25(\mathbf{p}_1\cdot\mathbf{p}_2)^2+371\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2 m_2^2} + \frac{17}{16}\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^3} + \frac{5}{12}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^3}\\
& - \frac{1}{8}m_1\frac{(15\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)+11(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1))(\mathbf{n}_{12}\cdot\mathbf{p}_1)}{m_1^3 m_2} - \frac{3}{2}m_1\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3 m_2}\\
& + \frac{125}{12}m_2\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2 m_2^2} + \frac{10}{3}m_2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2}\\
& \left. - \frac{1}{48}(220m_1+193m_2)\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2}\right) + \frac{G^3 m_1 m_2}{r_{12}^3}\left(-\frac{1}{48}\left(425m_1^2+\left(473-\frac{3}{4}\pi^2\right)m_1 m_2+150m_2^2\right)\frac{\mathbf{p}_1^2}{m_1^2}\right.\\
& + \frac{1}{16}\left(77(m_1^2+m_2^2)+\left(143-\frac{1}{4}\pi^2\right)m_1 m_2\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1 m_2} + \frac{1}{16}\left(20m_1^2-\left(43+\frac{3}{4}\pi^2\right)m_1 m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2}\\
& \left. + \frac{1}{16}\left(21(m_1^2+m_2^2)+\left(119+\frac{3}{4}\pi^2\right)m_1 m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1 m_2}\right)\\
& + \frac{1}{8}\frac{G^4 m_1 m_2^3}{r_{12}^4}\left(\left(\frac{227}{3}-\frac{21}{4}\pi^2\right)m_1+m_2\right) + (1\leftrightarrow 2).
\end{aligned}
$$

2-body Taylor-expanded 4PN Hamiltonian [DJS, 2014, JS 2015]

$$
\begin{aligned}
c^8 H^{\rm local}_{\rm 4PN}(\mathbf{x}_a,\mathbf{p}_a) = {} & \frac{7(\mathbf{p}_1^2)^5}{256 m_1^9} + \frac{G m_1 m_2}{r_{12}} H_{48}(\mathbf{x}_a,\mathbf{p}_a) + \frac{G^2 m_1 m_2}{r_{12}^2} m_1 H_{46}(\mathbf{x}_a,\mathbf{p}_a) \\
& + \frac{G^3 m_1 m_2}{r_{12}^3}\left(m_1^2 H_{441}(\mathbf{x}_a,\mathbf{p}_a) + m_1 m_2 H_{442}(\mathbf{x}_a,\mathbf{p}_a)\right) \\
& + \frac{G^4 m_1 m_2}{r_{12}^4}\left(m_1^3 H_{421}(\mathbf{x}_a,\mathbf{p}_a) + m_1^2 m_2 H_{422}(\mathbf{x}_a,\mathbf{p}_a)\right) \\
& + \frac{G^5 m_1 m_2}{r_{12}^5} H_{40}(\mathbf{x}_a,\mathbf{p}_a) + (1 \leftrightarrow 2), \qquad \text{(A3)}
\end{aligned}
$$

$$
\begin{aligned}
H_{48}(\mathbf{x}_a,\mathbf{p}_a) = {} & \frac{45(\mathbf{p}_1^2)^4}{128 m_1^8} - \frac{9(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{64 m_1^6 m_2^2} + \frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^3}{64 m_1^6 m_2^2} - \frac{9(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{16 m_1^6 m_2^2} \\
& - \frac{3(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{32 m_1^6 m_2^2} + \frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{64 m_1^6 m_2^2} - \frac{21(\mathbf{p}_1^2)^3\mathbf{p}_2^2}{64 m_1^6 m_2^2} - \frac{35(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{256 m_1^5 m_2^3} \\
& + \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2}{128 m_1^5 m_2^3} + \frac{23(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3(\mathbf{p}_1^2)^2}{256 m_1^5 m_2^3} - \frac{85(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2^3} \\
& - \frac{45(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{128 m_1^5 m_2^3} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2^3} + \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64 m_1^5 m_2^3} \\
& + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64 m_1^5 m_2^3} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{64 m_1^5 m_2^3} + \frac{3\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{64 m_1^5 m_2^3} + \frac{55(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256 m_1^5 m_2^3} \\
& - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{128 m_1^5 m_2^3} - \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{256 m_1^5 m_2^3} - \frac{23(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256 m_1^5 m_2^3} \\
& + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{128 m_1^5 m_2^3} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256 m_1^5 m_2^3} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4\mathbf{p}_1^2}{64 m_1^4 m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4(\mathbf{p}_1^2)^2}{64 m_1^4 m_2^4} \\
& - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{4 m_1^4 m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{16 m_1^4 m_2^4} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{64 m_1^4 m_2^4} + \frac{21(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{64 m_1^4 m_2^4} \\
& - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{32 m_1^4 m_2^4} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{4 m_1^4 m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{16 m_1^4 m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{16 m_1^4 m_2^4} \\
& - \frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{32 m_1^4 m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_2^2)^2}{64 m_1^4 m_2^4} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{32 m_1^4 m_2^4} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_2^2)^2}{128 m_1^4 m_2^4}, \qquad \text{(A4a)}
\end{aligned}
$$

$$
\begin{aligned}
H_{46}(\mathbf{x}_a,\mathbf{p}_a) = {} & \frac{369(\mathbf{n}_{12}\cdot\mathbf{p}_1)^6}{160 m_1^6} - \frac{889(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4\mathbf{p}_1^2}{192 m_1^6} + \frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1^2)^2}{16 m_1^6} - \frac{63(\mathbf{p}_1^2)^3}{64 m_1^6} - \frac{549(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{128 m_1^5 m_2} \\
& + \frac{67(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{16 m_1^5 m_2} - \frac{167(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2}{128 m_1^5 m_2} + \frac{1547(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2} - \frac{851(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{128 m_1^5 m_2} \\
& + \frac{1099(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2} + \frac{3263(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{1280 m_1^4 m_2^2} + \frac{1067(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{480 m_1^4 m_2^2} - \frac{4567(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{3840 m_1^4 m_2^2} \\
& - \frac{3571(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{320 m_1^4 m_2^2} + \frac{3073(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{480 m_1^4 m_2^2} + \frac{4349(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{1280 m_1^4 m_2^2} \\
& - \frac{3461\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{3840 m_1^4 m_2^2} + \frac{1673(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4\mathbf{p}_2^2}{1920 m_1^4 m_2^2} - \frac{1999(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2\mathbf{p}_2^2}{3840 m_1^4 m_2^2} + \frac{2081(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{3840 m_1^4 m_2^2} - \frac{13(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{8 m_1^3 m_2^3} \\
& + \frac{191(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2}{192 m_1^3 m_2^3} - \frac{19(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384 m_1^3 m_2^3} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384 m_1^3 m_2^3} \\
& + \frac{11(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{192 m_1^3 m_2^3} + \frac{77(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{96 m_1^3 m_2^3} + \frac{233(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{96 m_1^3 m_2^3} - \frac{47(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{32 m_1^3 m_2^3} \\
& + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{384 m_1^3 m_2^3} - \frac{185\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{384 m_1^3 m_2^3} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4}{4 m_1^2 m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4\mathbf{p}_1^2}{4 m_1^2 m_2^4} \\
& - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3(\mathbf{p}_1\cdot\mathbf{p}_2)}{2 m_1^2 m_2^4} + \frac{21(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{16 m_1^2 m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{6 m_1^2 m_2^4} + \frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{48 m_1^2 m_2^4} \\
& - \frac{133(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{24 m_1^2 m_2^4} - \frac{77(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{96 m_1^2 m_2^4} + \frac{197(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_2^2)^2}{96 m_1^2 m_2^4} - \frac{173\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{48 m_1^2 m_2^4} + \frac{13(\mathbf{p}_2^2)^3}{8 m_2^6}, \qquad \text{(A4b)}
\end{aligned}
$$

$$
\begin{aligned}
H_{441}(\mathbf{x}_a,\mathbf{p}_a) = {} & \frac{5027(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{384 m_1^4} - \frac{22993(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2}{960 m_1^4} - \frac{6695(\mathbf{p}_1^2)^2}{1152 m_1^4} - \frac{3191(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{640 m_1^3 m_2} \\
& + \frac{28561(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{1920 m_1^3 m_2} + \frac{8777(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384 m_1^3 m_2} - \frac{752969\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{28800 m_1^3 m_2} \\
& - \frac{16481(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{960 m_1^2 m_2^2} + \frac{94433(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{4800 m_1^2 m_2^2} - \frac{103957(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{2400 m_1^2 m_2^2} \\
& + \frac{791(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{400 m_1^2 m_2^2} + \frac{26627(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_2^2}{1600 m_1^2 m_2^2} - \frac{118261\mathbf{p}_1^2\mathbf{p}_2^2}{4800 m_1^2 m_2^2} + \frac{105(\mathbf{p}_2^2)^2}{32 m_2^4}, \qquad \text{(A4c)}
\end{aligned}
$$

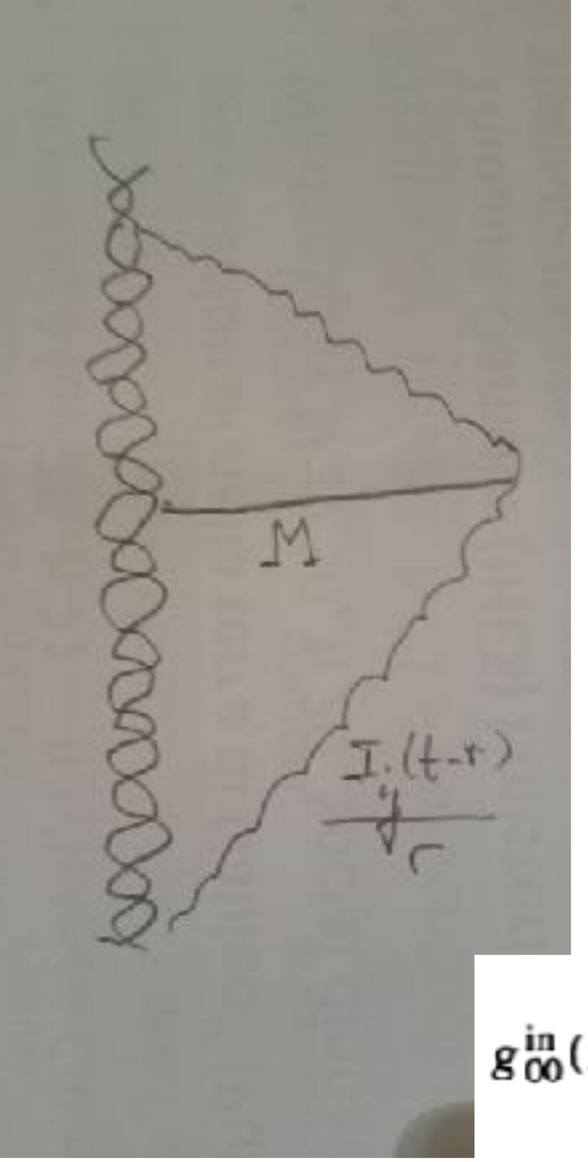
$$
\begin{aligned}
H_{442}(\mathbf{x}_a,\mathbf{p}_a) = {} & \left(\frac{2749\pi^2}{8192} - \frac{211189}{19200}\right)\frac{(\mathbf{p}_1^2)^2}{m_1^4} + \left(\frac{63347}{1600} - \frac{1059\pi^2}{1024}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2}{m_1^4} + \left(\frac{375\pi^2}{8192} - \frac{23533}{1280}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^4} \\
& + \left(\frac{10631\pi^2}{8192} - \frac{1918349}{57600}\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2} + \left(\frac{13723\pi^2}{16384} - \frac{2492417}{57600}\right)\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2 m_2^2} \\
& + \left(\frac{1411429}{19200} - \frac{1059\pi^2}{512}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{m_1^2 m_2^2} + \left(\frac{248991}{6400} - \frac{6153\pi^2}{2048}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^2 m_2^2} \\
& - \left(\frac{30383}{960} + \frac{36405\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2} + \left(\frac{1243717}{14400} - \frac{40483\pi^2}{16384}\right)\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3 m_2} \\
& + \left(\frac{2369}{60} + \frac{35655\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3 m_2} + \left(\frac{43101\pi^2}{16384} - \frac{391711}{6400}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{m_1^3 m_2} \\
& + \left(\frac{56955\pi^2}{16384} - \frac{1646983}{19200}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3 m_2}, \qquad \text{(A4d)}
\end{aligned}
$$

$$
H_{421}(\mathbf{x}_a,\mathbf{p}_a) = \frac{64861\mathbf{p}_1^2}{4800 m_1^2} - \frac{91(\mathbf{p}_1\cdot\mathbf{p}_2)}{8 m_1 m_2} + \frac{105\mathbf{p}_2^2}{32 m_2^2} - \frac{9841(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{1600 m_1^2} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{2 m_1 m_2}, \qquad \text{(A4e)}
$$

$$
\begin{aligned}
H_{422}(\mathbf{x}_a,\mathbf{p}_a) = {} & \left(\frac{1937033}{57600} - \frac{199177\pi^2}{49152}\right)\frac{\mathbf{p}_1^2}{m_1^2} + \left(\frac{176033\pi^2}{24576} - \frac{2864917}{57600}\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1 m_2} + \left(\frac{282361}{19200} - \frac{21837\pi^2}{8192}\right)\frac{\mathbf{p}_2^2}{m_2^2} \\
& + \left(\frac{698723}{19200} + \frac{21745\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2} - \left(\frac{63641\pi^2}{24576} - \frac{2712013}{19200}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1 m_2} \\
& + \left(\frac{3200179}{57600} - \frac{28691\pi^2}{24576}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_2^2}, \qquad \text{(A4f)}
\end{aligned}
$$

$$
H_{40}(\mathbf{x}_a,\mathbf{p}_a) = -\frac{m_1^4}{16} + \left(\frac{6237\pi^2}{1024} - \frac{169799}{2400}\right)m_1^3 m_2 + \left(\frac{44825\pi^2}{6144} - \frac{609427}{7200}\right)m_1^2 m_2^2. \qquad \text{(A4g)}
$$

$$
H^{\rm nonloc}_{\rm 4PN}(t) = -\frac{1}{5}\frac{G^2 M}{c^8} I^{(3)}_{ij}(t) \times {\rm Pf}_{2r_{12}/c}\int_{-\infty}^{+\infty}\frac{dv}{|v|} I^{(3)}_{ij}(t+v).
$$



Nonlocality in time: Tail-transported hereditary effects

(Blanchet-Damour ’88)

Hereditary (time-dissymetric) modification of the quadrupolar radiation-damping force, signalling a breakdown of a basic tenet of PN expansion at the 4PN level: (v/c)^8 fractional

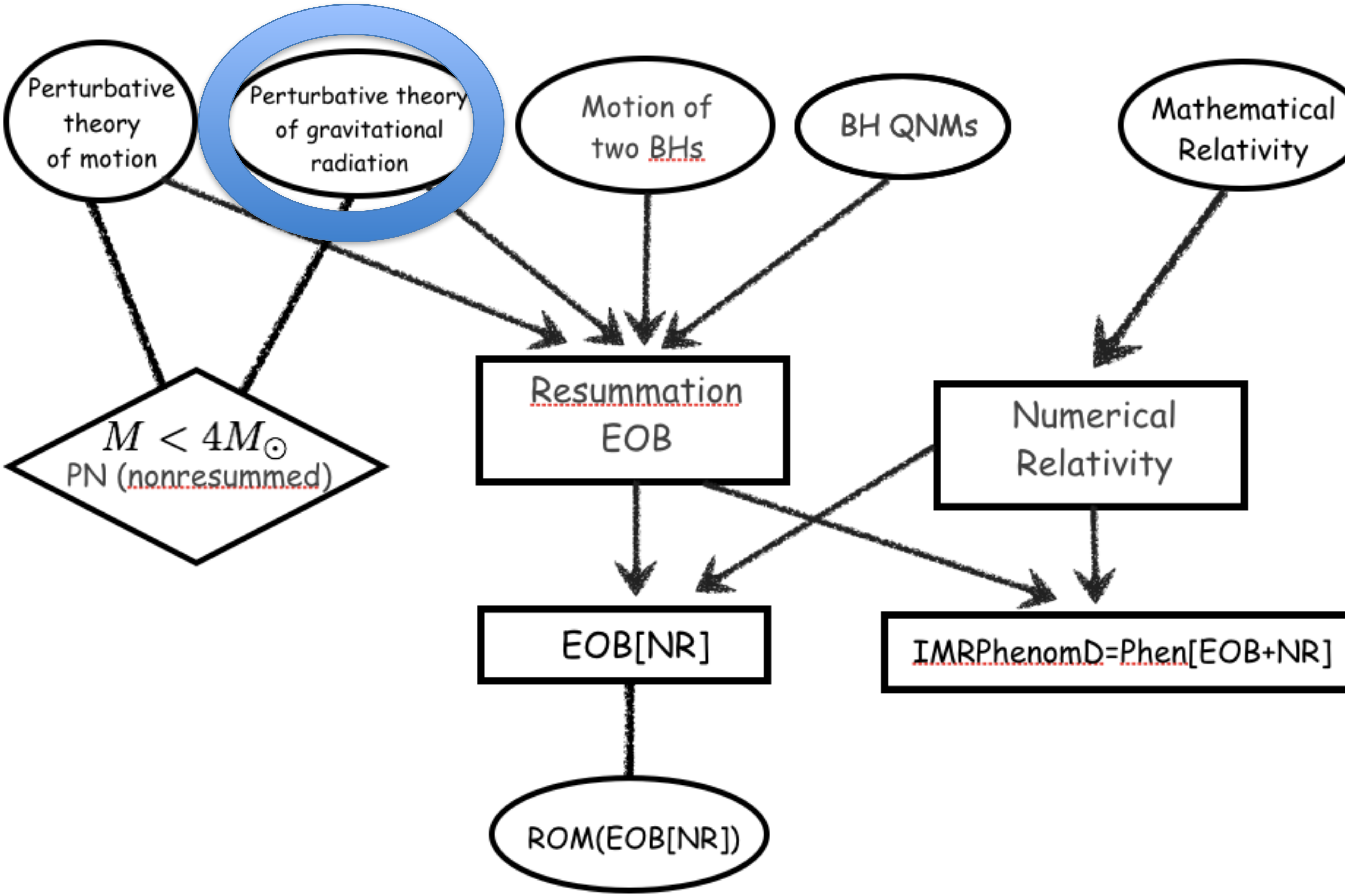
$$g_{00}^{\text{in}}(\mathbf{x},t) = -1 + \frac{1}{c^2}\left[2\int \frac{d^3\mathbf{y}\,\rho(\mathbf{y},t)}{|\mathbf{x}-\mathbf{y}|}\right] + \frac{1}{c^4}\left[\partial_t^2 X - 2U^2 + 4\int \frac{d^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\rho\left[\mathbf{v}^2 + U + \frac{\Pi}{2} + \frac{3p}{2\rho}\right]\right]$$

$$+ \frac{1}{c^6}\,{}_6\hat{\Phi}_{00} + \frac{1}{c^7}\left[-\tfrac{2}{5}x_{ab}\,{}^{(5)}I_{ab}(t)\right] + \frac{1}{c^8}\,{}_8\hat{\Phi}_{00} + \frac{1}{c^9}\,{}_9\hat{\Phi}_{00}$$

$$+ \frac{1}{c^{10}}\left[-\tfrac{8}{5}x_{ab}I(t)\int_0^{+\infty} dv\,\ln\left[\frac{v}{2P}\right]{}^{(7)}I_{ab}(t-v) + {}_{10}\hat{\Phi}_{00}\right] + \cdots .$$

generates a time-symmetric nonlocal-in-time 4PN-level action (Damour-Jaranowski-Schaefer’14) which was uniquely matched to the local-zone metric via the Regge-Wheeler-Zerilli-Mano-Suzuki-Takasugi- based work of Bini-Damour’13

$$H_{4\mathrm{PN}}^{\text{nonloc}}(t) = -\frac{1}{5}\frac{G^2 M}{c^8} I_{ij}^{(3)}(t) \times \mathrm{Pf}_{2r_{12}/c}\int_{-\infty}^{+\infty}\frac{\mathrm{d}v}{|v|} I_{ij}^{(3)}(t+v),$$


Perturbative theory of motion
Perturbative theory of gravitational radiation
Motion of two BHs
BH QNMs
Mathematical Relativity
$M < 4M_\odot$
PN (nonresummed)
Resummation EOB
Numerical Relativity
EOB[NR]
IMRPhenomD=Phen[EOB+NR]
ROM(EOB[NR])

Perturbative Theory of the Generation of Gravitational Radiation

Einstein '16, '18 (+ Landau-Lifshitz 41, and Fock '55) : h_+, h_x and quadrupole formula

Relativistic, multipolar extensions of LO quadrupole radiation :

Sachs-Bergmann '58, Sachs '61, Mathews '62, Peters-Mathews '63, Pirani '64

Campbell-Morgan '71,

Campbell et al '75,

nonlinear effects:

Bonnor-Rotenberg '66,

Epstein-Wagoner-Will '75-76

Thorne '80, .., Will et al 00

MPM Formalism:

Blanchet-Damour '86,

Damour-Iyer '91,

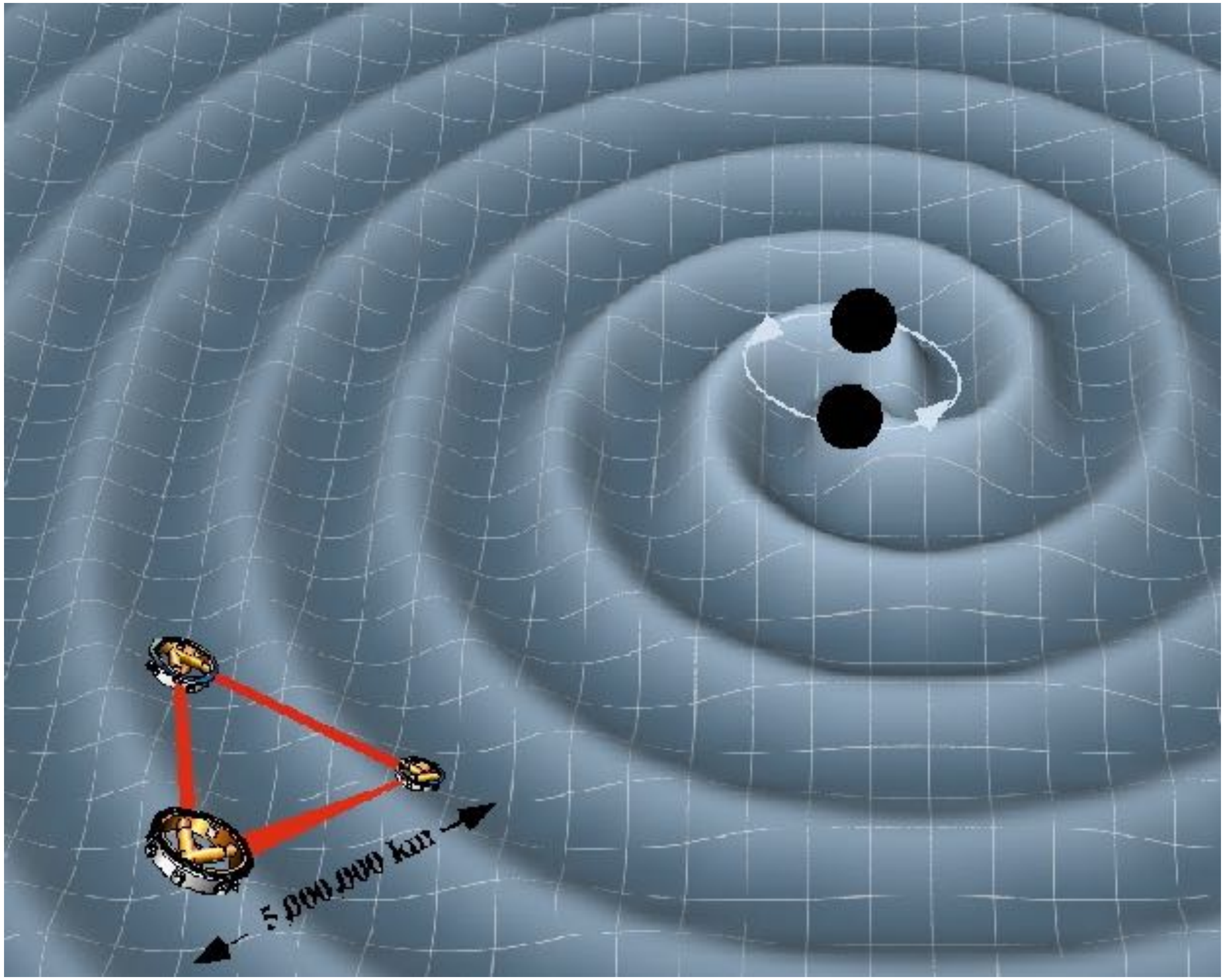
Blanchet '95 '98

Combines multipole exp. ,

Post Minkowkian exp.,

analytic continuation,

and PN matching



Multipolar Expansions Using STF Tensors

A convenient form for (2l+1)-dim irrep of SO(3): STF tensors multi-index notation (Blanchet-Damour '86)

$$\widehat{T}_L = T_{\langle i_1 i_2 \cdots i_l \rangle} \quad T_L S_L \quad \partial_L = \partial_{i_1 i_2 \cdots i_l}$$

Multipolar expansions with STF tensors

$$\Delta\phi = -4\pi\rho$$

can be written either as

$$\phi(\mathbf{X}) = 4\pi \sum_{l \geq 0} \sum_{-l \leq m \leq l} \frac{Q_{lm}}{2l+1} \frac{Y_{lm}(\Theta,\Phi)}{R^{l+1}}$$

or

$$\phi(\mathbf{X}) = \sum_{l \geq 0} \frac{(-)^l}{l!} Q_{i_1 \cdots i_l} \partial_{i_1 \cdots i_l} \left[\frac{1}{R} \right] ,$$

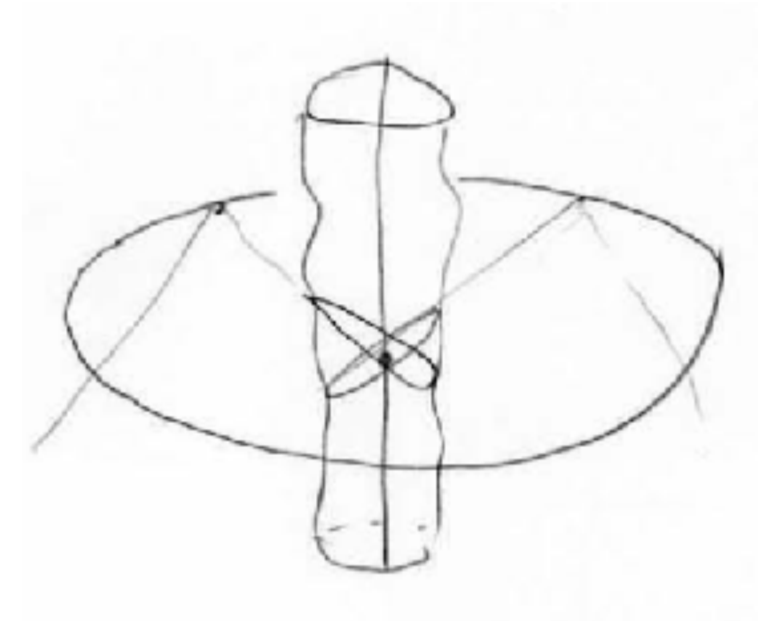
where, respectively,

$$Q_{lm} = \int d^3x \; Y^*_{lm}(\theta,\varphi) r^l \rho(\mathbf{x})$$

or

$$Q_{i_1 \cdots i_l} = \int d^3x \, x^{\langle i_1} \cdots x^{i_l \rangle} \rho(\mathbf{x}) ,$$

STF Multipolar Analysis of Linearized Gravity (Damour-Iyer '91)



$$\Box \bar{h}_{\mu\nu}(\mathbf{X},T) = -\frac{16\pi G}{c^4} T_{\mu\nu}(\mathbf{X},T) \,. \qquad (\partial_\nu \bar{h}^{\mu\nu} = 0$$

$$\bar{h}^{\alpha\beta}_{\text{can}}[\mathbb{M}] = \bar{h}^{\alpha\beta}[\mathbb{M},\mathbb{W}] + \partial^\alpha w^\beta[\mathbb{W}] + \partial^\beta w^\alpha[\mathbb{W}] - f^{\alpha\beta}\partial_\mu w^\mu[\mathbb{W}] \,,$$

$$\bar{h}^{00}_{\text{can}}(\mathbf{X},T) = +\frac{4}{c^2}\sum_{l\geq 0}\frac{(-)^l}{l!}\partial_L[R^{-1}M_L(U)] \,,$$

$$\bar{h}^{0i}_{\text{can}}(\mathbf{X},T) = -\frac{4}{c^3}\sum_{l\geq 1}\frac{(-)^l}{l!}\partial_{L-1}[R^{-1}\dot{M}_{iL-1}(U)] - \frac{4}{c^3}\sum_{l\geq 1}\frac{(-)^l l}{(l+1)!}\epsilon_{iab}\partial_{aL-1}[R^{-1}S_{bL-1}(U)] \,,$$

$$\bar{h}^{ij}_{\text{can}}(\mathbf{X},T) = +\frac{4}{c^4}\sum_{l\geq 2}\frac{(-)^l}{l!}\partial_{L-2}[R^{-1}\ddot{M}_{ijL-2}(U)] + \frac{8}{c^4}\sum_{l\geq 2}\frac{(-)^l l}{(l+1)!}\partial_{aL-2}[R^{-1}\epsilon_{ab(i}\dot{S}_{j)bL-2}(U)] \,.$$

$$M_L(U) = G\int d^3x \int_{-1}^{1} dz \left[\delta_l \hat{x}_L \tilde{\sigma} - \frac{4(2l+1)}{c^2(l+1)(2l+3)}\delta_{l+1}\hat{x}_{aL}\frac{\partial}{\partial U}\tilde{\sigma}^a + \frac{2(2l+1)}{c^4(l+1)(l+2)(2l+5)}\delta_{l+2}\hat{x}_{abL}\frac{\partial^2}{\partial U^2}\tilde{T}^{ab}\right] ;$$

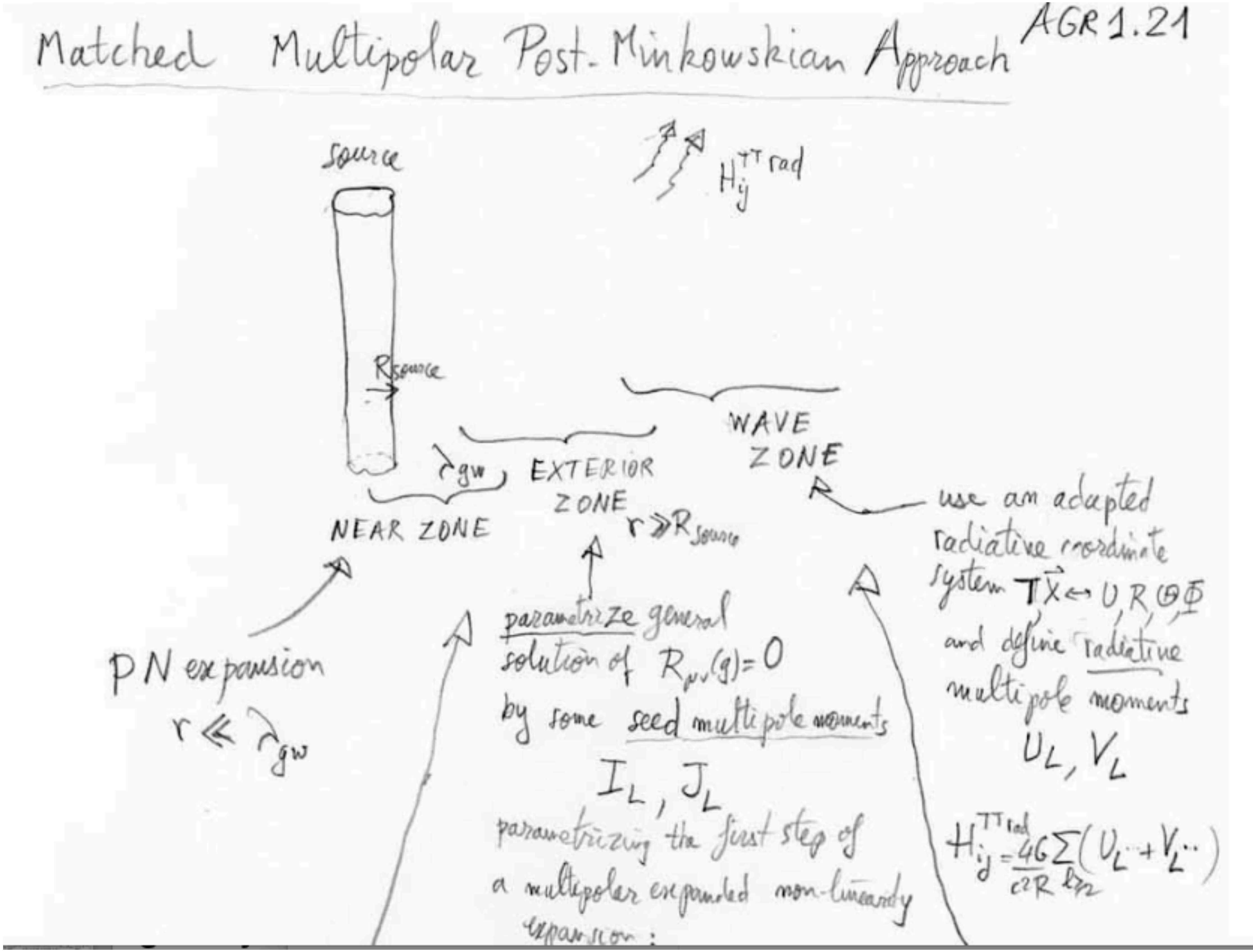
$$S_L(U) = G\,\underset{L}{\text{STF}}\int d^3x \int_{-1}^{1} dz \left[\delta_l \hat{x}_{L-1}\epsilon_{i_l ab}x^a\tilde{\sigma}^b - \frac{2l+1}{c^2(l+2)(2l+3)}\delta_{l+1}\epsilon_{i_l ab}\hat{x}_{acL-1}\frac{\partial}{\partial U}\tilde{T}^{bc}\right] ,$$

$$\sigma \equiv \frac{T^{00}+T^{ss}}{c^2} \,, \qquad \tilde{T}^{\mu\nu} = T^{\mu\nu}\left(\vec{x}, U + \frac{rz}{c}\right)$$

$$\sigma^a \equiv \frac{T^{0a}}{c} \,,$$

linearized gravity

generalizing the scalar field case (Blanchet-Damour'89)



Radiative multipole moments

$$H_{ij}^{\mathrm{TT}}(U,\mathbf{X}) = \frac{4G}{c^2R}\mathcal{P}_{ijab}(\boldsymbol{N})\sum_{\ell=2}^{+\infty}\frac{1}{c^\ell \ell!}\left\{N_{L-2}\mathrm{U}_{abL-2}(U) - \frac{2\ell}{c(\ell+1)}N_{cL-2}\epsilon_{cd(a}\mathrm{V}_{b)dL-2}(U)\right\} + \mathcal{O}\left(\frac{1}{R^2}\right).$$

MULTIPOLAR POST-MINKOWSKIAN FORMALISM (BLANCHET-DAMOUR-IYER)

Decomposition of space-time in various overlapping regions:

1. near-zone: r << lambda : PN theory
2. exterior zone: r >> r_source: MPM expansion
3. far wave-zone: Bondi-type expansion

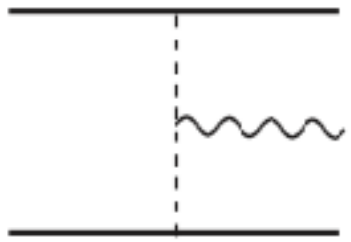
followed by matching between the zones

in exterior zone, iterative solution of Einstein's vacuum field equations by means of a double expansion in non-linearity and in multipoles, with crucial use of analytic continuation (complex B) for dealing with formal UV divergences at r=0

$$
\begin{aligned}
g &= \eta + G h_1 + G^2 h_2 + G^3 h_3 + \dots,\\
\Box h_1 &= 0,\\
\Box h_2 &= \partial\partial h_1 h_1,\\
\Box h_3 &= \partial\partial h_1 h_1 h_1 + \partial\partial h_1 h_2,\\
h_1 &= \sum_\ell \partial_{i_1 i_2 \dots i_\ell}\left(\frac{M_{i_1 i_2 \dots i_\ell}(t-r/c)}{r}\right) + \partial\partial \dots \partial\left(\frac{\epsilon_{j_1 j_2 k} S_{k j_3 \dots j_\ell}(t-r/c)}{r}\right)\\
h_2 &= FP_B \Box^{-1}_{\mathrm{ret}}\left(\left(\frac{r}{r_0}\right)^B \partial\partial h_1 h_1\right) + \dots,\\
h_3 &= FP_B \Box^{-1}_{\mathrm{ret}} \dots .
\end{aligned}
$$

Nonlinearities in harmonic coordinates

(Blanchet-Damour'88,'89,'92)



$$h_2^{\alpha\beta} = \Box_R^{-1}(r^{-2}Q^{\alpha\beta}) + \Box_R^{-1}(r^{-3}N_3^{\alpha\beta} + \cdots) + \sum_{\ell=0,1} \partial_L(r^{-1}T_L^{\alpha\beta})$$

$$Q^{\alpha\beta}(u,\mathbf{n}) = \frac{k^\alpha k^\beta}{c^2}\,\Pi + \frac{4M}{c^4}\,\frac{d^2 z^{\alpha\beta}}{du^2}$$

$$\Pi(u,\mathbf{n}) = \frac{16\pi}{Gc^3}\,\frac{dL^{\text{grav}}}{d\Omega}\bigg|_{h_1} = \frac{16\pi}{Gc^3}\,\frac{dE^{\text{grav}}}{du\,d\Omega}\bigg|_{h_1}$$

M/r light-cone deviation

Both cured by coord. transformation $X^\alpha = x^\alpha + G\xi^\alpha + G^2\lambda^\alpha + O(G^3)$

$$\xi^\alpha \equiv -\frac{2M}{c^2}\,\delta_0^\alpha \ln(r/cP^{\text{rad}}) \qquad \lambda^\alpha \equiv \Box_R^{-1}\left(\frac{k^\alpha}{2cr^2}\,\Pi^{(-1)}(u,\mathbf{n})\right)$$

Hereditary (tail and memory) effects in GW reaction and generation (Blanchet-Damour '88,'89,'92,….)

Hereditary tail effect (mass x quadrupole) in near-zone

$$(\delta g_{00}^{\mathrm{near-zone}})^{\mathrm{hereditary}} = \frac{1}{c^{10}} \left[-\frac{8}{5} x_{ab} I(t) \int_0^{+\infty} dv \ln\left[\frac{v}{2P}\right] {}^{(7)}I_{ab}(t-v) +$$

dependence on infinite past

Memory and tail effects in wave-zone (radiative) GW multipoles

(Blanchet-Damour '89,92, Christodoulou'91,….)

depends on multipoles of energy flux

tail-transported hereditary effect

$$I_L^{\mathrm{rad}[\ell]}(U) = M_L^{(\ell)}(U) + \frac{G c^{\ell+1} \ell!}{2(\ell+1)(\ell+2)} \int_{-\infty}^{U} dV\, \Pi_L(V) + \frac{2GM}{c^3} \int_0^{+\infty} dY \ln\left(\frac{Y}{2P^{\mathrm{rad}}}\right) M_L^{(\ell+2)}(U-Y)$$
$$+ G S'_{2L}(U) + O(G^2)\,,$$
$$J_L^{\mathrm{rad}[\ell]}(U) = S_L^{(\ell)}(U) + \frac{2GM}{c^3} \int_0^{+\infty} dY \ln\left(\frac{Y}{2P^{\mathrm{rad}}}\right) S_L^{(\ell+2)}(U-Y) + G S''_{2L}(U) + O(G^2)\,,$$

Link radiative multipoles <-> source variables

(Blanchet-Damour ’89’92, Damour-Iyer’91, Blanchet ’95…)

$$\mathrm{U}_{ij}(U) = \mathrm{M}^{(2)}_{ij}(U) + \frac{2G\mathrm{M}}{c^3}\int_0^{+\infty} d\tau\, \mathrm{M}^{(4)}_{ij}(U-\tau)\left[\ln\left(\frac{c\tau}{2r_0}\right)+\frac{11}{12}\right]$$

tail

$$+\frac{G}{c^5}\left\{-\frac{2}{7}\int_0^{+\infty} d\tau\, \mathrm{M}^{(3)}_{a\langle i}(U-\tau)\mathrm{M}^{(3)}_{j\rangle a}(U-\tau)\right.$$

memory

$$\left.-\frac{2}{7}\mathrm{M}^{(3)}_{a\langle i}\mathrm{M}^{(2)}_{j\rangle a} - \frac{5}{7}\mathrm{M}^{(4)}_{a\langle i}\mathrm{M}^{(1)}_{j\rangle a} + \frac{1}{7}\mathrm{M}^{(5)}_{a\langle i}\mathrm{M}_{j\rangle a} + \frac{1}{3}\varepsilon_{ab\langle i}\mathrm{M}^{(4)}_{j\rangle a}\mathrm{S}_b\right\}$$

instant.

$$+\frac{2G^2\mathrm{M}^2}{c^6}\int_0^{+\infty} d\tau\, \mathrm{M}^{(5)}_{ij}(U-\tau)\left[\ln^2\left(\frac{c\tau}{2r_0}\right)+\frac{57}{70}\ln\left(\frac{c\tau}{2r_0}\right)+\frac{124627}{44100}\right]$$

tail-of-tail

$$+\mathcal{O}\left(\frac{1}{c^7}\right).$$

$$\mathrm{M}_{ij} = \mathrm{I}_{ij} - \frac{4G}{c^5}\left[\mathrm{W}^{(2)}\mathrm{I}_{ij} - \mathrm{W}^{(1)}\mathrm{I}^{(1)}_{ij}\right] + \mathcal{O}\left(\frac{1}{c^7}\right)$$

$$\Sigma = \frac{\overline{\tau}^{00}+\overline{\tau}^{ii}}{c^2},$$

$$\Sigma_i = \frac{\overline{\tau}^{0i}}{c},$$

$$\Sigma_{ij} = \overline{\tau}^{ij}$$

$$\mathrm{I}_L(u) = \mathcal{FP}\int d^3\mathbf{x}\int_{-1}^{1} dz\left\{\delta_l\hat{x}_L\Sigma - \frac{4(2l+1)}{c^2(l+1)(2l+3)}\delta_{l+1}\hat{x}_{iL}\Sigma^{(1)}_i\right.$$

$$\left.+\frac{2(2l+1)}{c^4(l+1)(l+2)(2l+5)}\delta_{l+2}\hat{x}_{ijL}\Sigma^{(2)}_{ij}\right\}(\mathbf{x}, u+z|\mathbf{x}|/c), \qquad (85)$$

$$\mathrm{J}_L(u) = \mathcal{FP}\int d^3\mathbf{x}\int_{-1}^{1} dz\,\varepsilon_{ab\langle i_l}\left\{\delta_l\hat{x}_{L-1\rangle a}\Sigma_b - \frac{2l+1}{c^2(l+2)(2l+3)}\delta_{l+1}\hat{x}_{L-1\rangle ac}\Sigma^{(1)}_{bc}\right\}(\mathbf{x}, u+z|\mathbf{x}|/c).$$

Explicit Source Quadrupole Moment at 3.5 PN for a binary system

(Blanchet-Damour-Esposito-Farese-Iyer’05; Blanchet et al;Faye-Marsat-Blanchet-Iyer’12)

$$\mathrm{I}_{ij} = \mu \left(A\, x_{\langle ij \rangle} + B\, \frac{r^2}{c^2} v^{\langle ij \rangle} + \frac{48}{7}\, \frac{G^2 m^2 \nu}{c^5 r}\, C\, x_{\langle i} v_{j \rangle} \right) + \mathcal{O}\left(\frac{1}{c^8} \right)$$

$$A = 1 + \gamma \left(-\frac{1}{42} - \frac{13}{14}\nu \right) + \gamma^2 \left(-\frac{461}{1512} - \frac{18395}{1512}\nu - \frac{241}{1512}\nu^2 \right)$$

$$+ \gamma^3 \left(\frac{395899}{13200} - \frac{428}{105} \ln\left(\frac{r_{12}}{r_0} \right) + \left[\frac{3304319}{166320} - \frac{44}{3} \ln\left(\frac{r_{12}}{r'_0} \right) \right] \nu + \frac{162539}{16632}\nu^2 + \frac{2351}{33264}\nu^3 \right)$$

$$B = \frac{11}{21} - \frac{11}{7}\nu + \gamma \left(\frac{1607}{378} - \frac{1681}{378}\nu + \frac{229}{378}\nu^2 \right)$$

$$+ \gamma^2 \left(-\frac{357761}{19800} + \frac{428}{105} \ln\left(\frac{r_{12}}{r_0} \right) - \frac{92339}{5544}\nu + \frac{35759}{924}\nu^2 + \frac{457}{5544}\nu^3 \right),$$

$$C = 1 + \gamma \left(-\frac{256}{135} - \frac{1532}{405}\nu \right).$$

Perturbative computation of GW flux from binary system

- lowest order : Einstein 1918 Peters-Mathews 63
- 1 + (v^2/c^2) : Wagoner-Will 76
- … + (v^3/c^3) : Blanchet-Damour 92, Wiseman 93
- … + (v^4/c^4) : Blanchet-Damour-Iyer Will-Wiseman 95
- … + (v^5/c^5) : Blanchet 96
- … + (v^6/c^6) : Blanchet-Damour-Esposito-Farèse-Iyer 2004
- … + (v^7/c^7) : Blanchet

$$\nu = \frac{m_1 m_2}{(m_1+m_2)^2}$$

$$x = \left(\frac{v}{c}\right)^2 = \left(\frac{G(m_1+m_2)\Omega}{c^3}\right)^{\frac{2}{3}} = \left(\frac{\pi G(m_1+m_2)f}{c^3}\right)^{\frac{2}{3}}$$

$$\begin{aligned}\mathcal{F} = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ 1 &+ \left(-\frac{1247}{336} - \frac{35}{12}\nu\right) x + 4\pi x^{3/2} \\ &+ \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right) x^2 + \left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2} \\ &+ \left[\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_E - \frac{856}{105}\ln(16\,x) \right. \\ &\qquad \left. + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\right] x^3 \\ &+ \left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2} + \mathcal{O}\left(\frac{1}{c^8}\right) \Bigg\}.\end{aligned}$$

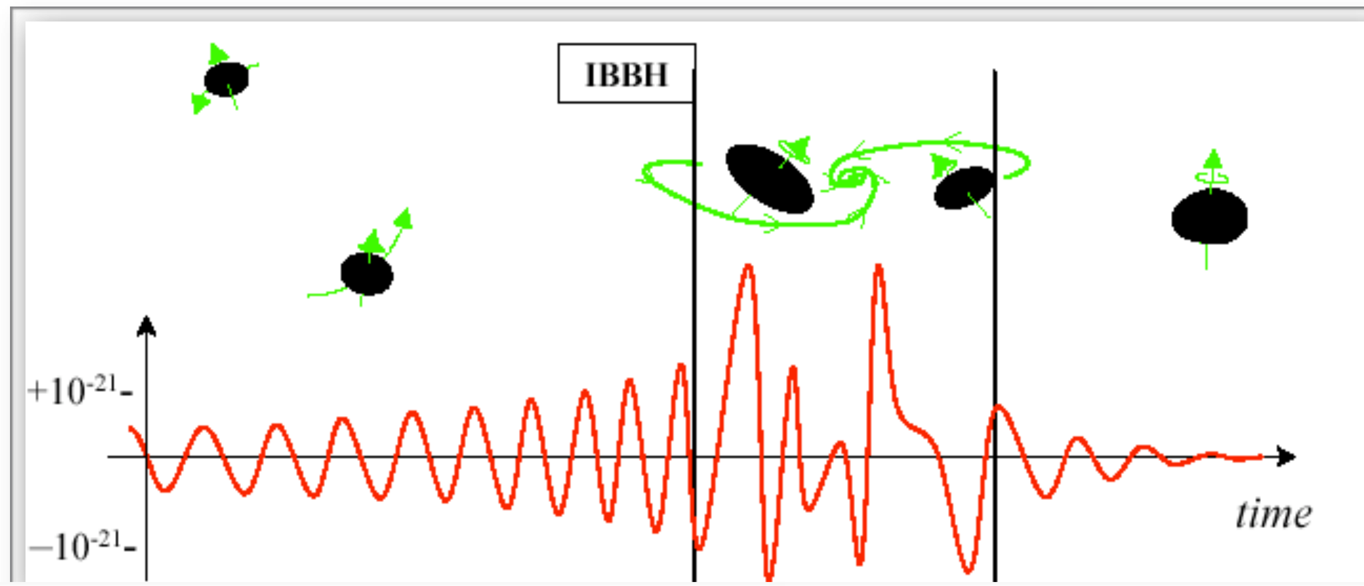
Analytical GW Templates for BBH Coalescences ?

PN corrections to Einstein's quadrupole frequency « chirping »
from PN-improved balance equation dE(f)/dt = - F(f)

$$\frac{d\phi}{d\ln f} = \frac{\omega^2}{d\omega/dt} = Q_\omega^N \widehat{Q}_\omega \qquad \frac{v}{c} = \left(\frac{\pi G(m_1+m_2)f}{c^3}\right)^{\frac{1}{3}}$$

$$Q_\omega^N = \frac{5\,c^5}{48\,\nu\,v^5}\,;\ \widehat{Q}_\omega = 1 + c_2\left(\frac{v}{c}\right)^2 + c_3\left(\frac{v}{c}\right)^3 + \cdots \qquad \nu = \frac{m_1 m_2}{(m_1+m_2)^2}$$

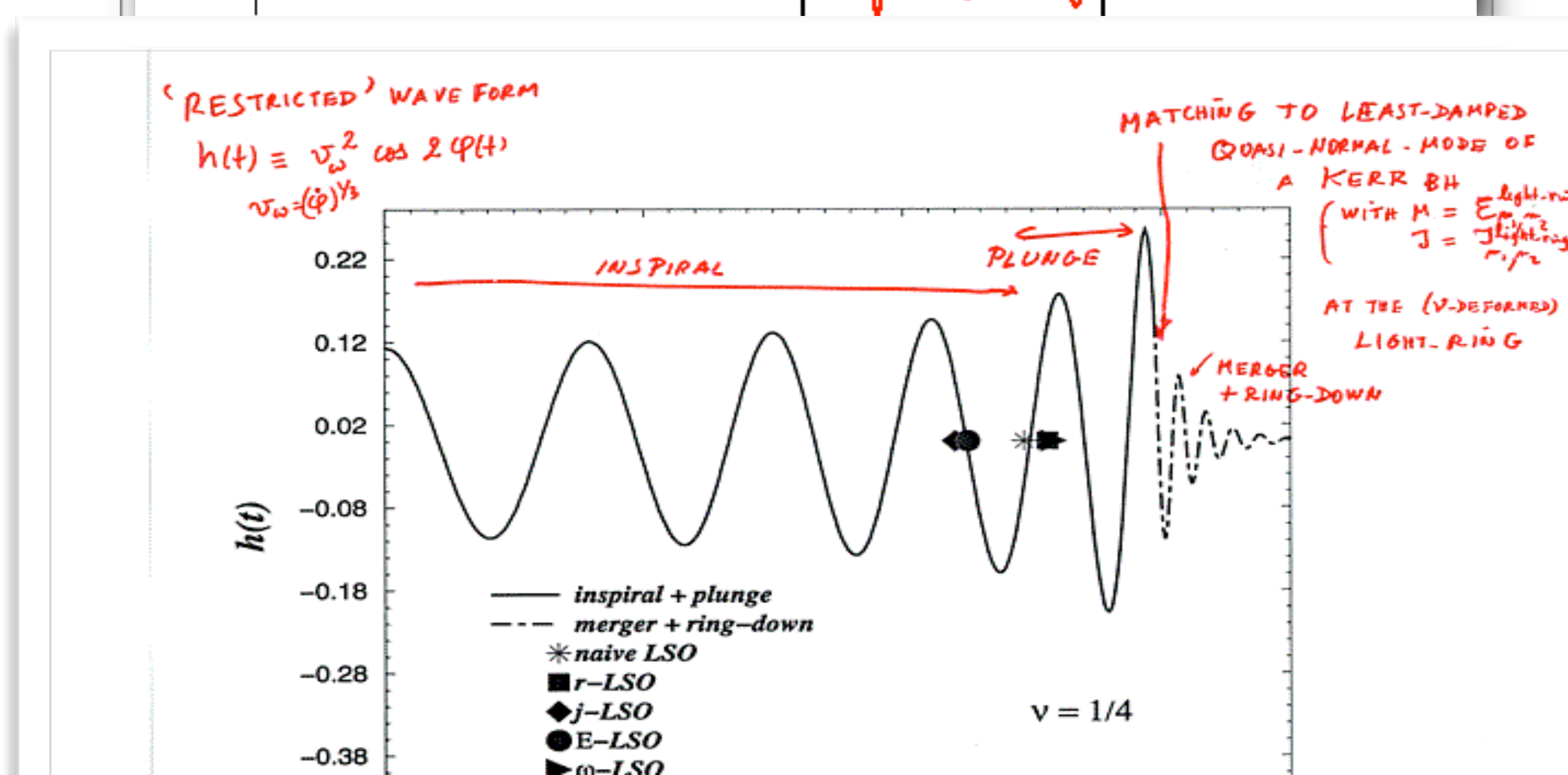
Cutler et al. '93:
« slow convergence of PN »



Brady-Creighton-Thorne'98:
« inability of current computational techniques to evolve a BBH through its last ~10 orbits of inspiral » and to compute the merger

Damour-Iyer-Sathyaprakash'98:
use resummation methods for E and F

Buonanno-Damour '99-00: novel, resummed approach: Effective-One-Body analytical formalism

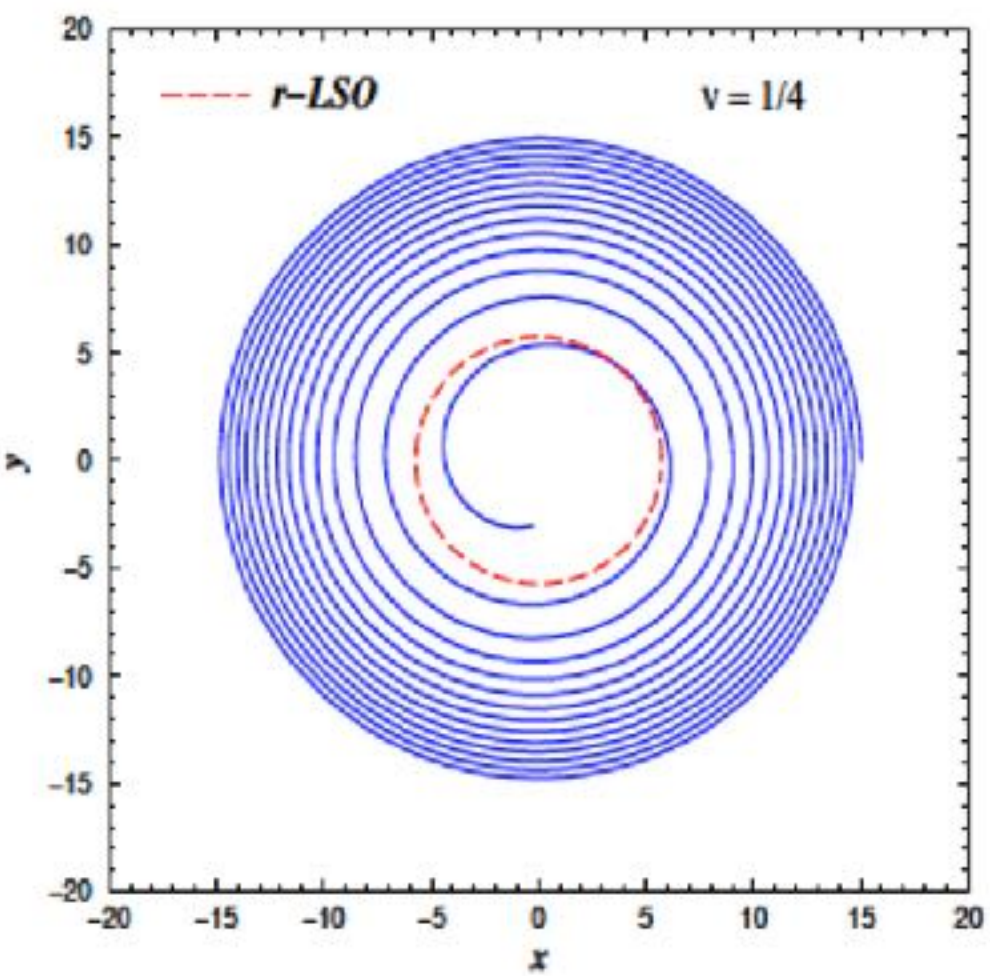


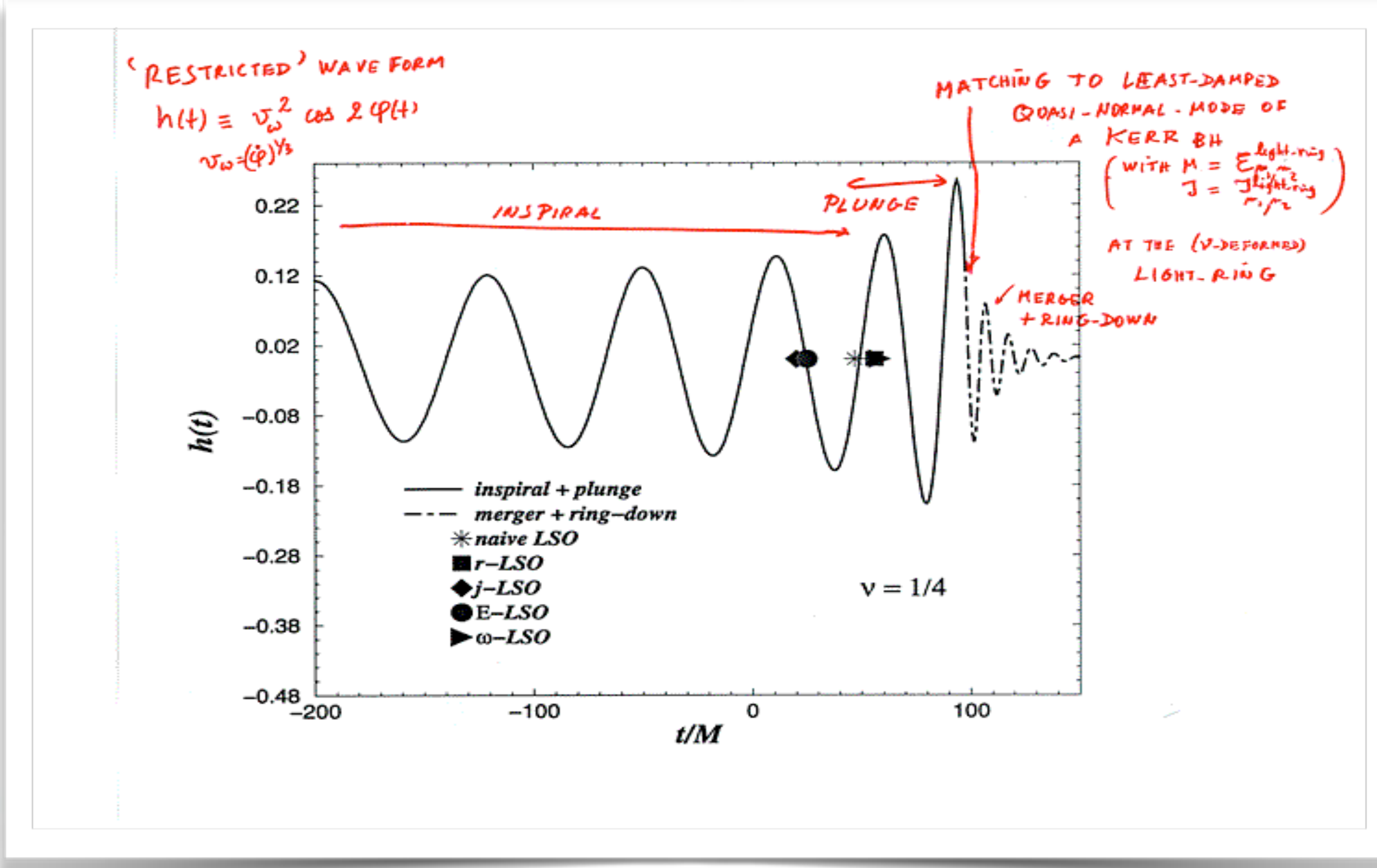
Effective One Body (EOB) Method)

Buonanno-Damour 1999, 2000; Damour-Jaranowski-Schaefer 2000; Damour 2001; Damour-Nagar 2007; Damour-Iyer-Nagar 2009

Resummation of both the Hamiltonian, the waveform and radiation-reaction
—> description of the coalescence + addition of ringdown (Vishveshwara 70, Davis-Ruffini-Tiomno 72)

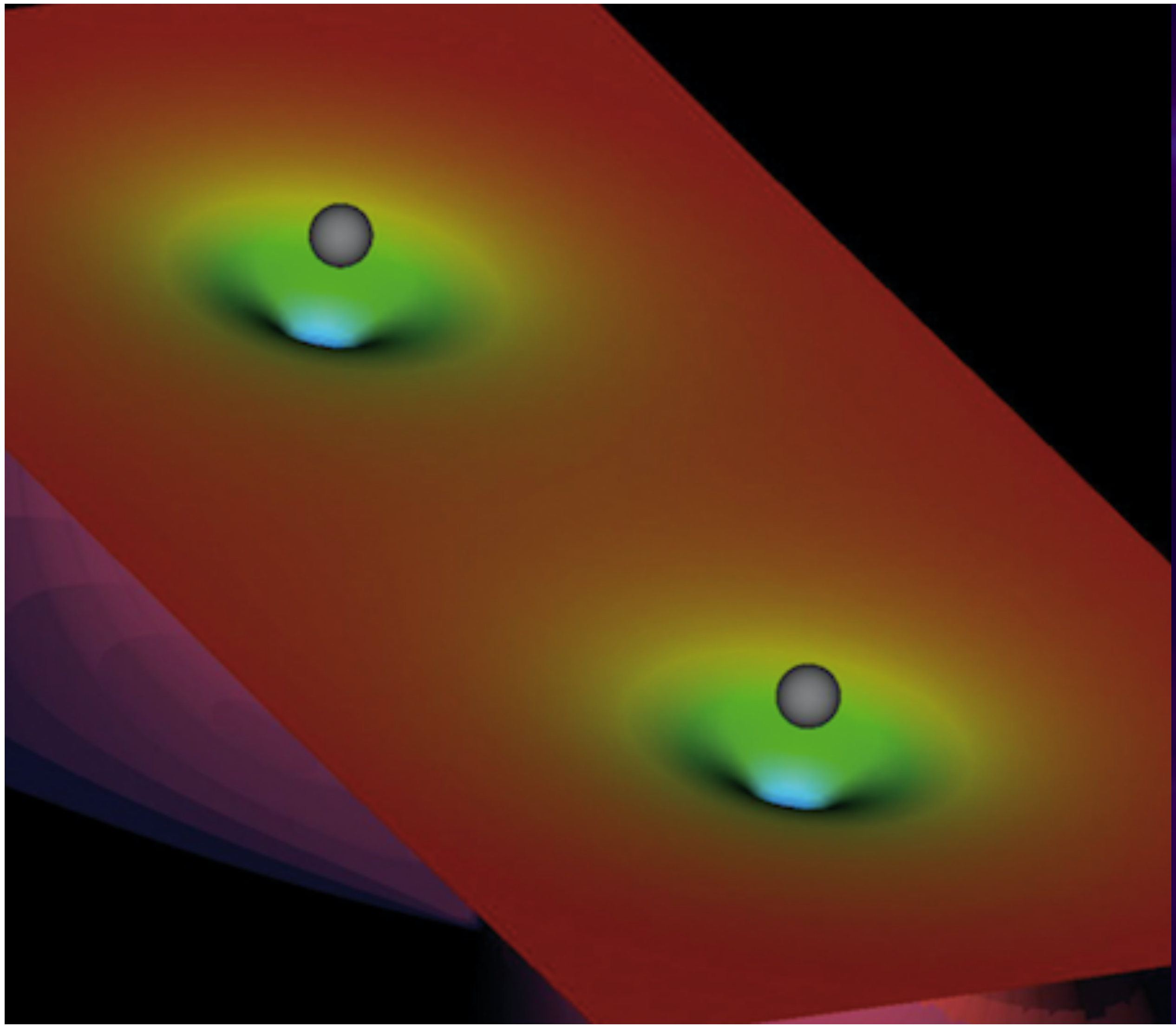
Buonanno-Damour 2000

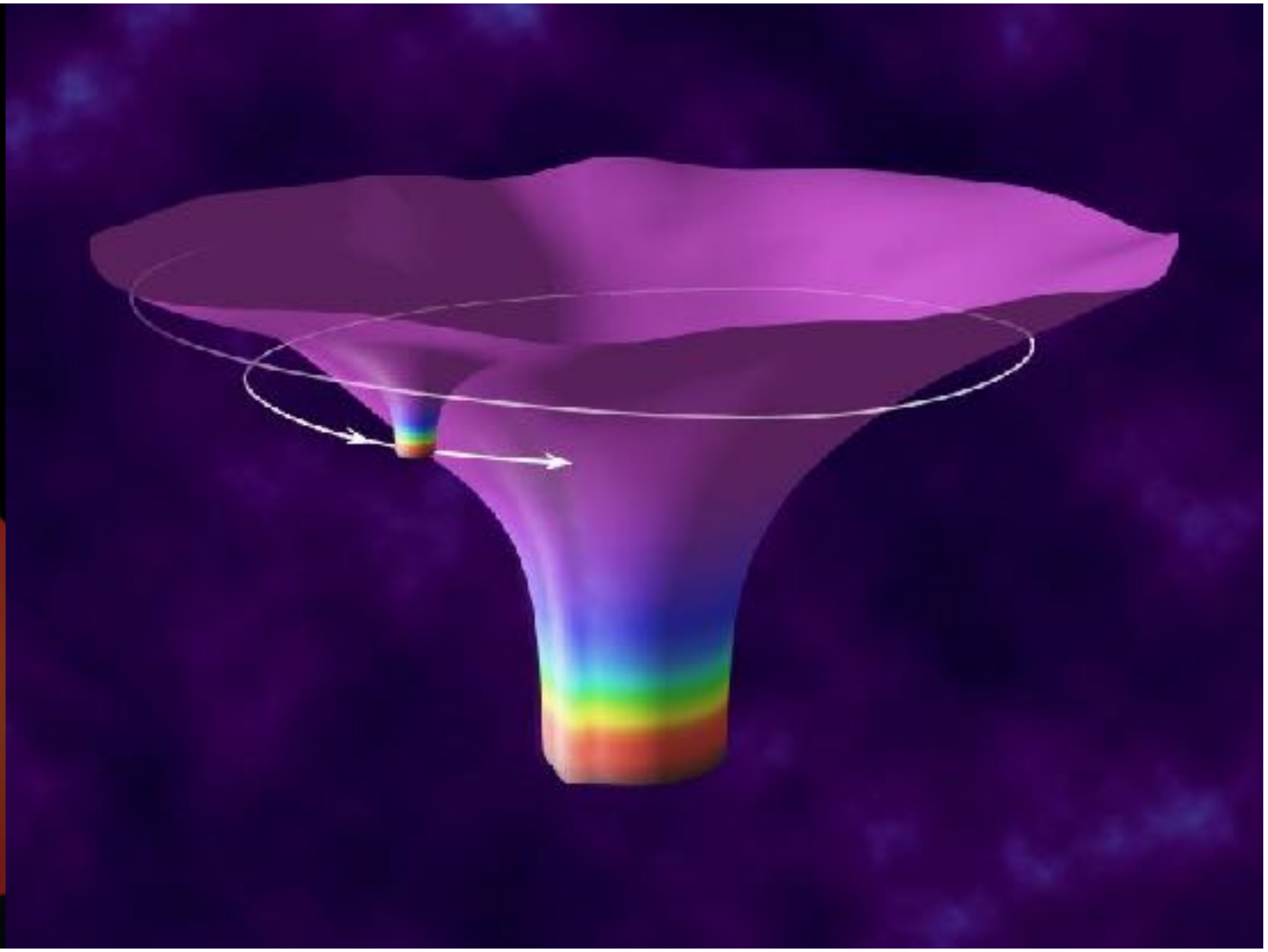




Predictions as early as 2000 :
continued transition, non adiabaticity, first complete waveform, final spin (OK within 10%), final mass

EOB: resumming the dynamics of a two-body system (m_1,m_2,S_1,S_2) in terms of the dynamics of a particle of mass mu and spin S* moving in some effective metric g(M,S)





Effective metric for non-spinning bodies: a nu-deformation of Schwarzschild

$$M = m_1 + m_2 \qquad \mu = \frac{m_1 m_2}{m_1 + m_2} \qquad \nu = \frac{\mu}{M} = \frac{m_1 m_2}{(m_1 + m_2)^2}$$

$$ds^2_{\text{eff}} = -A(r;\nu)\, dt^2 + B(r;\nu)\, dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

TWO-BODY/EOB "CORRESPONDENCE":

THINK QUANTUM-MECHANICALLY (J.A. WHEELER)

Real 2-body system
(in the c.o.m. frame)
(m_1, m_2)

1:1 map

An effective particle
in some effective metric

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$g_{\mu\nu}^{\mathrm{eff}}$$

$$\mathcal{E} = f(E)$$

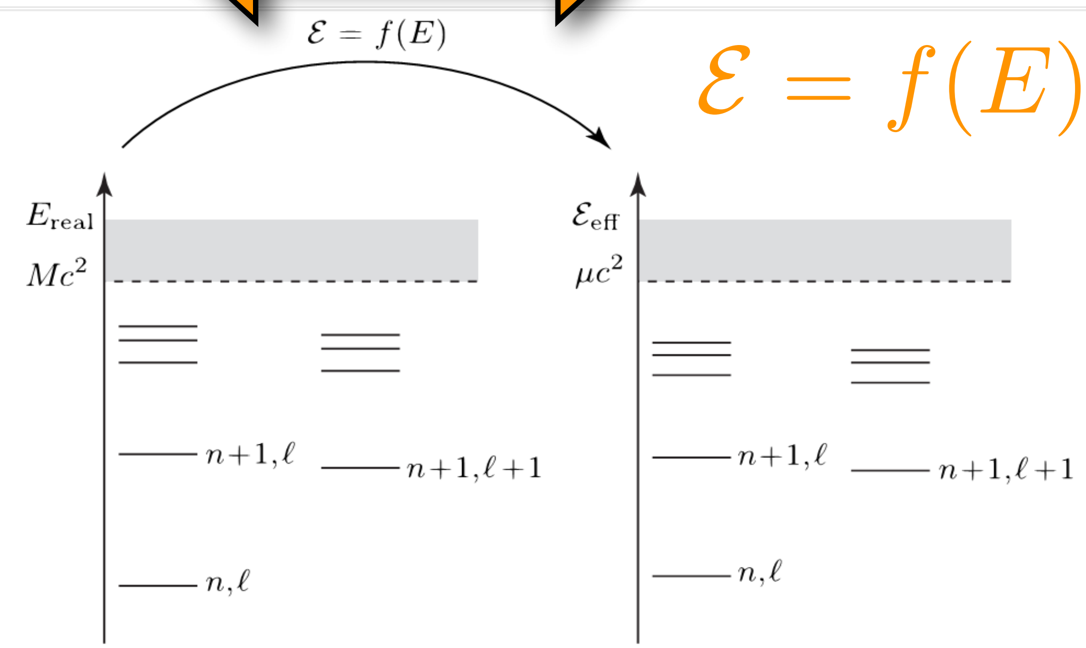


Figure 1: Sketch of the correspondence between the quantized energy levels of the real and effective conservative dynamics. n denotes the 'principal quantum

$$\mu^2 + g_{\mathrm{eff}}^{\mu\nu} \frac{\partial S_{\mathrm{eff}}}{\partial x^\mu} \frac{\partial S_{\mathrm{eff}}}{\partial x^\nu} + \mathcal{O}(p^4) = 0$$

Bohr-Sommerfeld's
Quantization Conditions
(action-angle variables &
Delaunay Hamiltonian)

$$J = \ell \hbar = \frac{1}{2\pi} \oint p_\varphi d\varphi$$

$$N = n\hbar = I_r + J$$

$$I_r = \frac{1}{2\pi} \oint p_r dr$$

$$\mathcal{E}_{\mathrm{eff}} = \frac{(\mathcal{E}_{\mathrm{real}})^2 - m_1^2 - m_2^2}{2(m_1 + m_2)}$$

$$H^{\mathrm{classical}}(q,p) \longrightarrow H^{\mathrm{classical}}(I_a) \longrightarrow E^{\mathrm{quantum}}(I_a = n_a h) = f^{-1}[\mathcal{E}_{\mathrm{eff}}^{\mathrm{quantum}}(I_a^{\mathrm{eff}} = n_a h)]$$

2-body Taylor-expanded N + 1PN + 2PN Hamiltonian

$$H_{\mathrm{N}}(\mathbf{x}_a, \mathbf{p}_a) = \frac{\mathbf{p}_1^2}{2m_1} - \frac{1}{2}\frac{Gm_1m_2}{r_{12}} + (1 \leftrightarrow 2)$$

$$c^2 H_{1\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) = -\frac{1}{8}\frac{(\mathbf{p}_1^2)^2}{m_1^3} + \frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left(-12\frac{\mathbf{p}_1^2}{m_1^2} + 14\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2} + 2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right)$$
$$+\frac{1}{4}\frac{Gm_1m_2}{r_{12}}\frac{G(m_1+m_2)}{r_{12}} + (1 \leftrightarrow 2),$$

$$c^4 H_{2\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) = \frac{1}{16}\frac{(\mathbf{p}_1^2)^3}{m_1^5} + \frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left(5\frac{(\mathbf{p}_1^2)^2}{m_1^4} - \frac{11}{2}\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2} - \frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2m_2^2} + 5\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right.$$
$$\left. - 6\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2} - \frac{3}{2}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right)$$
$$+\frac{1}{4}\frac{G^2m_1m_2}{r_{12}^2}\left(m_2\left(10\frac{\mathbf{p}_1^2}{m_1^2} + 19\frac{\mathbf{p}_2^2}{m_2^2}\right) - \frac{1}{2}(m_1+m_2)\frac{27(\mathbf{p}_1\cdot\mathbf{p}_2) + 6(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right)$$
$$-\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\frac{G^2(m_1^2 + 5m_1m_2 + m_2^2)}{r_{12}^2} + (1 \leftrightarrow 2),$$

EOB results at 1PN and 2PN

Theorem 1: The Lorentz-Droste-Einstein-Infeld-Hoffmann 1PN dynamics, considered in the center-of-mass frame, is mapped (at 1PN accuracy) onto the geodesic motion of a particle of mass $\mu = m_1 m_2/(m_1+m_2)$ in a Schwarzschild background of mass $M = m_1 + m_2$, modulo the very simple (but non trivial) energy map

$$\frac{\mathcal{E}_{\rm eff}}{\mu c^2} = \frac{(\mathcal{E}^{\rm tot}_{\rm real})^2 - m_1^2\, c^4 - m_2^2\, c^4}{2\, m_1\, m_2\, c^4} \quad \text{(at the 1PN, 2PN, 3PN, and 4PN levels).}$$

$$ds^2_{\rm eff} = -(1 - 2\frac{GM}{c^2 R})c^2 dt^2 + \frac{dR^2}{1 - 2\frac{GM}{c^2 R}} + R^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Theorem 2: The full 2PN dynamics (whose general-frame Hamiltonian contains thirteen 2PN-level independent terms besides the five 1PN-level ones), when considered in the center-of-mass frame, is mapped (at 2PN accuracy) onto the geodesic motion of a particle of mass μ in the following simple ν-deformation of the Schwarzschild metric of mass M,

$$(ds^2_{\rm eff})^{\rm 2PN} = -\left(1 - 2\,\frac{GM}{c^2 R} + 2\nu\left(\frac{GM}{c^2 R}\right)^3\right) c^2 dt^2 + \frac{1 - 6\nu\left(\frac{GM}{c^2 R}\right)^2}{1 - 2\frac{GM}{c^2 R}} dR^2 + R^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \quad (65)$$

modulo the same energy map (54) that appeared at the 1PN level.

2-body Taylor-expanded 3PN Hamiltonian [JS 98, DJS 01]

$$
\begin{aligned}
c^6 H_{3\mathrm{PN}}(\mathbf{x}_a,\mathbf{p}_a) = & -\frac{5}{128}\frac{(\mathbf{p}_1^2)^4}{m_1^7} + \frac{1}{32}\frac{Gm_1m_2}{r_{12}}\Bigg(-14\frac{(\mathbf{p}_1^2)^3}{m_1^6} + 4\frac{((\mathbf{p}_1\cdot\mathbf{p}_2)^2+4\mathbf{p}_1^2\mathbf{p}_2^2)\mathbf{p}_1^2}{m_1^4m_2^2} + 6\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^4m_2^2} \\
& - 10\frac{(\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2+\mathbf{p}_2^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2)\mathbf{p}_1^2}{m_1^4m_2^2} + 24\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^4m_2^2} \\
& + 2\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3} + \frac{(7\mathbf{p}_1^2\mathbf{p}_2^2-10(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2^3} \\
& + \frac{(\mathbf{p}_1^2\mathbf{p}_2^2-2(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2^3} + 15\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3} \\
& - 18\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3} + 5\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3}\Bigg) + \frac{G^2m_1m_2}{r_{12}^2}\Bigg(\frac{1}{16}(m_1-27m_2)\frac{(\mathbf{p}_1^2)^2}{m_1^4} \\
& - \frac{115}{16}m_1\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2} + \frac{1}{48}m_2\frac{25(\mathbf{p}_1\cdot\mathbf{p}_2)^2+371\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2} + \frac{17}{16}\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^3} + \frac{5}{12}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^3} \\
& - \frac{1}{8}m_1\frac{(15\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)+11(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1))(\mathbf{n}_{12}\cdot\mathbf{p}_1)}{m_1^3m_2} - \frac{3}{2}m_1\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2} \\
& + \frac{125}{12}m_2\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2} + \frac{10}{3}m_2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2} \\
& - \frac{1}{48}(220m_1+193m_2)\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\Bigg) + \frac{G^3m_1m_2}{r_{12}^3}\Bigg(-\frac{1}{48}\left(425m_1^2+\left(473-\frac{3}{4}\pi^2\right)m_1m_2+150m_2^2\right)\frac{\mathbf{p}_1^2}{m_1^2} \\
& + \frac{1}{16}\left(77(m_1^2+m_2^2)+\left(143-\frac{1}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2} + \frac{1}{16}\left(20m_1^2-\left(43+\frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2} \\
& + \frac{1}{16}\left(21(m_1^2+m_2^2)+\left(119+\frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\Bigg) \\
& + \frac{1}{8}\frac{G^4m_1m_2^3}{r_{12}^4}\left(\left(\frac{227}{3}-\frac{21}{4}\pi^2\right)m_1+m_2\right) + (1\leftrightarrow 2).
\end{aligned}
$$

EOB results at 3PN (DJS'00)

Less eqs than unknowns ? Need to introduce higher-in-momenta contributions

$$S = -\mu \int ds_{\text{eff}} \left[1 + A_{\mu\nu\kappa\lambda}(x)\, u^\mu\, u^\nu\, u^\kappa\, u^\lambda + \ldots\right].$$

post-geodesic effective mass-shell condition

$$\boxed{g_{\text{eff}}^{\mu\nu}\, P'_\mu\, P'_\nu + \mu^2\, c^2 + Q(P'_\mu) = 0\,,}$$

at 3PN

$$Q(P'_\mu) = \frac{1}{c^6}\,\frac{1}{\mu^2}\left(\frac{GM}{R'}\right)^2 \left[z_1\, \boldsymbol{P}'^4 + z_2\, \boldsymbol{P}'^2(\boldsymbol{n}'\cdot\boldsymbol{P}')^2 + z_3(\boldsymbol{n}'\cdot\boldsymbol{P}')^4\right]$$

$$8\, z_1 + 4\, z_2 + 3\, z_3 = 6\nu(4 - 3\nu).$$

DJS gauge: z1=z2=0 to have only pr^4 terms

3PN EOB

Theorem 3: The full 3PN dynamics (whose general-frame Hamiltonian contains ~ 40 3PN-level terms), when considered in the center-of-mass frame, is mapped (at 3PN accuracy), via the simple energy map (54) that appeared at 1PN, onto the motion of a particle of mass μ submitted to the mass-shell condition (38) where $g_{\mu\nu}^{\rm eff}$ is given by Eq. , with

$$A^{\rm 3PN}(u) = 1 - 2u + 2\nu\, u^3 + \left(\frac{94}{3} - \frac{41}{32}\,\pi^2\right)\nu u^4, \tag{71}$$

$$\overline{D}^{\rm 3PN}(u) = 1 + 6\nu\, u^2 + (52\nu - 6\nu^2)\, u^3, \tag{72}$$

and where

$$\widehat{Q}^{\rm 3PN} \equiv \frac{Q}{\mu^2 c^2} = (8\nu - 6\nu^2)\, u^2\, \frac{p_r^4}{c^4}. \tag{73}$$

Here $\overline{D} \equiv (AB)^{-1}$, we used the scaled momentum $p_r \equiv P_R^{\rm EOB}/\mu$ (with dimension $[p] = [\text{velocity}]$ so that p/c is dimensionless, and we introduced the convenient dimensionless EOB variable

$$\boxed{u \equiv \frac{GM}{c^2 R_{\rm EOB}}.} \tag{74}$$

2-body Taylor-expanded 4PN Hamiltonian [DJS, 2014]

$$c^8 H^{\rm loc}_{\rm 4PN}(\mathbf{x}_a,\mathbf{p}_a) = \frac{7(\mathbf{p}_1^2)^5}{256 m_1^9} + \frac{G m_1 m_2}{r_{12}} H_{48}(\mathbf{x}_a,\mathbf{p}_a) + \frac{G^2 m_1 m_2}{r_{12}^2} m_1 H_{46}(\mathbf{x}_a,\mathbf{p}_a)$$
$$+ \frac{G^3 m_1 m_2}{r_{12}^3}\left(m_1^2 H_{441}(\mathbf{x}_a,\mathbf{p}_a) + m_1 m_2 H_{442}(\mathbf{x}_a,\mathbf{p}_a)\right)$$
$$+ \frac{G^4 m_1 m_2}{r_{12}^4}\left(m_1^3 H_{421}(\mathbf{x}_a,\mathbf{p}_a) + m_1^2 m_2 H_{422}(\mathbf{x}_a,\mathbf{p}_a)\right)$$
$$+ \frac{G^5 m_1 m_2}{r_{12}^5} H_{40}(\mathbf{x}_a,\mathbf{p}_a) + (1 \leftrightarrow 2), \qquad \text{(A3)}$$

$$H_{48}(\mathbf{x}_a,\mathbf{p}_a) = \frac{45(\mathbf{p}_1^2)^4}{128 m_1^8} - \frac{9(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{64 m_1^6 m_2^2} + \frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^3}{64 m_1^6 m_2^2} - \frac{9(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{16 m_1^6 m_2^2}$$
$$- \frac{3(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{32 m_1^6 m_2^2} + \frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{64 m_1^6 m_2^2} - \frac{21(\mathbf{p}_1^2)^3\mathbf{p}_2^2}{64 m_1^6 m_2^2} - \frac{35(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{256 m_1^5 m_2^3}$$
$$+ \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2}{128 m_1^5 m_2^3} + \frac{33(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3(\mathbf{p}_1^2)^2}{256 m_1^5 m_2^3} - \frac{85(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2^3}$$
$$- \frac{45(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{128 m_1^5 m_2^3} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2^3} + \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64 m_1^5 m_2^3}$$
$$+ \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64 m_1^5 m_2^3} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{64 m_1^5 m_2^3} + \frac{3\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{64 m_1^5 m_2^3} + \frac{55(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256 m_1^5 m_2^3}$$
$$- \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{128 m_1^5 m_2^3} - \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{256 m_1^5 m_2^3} - \frac{23(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256 m_1^5 m_2^3}$$
$$+ \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{128 m_1^5 m_2^3} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256 m_1^5 m_2^3} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4\mathbf{p}_1^2}{64 m_1^4 m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4(\mathbf{p}_1^2)^2}{64 m_1^4 m_2^4}$$
$$- \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{4 m_1^4 m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{16 m_1^4 m_2^4} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{64 m_1^4 m_2^4} + \frac{21(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{64 m_1^4 m_2^4}$$
$$- \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{32 m_1^4 m_2^4} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{4 m_1^4 m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{16 m_1^4 m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{16 m_1^4 m_2^4}$$
$$- \frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{32 m_1^4 m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_2^2)^2}{64 m_1^4 m_2^4} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{32 m_1^4 m_2^4} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_2^2)^2}{128 m_1^4 m_2^4}, \qquad \text{(A4a)}$$

$$H_{46}(\mathbf{x}_a,\mathbf{p}_a) = \frac{369(\mathbf{n}_{12}\cdot\mathbf{p}_1)^6}{160 m_1^6} - \frac{889(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4\mathbf{p}_1^2}{192 m_1^6} + \frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1^2)^2}{16 m_1^6} - \frac{63(\mathbf{p}_1^2)^3}{64 m_1^6} - \frac{549(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{128 m_1^5 m_2}$$
$$+ \frac{67(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{16 m_1^5 m_2} - \frac{167(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2}{128 m_1^5 m_2} + \frac{1547(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2} - \frac{851(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{128 m_1^5 m_2}$$
$$+ \frac{1099(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2} + \frac{3263(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{1280 m_1^4 m_2^2} + \frac{1067(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{480 m_1^4 m_2^2} - \frac{4567(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{3840 m_1^4 m_2^2}$$
$$- \frac{3571(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{320 m_1^4 m_2^2} + \frac{3073(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{480 m_1^4 m_2^2} + \frac{4349(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{1280 m_1^4 m_2^2}$$
$$- \frac{3461\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{3840 m_1^4 m_2^2} + \frac{1673(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4\mathbf{p}_2^2}{1920 m_1^4 m_2^2} - \frac{1999(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2\mathbf{p}_2^2}{3840 m_1^4 m_2^2} + \frac{2081(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{3840 m_1^4 m_2^2} - \frac{13(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{8 m_1^3 m_2^3}$$
$$+ \frac{191(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2}{192 m_1^3 m_2^3} - \frac{19(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384 m_1^3 m_2^3} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384 m_1^3 m_2^3}$$
$$+ \frac{11(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{192 m_1^3 m_2^3} + \frac{77(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{96 m_1^3 m_2^3} + \frac{233(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{96 m_1^3 m_2^3} - \frac{47(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{32 m_1^3 m_2^3}$$
$$+ \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{384 m_1^3 m_2^3} - \frac{185\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{384 m_1^3 m_2^3} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4}{4 m_1^2 m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4\mathbf{p}_1^2}{4 m_1^2 m_2^4}$$
$$- \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3(\mathbf{p}_1\cdot\mathbf{p}_2)}{2 m_1^2 m_2^4} + \frac{21(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{16 m_1^2 m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{6 m_1^2 m_2^4} + \frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{48 m_1^2 m_2^4}$$
$$- \frac{133(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{24 m_1^2 m_2^4} - \frac{77(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{96 m_1^2 m_2^4} + \frac{197(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_2^2)^2}{96 m_1^2 m_2^4} - \frac{173\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{48 m_1^2 m_2^4} + \frac{13(\mathbf{p}_2^2)^3}{8 m_2^6}, \qquad \text{(A4b)}$$

$$H_{441}(\mathbf{x}_a,\mathbf{p}_a) = \frac{5027(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{384 m_1^4} - \frac{22993(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2}{960 m_1^4} - \frac{6695(\mathbf{p}_1^2)^2}{1152 m_1^4} - \frac{3191(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{640 m_1^3 m_2}$$
$$+ \frac{28561(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{1920 m_1^3 m_2} + \frac{8777(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384 m_1^3 m_2} - \frac{752969\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{28800 m_1^3 m_2}$$
$$- \frac{16481(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{960 m_1^2 m_2^2} + \frac{94433(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{4800 m_1^2 m_2^2} - \frac{103957(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{2400 m_1^2 m_2^2}$$
$$+ \frac{791(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{400 m_1^2 m_2^2} + \frac{26627(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_2^2}{1600 m_1^2 m_2^2} - \frac{118261\mathbf{p}_1^2\mathbf{p}_2^2}{4800 m_1^2 m_2^2} + \frac{105(\mathbf{p}_2^2)^2}{32 m_2^4}, \qquad \text{(A4c)}$$

$$H_{442}(\mathbf{x}_a,\mathbf{p}_a) = \left(\frac{2749\pi^2}{8192} - \frac{211189}{19200}\right)\frac{(\mathbf{p}_1^2)^2}{m_1^4} + \left(\frac{63347}{1600} - \frac{1059\pi^2}{1024}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2}{m_1^4} + \left(\frac{375\pi^2}{8192} - \frac{23533}{1280}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^4}$$
$$+ \left(\frac{10631\pi^2}{8192} - \frac{1918349}{57600}\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2} + \left(\frac{13723\pi^2}{16384} - \frac{2492417}{57600}\right)\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2 m_2^2}$$
$$+ \left(\frac{1411429}{19200} - \frac{1059\pi^2}{512}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{m_1^2 m_2^2} + \left(\frac{248991}{6400} - \frac{6153\pi^2}{2048}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^2 m_2^2}$$
$$- \left(\frac{30383}{960} + \frac{36405\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2} + \left(\frac{1243717}{14400} - \frac{40483\pi^2}{16384}\right)\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3 m_2}$$
$$+ \left(\frac{2369}{60} + \frac{35655\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3 m_2} + \left(\frac{43101\pi^2}{16384} - \frac{391711}{6400}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{m_1^3 m_2}$$
$$+ \left(\frac{56955\pi^2}{16384} - \frac{1646983}{19200}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3 m_2}, \qquad \text{(A4d)}$$

$$H_{421}(\mathbf{x}_a,\mathbf{p}_a) = \frac{64861\mathbf{p}_1^2}{4800 m_1^2} - \frac{91(\mathbf{p}_1\cdot\mathbf{p}_2)}{8 m_1 m_2} + \frac{105\mathbf{p}_2^2}{32 m_2^2} - \frac{9841(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{1600 m_1^2} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{2 m_1 m_2}, \qquad \text{(A4e)}$$

$$H_{422}(\mathbf{x}_a,\mathbf{p}_a) = \left(\frac{1937033}{57600} - \frac{199177\pi^2}{49152}\right)\frac{\mathbf{p}_1^2}{m_1^2} + \left(\frac{176033\pi^2}{24576} - \frac{2864917}{57600}\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1 m_2} + \left(\frac{282361}{19200} - \frac{21837\pi^2}{8192}\right)\frac{\mathbf{p}_2^2}{m_2^2}$$
$$+ \left(\frac{698723}{19200} + \frac{21745\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2} - \left(\frac{63641\pi^2}{24576} - \frac{2712013}{19200}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1 m_2}$$
$$+ \left(\frac{3200179}{57600} - \frac{28691\pi^2}{24576}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_2^2}, \qquad \text{(A4f)}$$

$$H_{40}(\mathbf{x}_a,\mathbf{p}_a) = -\frac{m_1^4}{16} + \left(\frac{6237\pi^2}{1024} - \frac{169799}{2400}\right)m_1^3 m_2 + \left(\frac{44825\pi^2}{6144} - \frac{609427}{7200}\right)m_1^2 m_2^2. \qquad \text{(A4g)}$$

$$H^{\rm nonloc}_{\rm 4PN}(t) = -\frac{1}{5}\frac{G^2 M}{c^8} I^{(3)}_{ij}(t) \times {\rm Pf}_{2r_{12}/c}\int_{-\infty}^{+\infty}\frac{{\rm d}v}{|v|} I^{(3)}_{ij}(t+v),$$

Resummed (non-spinning) 4PN EOB interaction potentials

$$M = m_1 + m_2\,; \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}\,, \qquad \nu = \frac{m_1 m_2}{(m_1+m_2)^2} = \frac{\mu}{M} \qquad u \equiv \frac{GM}{R\,c^2}$$

$$ds^2_{\rm eff} = -A(r;\nu)\,dt^2 + B(r;\nu)\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right) \qquad \bar{D} \equiv (A\,B)^{-1}$$

DJS gauge: describing the main effects via the radial potential A(r)

$$A^{\rm PN}(u;\nu) = 1 - 2u + 2\nu u^3 + \nu a_4 u^4 + (\nu a_5^c + \nu^2 a_5' + \frac{64}{5}\nu \ln u)u^5$$

$$A^{\rm EOB}(u) = {\rm Pade}^1_4[A^{PN}(u)] \qquad a_4 = \frac{94}{3} - \frac{41\pi^2}{32}\,;\; a_5^c = \frac{2275\pi^2}{512} + \ldots\,;\; a_5' = \ldots$$

$$\bar{D}(u) = 1 + 6\nu u^2 + (52\nu - 6\nu^2)u^3 + \left(\left(-\frac{533}{45} - \frac{23761\pi^2}{1536} + \frac{1184}{15}\gamma_{\rm E} - \frac{6496}{15}\ln 2 + \frac{2916}{5}\ln 3\right)\nu + \left(\frac{123\pi^2}{16} - 260\right)\nu^2 + \frac{592}{15}\nu \ln u\right)u^4,$$

$$\hat{Q}(\mathbf{r}',\mathbf{p}') = \left(2(4-3\nu)\nu u^2 + \left(\left(-\frac{5308}{15} + \frac{496256}{45}\ln 2 - \frac{33048}{5}\ln 3\right)\nu - 83\nu^2 + 10\nu^3\right)u^3\right)(\mathbf{n}'\cdot\mathbf{p}')^4 + \left(\left(-\frac{827}{3} - \frac{2358912}{25}\ln 2 + \frac{1399437}{50}\ln 3 + \frac{390625}{18}\ln 5\right)\nu - \frac{27}{5}\nu^2 + 6\nu^3\right)u^2(\mathbf{n}'\cdot\mathbf{p}')^6 + \mathcal{O}[\nu u(\mathbf{n}'\cdot\mathbf{p}')^8].$$

Spinning EOB effective Hamiltonian

$$H_{\rm eff} = H_{\rm orb} + H_{\rm so} \quad \rightarrow H_{\rm EOB} = Mc^2\sqrt{1 + 2\nu\left(\frac{H_{\rm eff}}{\mu c^2} - 1\right)}$$

$$\hat{H}^{\rm eff}_{\rm orb} = \sqrt{A\left(1 + B_p \boldsymbol{p}^2 + B_{np}(\boldsymbol{n}\cdot\boldsymbol{p})^2 - \frac{1}{1 + \frac{(\boldsymbol{n}\cdot\boldsymbol{\chi}_0)^2}{r^2}}\frac{(r^2 + 2r + (\boldsymbol{n}\cdot\boldsymbol{\chi}_0)^2)}{\mathcal{R}^4 + \Delta(\boldsymbol{n}\cdot\boldsymbol{\chi}_0)^2}((\boldsymbol{n}\times\boldsymbol{p})\cdot\boldsymbol{\chi}_0)^2 + Q_4\right)}.$$

$$H_{\rm so} = G_S \boldsymbol{L}\cdot\boldsymbol{S} + G_{S^*}\boldsymbol{L}\cdot\boldsymbol{S}^*,$$

$$\mathbf{S} = \mathbf{S_1} + \mathbf{S_2}\,;\; \mathbf{S}_* = \frac{m_2}{m_1}\mathbf{S_1} + \frac{m_1}{m_2}\mathbf{S_2}\,,$$

Gyrogravitomagnetic ratios (when neglecting spin^2 effects)

$$r^3 G_S^{\rm PN} = 2 - \frac{5}{8}\nu u - \frac{27}{8}\nu p_r^2 + \nu\left(-\frac{51}{4}u^2 - \frac{21}{2}u p_r^2 + \frac{5}{8}p_r^4\right) + \nu^2\left(-\frac{1}{8}u^2 + \frac{23}{8}u p_r^2 + \frac{35}{8}p_r^4\right)$$

$$r^3 G_{S_*}^{\rm PN} = \frac{3}{2} - \frac{9}{8}u - \frac{15}{8}p_r^2 + \nu\left(-\frac{3}{4}u - \frac{9}{4}p_r^2\right) - \frac{27}{16}u^2 + \frac{69}{16}u p_r^2 + \frac{35}{16}p_r^4 + \nu\left(-\frac{39}{4}u^2 - \frac{9}{4}u p_r^2 + \frac{5}{2}p_r^4\right) + \nu^2\left(-\frac{3}{16}u^2 + \frac{57}{16}u p_r^2 + \frac{45}{16}p_r^4\right)$$

Resummed EOB waveform

(Damour-Iyer-Sathyaprakash ’98) Damour-Nagar ’07, Damour-Iyer -Nagar ’08, Pan et al. ‘10

$$h_{\ell m} \equiv h_{\ell m}^{(N,\epsilon)} \hat{h}_{\ell m}^{(\epsilon)} \hat{h}_{\ell m}^{\rm NQC}$$

$$\hat{h}_{\ell m}^{(\epsilon)} = \hat{S}_{\rm eff}^{(\epsilon)} T_{\ell m} e^{i\delta_{\ell m}} \rho_{\ell m}^{\ell}$$

$$T_{\ell m} = \frac{\Gamma(\ell + 1 - 2i\hat{\hat{k}})}{\Gamma(\ell+1)} e^{\pi\hat{\hat{k}}} e^{2i\hat{\hat{k}}\ln(2kr_0)}$$

NB: T_lm resums an infinite number of terms and already contains, eg, 4.5PN tail^3 terms (Messina-Nagar17)

$$\rho_{22}(x;\nu) = 1 + \left(\frac{55\nu}{84} - \frac{43}{42}\right)x + \left(\frac{19583\nu^2}{42336} - \frac{33025\nu}{21168} - \frac{20555}{10584}\right)x^2$$
$$+ \left(\frac{10620745\nu^3}{39118464} - \frac{6292061\nu^2}{3259872} + \frac{41\pi^2\nu}{192} - \frac{48993925\nu}{9779616} - \frac{428}{105}\mathrm{eulerlog}_2(x) + \frac{1556919113}{122245200}\right)x^3$$
$$+ \left(\frac{9202}{2205}\mathrm{eulerlog}_2(x) - \frac{387216563023}{160190110080}\right)x^4 + \left(\frac{439877}{55566}\mathrm{eulerlog}_2(x) - \frac{16094530514677}{533967033600}\right)x^5 + \mathcal{O}(x^6),$$

$$\mathcal{F}_\varphi \equiv -\frac{1}{8\pi\Omega}\sum_{\ell=2}^{\ell_{\rm max}}\sum_{m=1}^{\ell}(m\Omega)^2|Rh_{\ell m}^{(\epsilon)}|^2$$

EOB

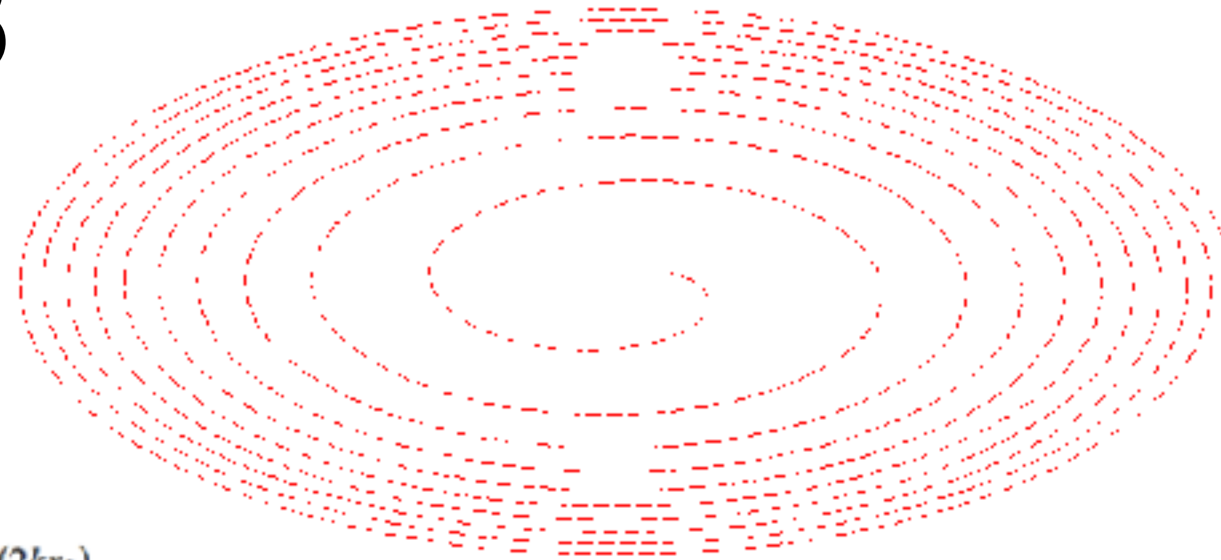
$$\frac{dr}{dt} = \left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{\rm EOB}}{\partial p_{r_*}} \,,$$

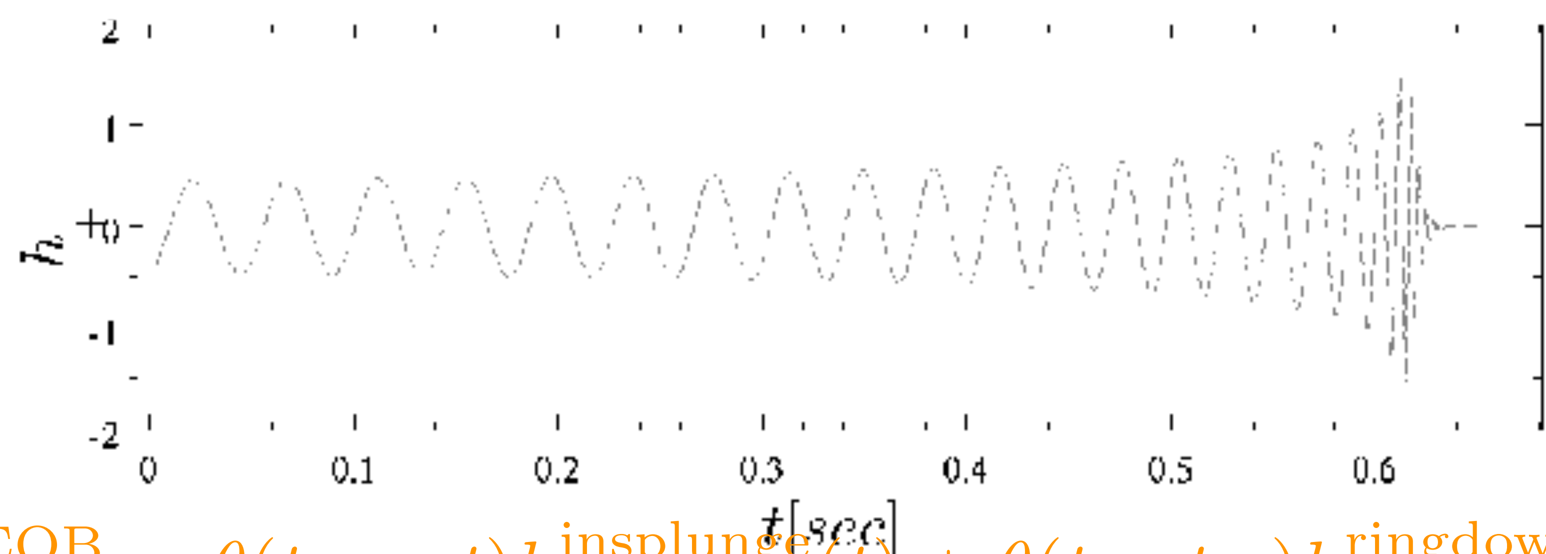
$$\frac{dp_{r_*}}{dt} = -\left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{\rm EOB}}{\partial r} \,,$$

$$\Omega \equiv \frac{d\varphi}{dt} = \frac{\partial \hat{H}_{\rm EOB}}{\partial p_\varphi}$$

$$\frac{dp_\varphi}{dt} = \hat{\mathcal{F}}_\varphi \,.$$

$$T_{\ell m} = \frac{\Gamma(\ell+1-2{\rm i}\hat{\hat{k}})}{\Gamma(\ell+1)} e^{\pi \hat{k}} e^{2{\rm i}\hat{k}\log(2kr_0)},$$





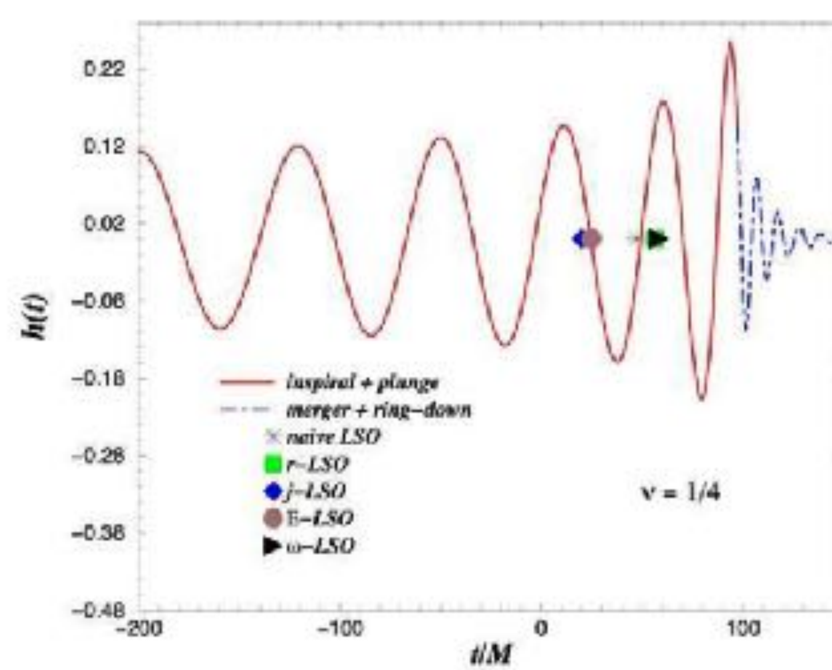
$$h_{\ell m} \equiv h_{\ell m}^{(N,\epsilon)} \hat{h}_{\ell m}^{(\epsilon)} \hat{h}_{\ell m}^{\rm NQC}$$

$$\hat{h}_{\ell m}^{(\epsilon)} = \hat{S}_{\rm eff}^{(\epsilon)} T_{\ell m} e^{i\delta_{\ell m}} \rho_{\ell m}^{\ell}$$

$$\mathcal{F}_\varphi \equiv -\frac{1}{8\pi\Omega} \sum_{\ell=2}^{\ell_{\rm max}} \sum_{m=1}^{\ell} (m\Omega)^2 |Rh_{\ell m}^{(\epsilon)}|^2$$

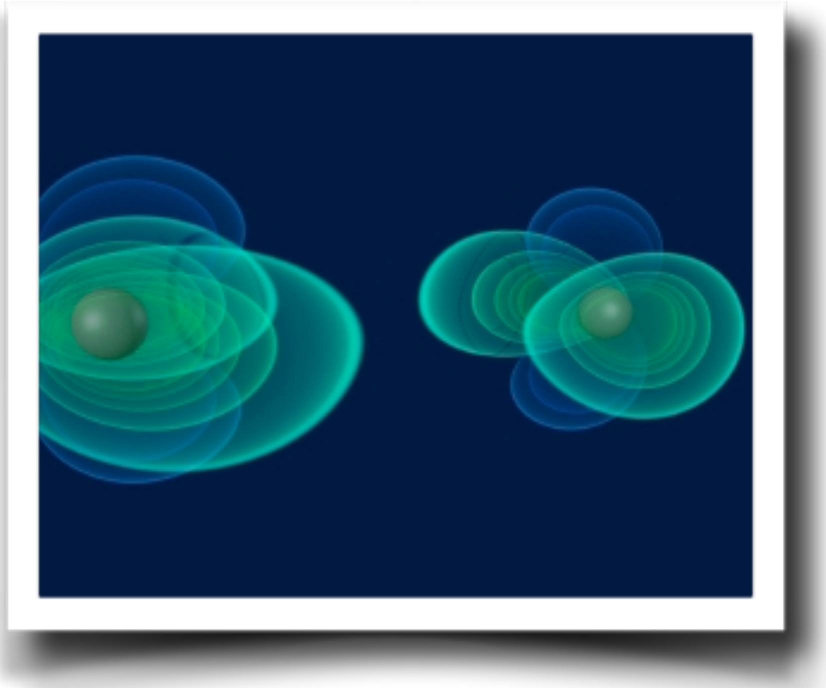
$$h_{\ell m}^{\rm EOB} = \theta(t_m - t) h_{\ell m}^{\rm insplunge}(t) + \theta(t - t_m) h_{\ell m}^{\rm ringdow}$$

$$h_{\ell m}^{\rm ringdown}(t) = \sum_N C_N^+ e^{-\sigma_N^+ (t-t_m)}$$

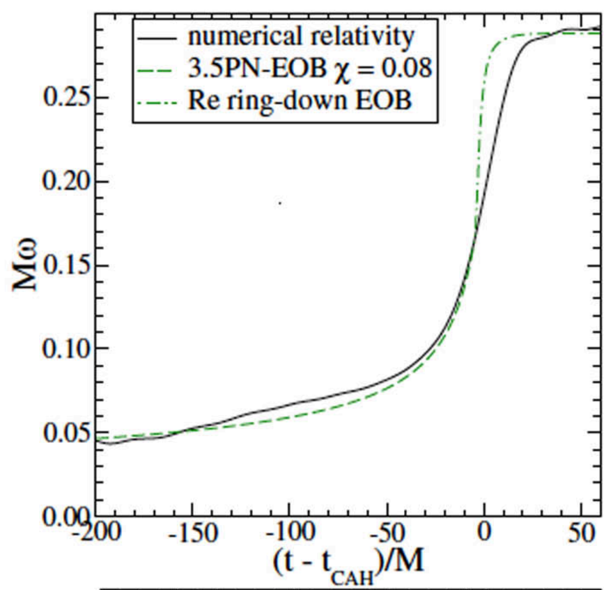


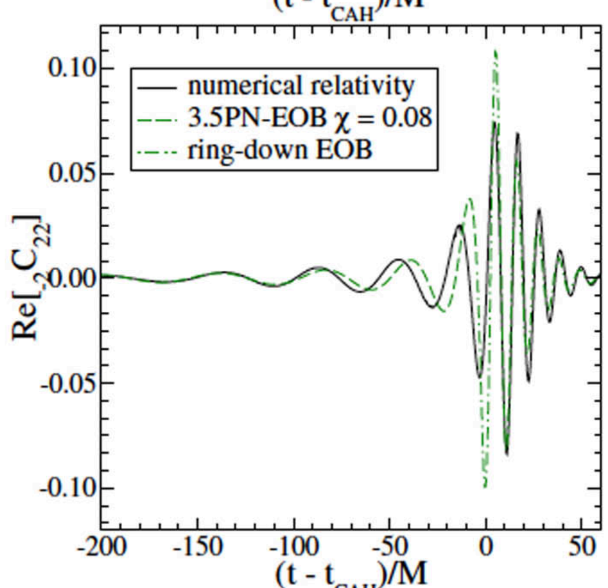
First complete waveforms for BBH coalescences: analytical EOB (Buonanno-Damour'00, Buonanno-Chen-Damour'05)

NR, EOB[NR] AND EOB MAIN RADIAL POTENTIAL A(R)



Buonanno-Cook-Pretorius 2007





Mathematical foundations : Darmois 27, Lichnerowicz 43,Choquet-Bruhat 52-

Breakthrough:

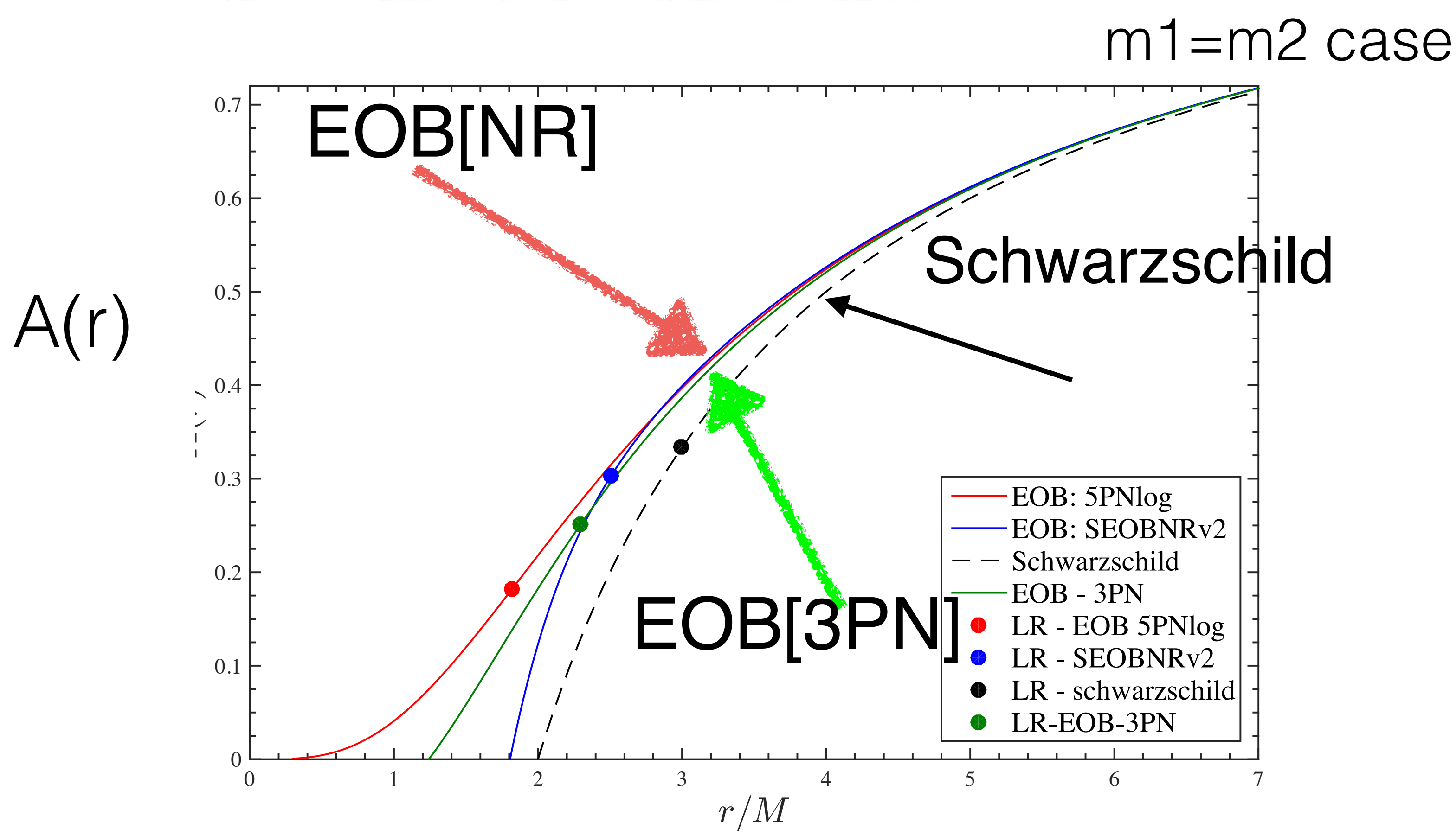
Pretorius 2005: generalized harmonic coordinates (Friedrich,Garfinkle); constraint damping (Brodbeck et al., Gundlach et al., Pretorius, Lindblom et al.); excision;

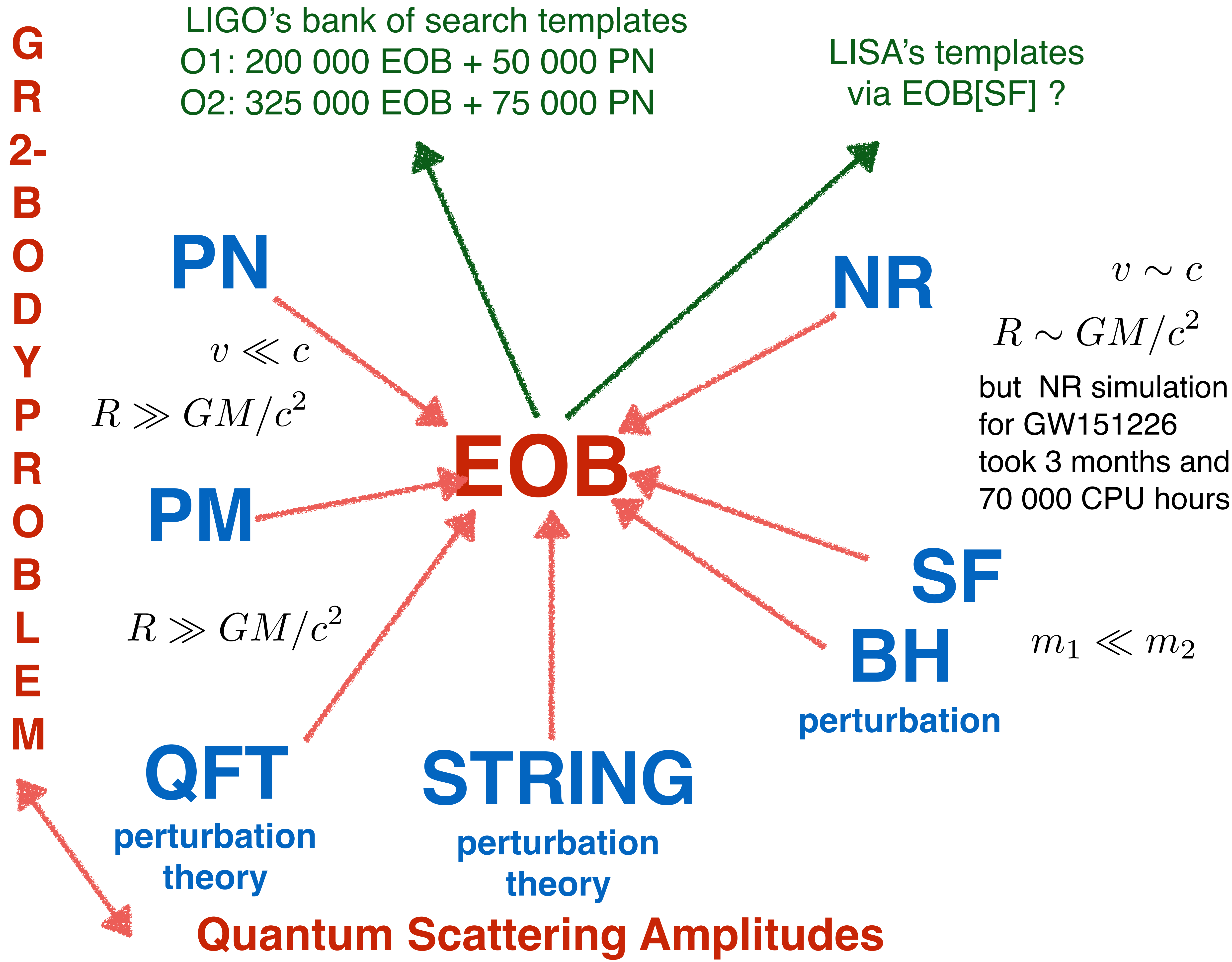
Moving punctures: Campanelli-Lousto-Maronetti-Zlochover 2006

Baker-Centrella-Choi-Koppitz-van Meter 2006

$$
\begin{aligned}
A(u;\nu,a_6^c) &= P_5^1\Big[1-2u+2\nu u^3+\nu\left(\frac{94}{3}-\frac{41}{32}\pi^2\right)u^4 \\
&+ \nu\left[-\frac{4237}{60}+\frac{2275}{512}\pi^2+\left(-\frac{221}{6}+\frac{41}{32}\pi^2\right)\nu+\frac{64}{5}\ln(16e^{2\gamma}u)\right]u^5 \\
&+ \nu\left[a_6^c(\nu)-\left(\frac{7004}{105}+\frac{144}{5}\nu\right)\ln u\right]u^6\Big]
\end{aligned}
$$

$$
a_6^{c\,\mathrm{NR\text{-}tuned}}(\nu) = 81.38 - 1330.6\,\nu + 3097.3\,\nu^2
$$




G R 2- B O D Y P R O B L E M
LIGO's bank of search templates
O1: 200 000 EOB + 50 000 PN
O2: 325 000 EOB + 75 000 PN
LISA's templates
via EOB[SF] ?
PN
$v \ll c$
$R \gg GM/c^2$
NR
$v \sim c$
$R \sim GM/c^2$
but NR simulation
for GW151226
took 3 months and
70 000 CPU hours
EOB
PM
$R \gg GM/c^2$
SF
BH
perturbation
$m_1 \ll m_2$
QFT
perturbation
theory
STRING
perturbation
theory
Quantum Scattering Amplitudes