Black Hole Entropy Function and AdS_2/CFT_1 Correspondence

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Motivation:

Low energy limit of string theory gives rise to gravity coupled to other fields.

These theories typically have black hole solutions.

Thus string theory gives a framework for studying classical and quantum properties of black holes.

One of the important properties characterizing a black hole is the Bekenstein-Hawking entropy S_{BH}.

In the low energy limit

$$S_{BH} = A/(4G_N)$$

For a wide class of extremal black holes

$$S_{BH} = S_{stat}, \qquad S_{stat} \equiv \ln(Degeneracy)$$

Strominger, Vafa; . . .

This gives a good understanding of this entropy from microscopic viewpoint.

Although string theory $\to$ gravity at low energy, the full theory contains higher derivative corrections and quantum corrections.

What are the effects of these corrections to S_{BH} and how do they affect the relation between S_{BH} and S_{stat}?

In order to address this question we need to understand both S_{BH} and S_{stat} to better precision.

In this lecture we shall try to gain a better understanding of S_{BH}.

A general frameork for computing higher derivative corrections to black hole entropy has been developed by Wald.

$$S_{BH} = -8\pi \int_H d\theta\, d\phi\, \frac{\delta \mathcal{S}}{\delta R_{rtrt}} \sqrt{-g_{rr}\, g_{tt}}\,,$$

for spherically symmetric black holes.

In computing $\delta\mathcal{S}/\delta R_{\mu\nu\rho\sigma}$

1. express the action $\mathcal{S}$ in terms of symmetrized covariant derivatives of fields

2. treat $R_{\mu\nu\rho\sigma}$ as independent variables.

We shall use this to study black hole entropy in the extremal limit.

How do we define extremal black holes in a higher derivative theory?

Take the clue from usual (super-)gravity.

Reissner-Nordstrom solution in $D = 4$:

$$
\begin{aligned}
ds^2 &= -(1-\rho_+/\rho)(1-\rho_-/\rho)d\tau^2 \\
&\quad + \frac{d\rho^2}{(1-\rho_+/\rho)(1-\rho_-/\rho)} \\
&\quad + \rho^2(d\theta^2 + \sin^2\theta d\phi^2)
\end{aligned}
$$

$\rho_\pm$: parameters related to mass and charge

Extremal black hole: $\rho_+ = \rho_-$

Instead of studying directly extremal black holes, for which Wald's formula is **not valid**, we shall study the **extremal limit** of regular black holes.

$$
\begin{aligned}
ds^2 &= -(1-\rho_+/\rho)(1-\rho_-/\rho)d\tau^2 + \frac{d\rho^2}{(1-\rho_+/\rho)(1-\rho_-/\rho)} \\
&\quad +\rho^2(d\theta^2 + \sin^2\theta d\phi^2)
\end{aligned}
$$

Define

$$
2\lambda = \rho_+ - \rho_-, \quad t = \frac{\lambda\,\tau}{\rho_+^2}, \quad r = \frac{2\rho - \rho_+ - \rho_-}{2\lambda}
$$

and take $\lambda \to 0$ limit.

$$
ds^2 = \rho_+^2\left[-(r^2-1)dt^2 + \frac{dr^2}{r^2-1}\right] + \rho_+^2(d\theta^2 + \sin^2\theta d\phi^2)
$$

$\rightarrow$ near horizon geometry $AdS_2 \times S^2$

The horizon is at $r = 1$.

The complete near horizon solution:

$$ds^2 = \rho_+^2 \left[-(r^2-1)dt^2 + \frac{dr^2}{r^2-1} \right] + \rho_+^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

$$F_{rt} = \frac{q}{4\pi}, \qquad F_{\theta\phi} = \frac{p}{4\pi} \sin\theta$$

$$\rho_+^2 = G_N \frac{q^2+p^2}{4\pi}$$

q, p: label electric and magnetic charges

The full background has $SO(2,1) \times SO(3)$ isometry.

In general, the near horizon geometry of all known extremal black holes in all dimensions have **time translation symmetry enhanced to** $SO(2,1)$

t and r form an AdS_2 space.

We shall take this as the definition of extremal black holes even in theories with higher derivative terms in the action.

(Partial proof by Kunduri, Lucietti, Reall; Figueras,Kunduri, Lucietti,Rangamani)

For simplicity, explicit discussion will be restricted to spherically symmetric black holes in $D = 4$.

$-$ near horizon geometry has $SO(2,1) \times SO(3)$ isometry.

However the results can be easily generalized to any extremal black hole, i.e. black holes with an enhanced $SO(2,1)$ isometry in the near horizon geometry.

Consider an arbitrary general coordinate invariant theory of gravity coupled to a set of Maxwell fields $A_\mu^{(i)}$ and neutral scalar fields $\{\phi_s\}$.

The most general form of the near horizon geometry of an extremal black hole consistent with $SO(2,1) \times SO(3)$ isometry:

$$
\begin{aligned}
ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = v_1 \left(-(r^2-1)dt^2 + \frac{dr^2}{r^2-1} \right) \\
+ v_2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right)
\end{aligned}
$$

$$\phi_s = u_s$$

$$F_{rt}^{(i)} = e_i, \qquad F_{\theta\phi}^{(i)} = \frac{p_i}{4\pi} \sin\theta$$

v_1, v_2, u_s, e_i, p_i are constants.

Equations of motion and Wald's formula gives a neat way of determining the near horizon parameters and the entropy.

Let $\sqrt{-\det g}\,\mathcal{L}$ be the Lagrangian density.

Define:

$$f(\vec{u},\vec{v},\vec{e},\vec{p}) \equiv \int d\theta\, d\phi\, \sqrt{-\det g}\,\mathcal{L}$$

$$\mathcal{E}(\vec{u},\vec{v},\vec{q},\vec{p}) \equiv 2\,\pi(e_i\, q_i - f(\vec{u},\vec{v},\vec{e},\vec{p}))$$

at $\quad \partial f/\partial e_i = q_i$.

$\mathcal{E}/(2\pi)$ is Legendre transform of f.

Results:

For an extremal black hole of electric charge $\vec{q}$ and magnetic charge $\vec{p}$,

1. the values of $\{u_s\}$, v_1 and v_2 and e_i are obtained by solving:

$$\frac{\partial \mathcal{E}}{\partial u_s} = 0, \quad \frac{\partial \mathcal{E}}{\partial v_1} = 0\,, \quad \frac{\partial \mathcal{E}}{\partial v_2} = 0, \quad e_i = \frac{1}{2\pi}\frac{\partial \mathcal{E}}{\partial q_i}$$

2. $S_{BH} = \mathcal{E}$ at the extremum.

These results come out of straightforward use of equations of motion and Wald's formula.

Example: For Reissner-Nordstrom black hole in ordinary gravity coupled to Maxwell theory

$$\begin{aligned}\mathcal{E} &= 2\pi\Bigg[-\frac{1}{4G_N}(2v_1 - 2v_2) \\ &\quad +2\pi\, v_1\, v_2^{-1}\left(\frac{q}{4\pi}\right)^2 + 2\pi\, v_1\, v_2^{-1}\left(\frac{p}{4\pi}\right)^2\Bigg]\end{aligned}$$

$\partial\mathcal{E}/\partial v_1 = \partial\mathcal{E}/\partial v_2 = 0$ give

$$v_1 = v_2 = G_N\,\frac{q^2 + p^2}{4\pi}$$

$$S_{BH} = \mathcal{E} = \frac{1}{4}(q^2 + p^2)$$

$\rightarrow$ correct answer for the entropy.

Two dimensional viewpoint

In $D = 4$ the near horizon geometry has the structure of $AdS_2 \times S^2 \times$ a compact space.

Treat S^2 also as part of a compact direction.

$\rightarrow AdS_2 \times$ a compact space.

This gives a uniform description of the near horizon geometry of all extremal black holes in all dimensions.

Two dimensional background

$$ds^2 \equiv g^{(2)}_{\mu\nu} dx^\mu dx^\nu = v_1 \left(-(r^2-1)dt^2 + \frac{dr^2}{r^2-1} \right)$$

$$\phi_s = u_s, \qquad F^{(i)}_{rt} = e_i$$

Let $\mathcal{L}^{(2)}$ be the two dimensional Lagrangian density.

Define

$$f \equiv v_1 \, \mathcal{L}^{(2)}, \quad \mathcal{E} \equiv 2\pi \, (e_i \, q_i - f) \quad \text{at } \partial f / \partial e_i = q_i$$

Then $S_{BH}(\vec{q}) = \mathcal{E}$ at the extremum of $\mathcal{E}$.

We shall now try to give a physical interpretation of S_{BH} using AdS_2/CFT_1 correspondence.

A.S., Gupta, A.S.

Strominger; Cadoni, Mignemi; Maldacena, Michelson, Strominger; Spradlin, Strominger; Navarro-Salas, Navarro; Caldarelli, Catelani, Vanzo; Cadoni, Carta, Klemm, Mignemi; Giveon, Sever; Azeyanagi, Nishioka, Takayanagi; Hartman, Strominger

Important equations

$$f = v_1 \mathcal{L}^{(2)}$$

$$2\pi f = 2\pi \vec{e}\cdot\vec{q} - S_{BH} \quad \text{at} \quad \partial f/\partial e_i = q_i$$

$$\rightarrow 2\pi e_i = \partial S_{BH}(\vec{q})/\partial q_i$$

$$ds^2 = v_1 \left(-(r^2-1)dt^2 + \frac{dr^2}{r^2-1} \right)$$

$$\phi_s = u_s, \qquad F_{rt}^{(i)} = e_i$$

Euclidean continuation:

$$t = -i\theta, \qquad r = \cosh\eta, \qquad \theta \equiv \theta + 2\pi$$

This gives

$$ds^2 = v_1 \left(d\eta^2 + \sinh^2\eta \, d\theta^2 \right),$$

$$\phi_s = u_s, \quad F_{\theta\eta}^{(i)} = i e_i \sinh\eta$$

$$\rightarrow \quad A_\theta^{(i)} = i\, e_i \left(\cosh\eta - 1 \right).$$

Classical supergravity partition function:

$$Z_{AdS_2} \simeq e^{-A}, \qquad A = \text{Euclidean action}$$

Since AdS_2 has infinite volume, A would be infinite.

We regularize by putting a cut-off at:

$$\eta = \eta_0 \quad \rightarrow \quad r = \cosh \eta_0 = r_0$$

This gives

$$\begin{aligned} A_{bulk} &= -2\pi v_1 \int_0^{\eta_0} d\eta \sinh \eta \, \mathcal{L}^{(2)} \\ &= -2\pi f (\cosh \eta_0 - 1) \\ &= -(r_0 - 1)(2\pi \vec{e} \cdot \vec{q} - S_{BH}) \end{aligned}$$

Besides this there may also be boundary contribution proportional to the length of the boundary.

$$A_{boundary} = -K \sinh \eta_0 = -K\, r_0 + \mathcal{O}(r_0^{-1})$$

This gives

$$\begin{aligned} Z_{AdS_2} &\simeq e^{-A_{bulk} - A_{boundary}} \\ &= e^{r_0(2\pi \vec{e}\cdot\vec{q} - S_{BH} + K) + S_{BH} - 2\pi\vec{e}\cdot\vec{q} + \mathcal{O}(r_0^{-1})} \end{aligned}$$

Note: In full quantum theory Z_{AdS_2} should be computed by the path integral of string theory on $AdS_2 \times$ compact space.

AdS_2/CFT_1 correspondence

By the usual AdS/CFT correspondence we would expect that string theory on AdS_2 should be equivalent to a conformal quantum mechanics (CQM) at the boundary $r = r_0$ of AdS_2.

$$Z_{CQM} = Z_{AdS_2}$$

We shall now analyze Z_{CQM}.

θ labels the coordinate along the boundary.

We shall use a rescaled boundary coordinate

$$w = \theta \sinh \eta_0$$

$$w \equiv w + 2\pi \sinh \eta_0 = w + 2\pi\, r_0 \left(1 + r_0^{-2}\right)$$

Boundary field configuration

$$ds_B^2 = v_1\, dw^2, \quad \phi_s = u_s,$$
$$A_w^{(i)} = i\, e_i\, (1 - r_0^{-1} + \mathcal{O}(r_0^{-2}))$$
$$w \equiv w + 2\pi\, r_0 \left(1 + r_0^{-2}\right)$$

Define

H: generator of w translation in CQM in the $r_0 \to \infty$ limit

Q_i: Conserved charge dual to $A_\mu^{(i)}$ in CQM

Then

$$Z_{CQM} = Tr\left[e^{-2\pi r_0 H - 2\pi e_i Q_i + \mathcal{O}(r_0^{-1})}\right]$$

$$Z_{CQM} = Tr\left[e^{-2\pi r_0 H - 2\pi e_i Q_i}\right]$$

In the $r_0 \to \infty$ limit only the ground states of H contribute.

$d(\vec{q})$: degeneracy of such states of charge $\vec{q}$

Then

$$Z_{CQM} = e^{-2\pi r_0 E_0} \sum_{\vec{q}} d(\vec{q})\, e^{-2\pi \vec{e}\cdot\vec{q}}$$

E_0: ground state energy

$$Z_{CQM} = e^{-2\pi r_0 E_0} \sum_{\vec{q}} d(\vec{q})\, e^{-2\pi \vec{e}\cdot\vec{q}}$$

For large charges one expects the summand to be sharpely peaked around the maximum

$$\partial \ln d(\vec{q})/\partial q_i = 2\pi e_i$$

$$Z_{CQM} \simeq e^{-2\pi r_0 E_0}\, d(\vec{q})\, e^{-2\pi \vec{e}\cdot\vec{q}}$$

around its maximum.

$$Z_{CQM} \simeq e^{-2\pi r_0 E_0}\, d(\vec{q})\, e^{-2\pi \vec{e}\cdot\vec{q}}$$

$$Z_{AdS_2} \simeq e^{-r_0(2\pi\vec{e}\cdot\vec{q} - S_{BH} + K) + S_{BH} - 2\pi\vec{e}\cdot\vec{q} + \mathcal{O}(r_0^{-1})}$$

This gives:

$$e^{S_{BH}} \simeq d(\vec{q})$$

in the semiclassical limit.

Thus in the semiclassical limit, when Wald's analysis is valid, Wald entropy computes the degeneracy of the ground state of the dual quantum mechanics.

The appropriate quantity that generalizes the Legendre transform of Wald's entropy to the full quantum theory is the partition function Z_{AdS_2} of string theory on AdS_2.

Comparison of S_{BH} and S_{stat} reduces to comparing

$$Z_{AdS_2}(\vec{e}) \leftrightarrow \sum_{\vec{q}} d_{microstate}(\vec{q}) e^{-2\pi \vec{q}\cdot\vec{e}}$$

Special case: Type IIA on CY_3

In this case Z_{AdS_2} may be computable.

Recall: In the semiclassical approximation

$$Z_{AdS_2} \simeq e^{-2\pi f}$$

after removing cut-off dependent terms.

If we evaluate $f = v_1 \mathcal{L}^{(2)}$ using one the F-type terms in the effective action then

$$Z_{AdS_2} \simeq e^{-2\pi f} = |Z_{top}|^2$$

Ooguri, Strominger, Vafa

Quantum corrections should be strongly constrained due to SUSY.

Expect

$$Z_{AdS_2} = |Z_{top}|^2 \times \text{simple measure factor}$$

It may not be impossible to calculate them completely.