Universal Vertex-IRF Transformation and Quantum Whittaker Vectors

E. Buffenoir

RAQIS - Annecy le Vieux - September 11-14, 2007
ENIGMA - Lalonde les Maures - October 14-19, 2007

- “Quantum Whittaker vectors and dynamical coboundary equation”
**8-vertex model:**
2-d square lattice model
link $\rightarrow \varepsilon_j = \pm$
vertex $\rightarrow$ Boltzmann weight
$z =$ spectral parameter,
$p =$ elliptic parameter

\[
R^{8V}(z_1/z_2)^{\varepsilon_1,\varepsilon_2}_{\varepsilon'_1,\varepsilon'_2} = z_2 \begin{array}{c}
\varepsilon'_2 \\
\varepsilon'_1 \\
z_1
\end{array}
\]

\[
a(z; p) = \frac{\Theta_{p^4}^4(p^2z)\Theta_{p^4}^4(p^2q^2)}{\Theta_{p^4}^4(p^2)\Theta_{p^4}^4(p^2q^{-2}z)}
\]
\[
b(z; p) = q^{-1} \frac{\Theta_{p^4}^4(z)\Theta_{p^4}^4(p^2q^2)}{\Theta_{p^4}^4(p^2)\Theta_{p^4}^4(q^{-2}z)}
\]
\[
c(z; p) = \frac{\Theta_{p^4}^4(p^2z)\Theta_{p^4}^4(q^{-2})}{\Theta_{p^4}^4(p^2)\Theta_{p^4}^4(q^{-2}z)}
\]
\[
d(z; p) = pq^{-1} \frac{\Theta_{p^4}^4(z)\Theta_{p^4}^4(q^2)}{\Theta_{p^4}^4(p^2)\Theta_{p^4}^2(q^{-2}z)}
\]

\[\Theta_p(x) := (x, px^{-1}, p; p)_\infty\]

satisfying the Quantum Yang-Baxter Equation (QYBE) with spectral param.
No charge conservation through a vertex $\rightarrow$ no direct Bethe Ansatz solution
Baxter’s solution (Ann.Phys.1973) $\rightarrow$ map onto an IRF model (SOS model)
SOS model (Interaction-Round-a-Face model):

2-d square lattice model
vertex → local height $n_j$

$n_j - n_k = \pm 1$ (adjacent)
face → Boltzmann weight $w$ dynamical parameter

$R^{SOS}(z_1; w_0 q^n)_{\varepsilon_1, \varepsilon_2} = \begin{pmatrix} n & | & n + \varepsilon_1' \\ \downarrow & | & \downarrow \\ n + \varepsilon_2 & | & n + \varepsilon_1 + \varepsilon_2 \\ \downarrow & | & \downarrow \\ z_1 & | & z_1 \end{pmatrix}$

$satisfying the Dynamical Quantum Yang-Baxter Equation (DQYBE):$

$R^{SOS}_{12}(z_1/z_2; wq^{h_3}) R^{SOS}_{13}(z_1; w) R^{SOS}_{23}(z_2; wq^{h_1}) = R^{SOS}_{23}(z_2; w) R^{SOS}_{13}(z_1; wq^{h_2}) R^{SOS}_{12}(z_1/z_2; w)$

with $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Charge conservation, solvable by Bethe Ansatz

$R^{SOS}(z; w) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(z; w, p) & c(z; w, p) & 0 \\ 0 & zc(z; \frac{p}{w}, p) & b(z; \frac{p}{w}, p) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$b(z; w, p) = q^{1 - \frac{1}{2} \Theta_{p^2}(q^2 p^2 w^{-2}, q^{-2} p^2 w^{-2}; p^2)_{\infty}} \Theta_{p^2}(q^{-2} z)(p^2 w^{-2}; p^2)_{\infty}$

$c(z; w, p) = \frac{\Theta_{p^2}(q^{-2}) \Theta_{p^2}(w^2 z)}{\Theta_{p^2}(w^2) \Theta_{p^2}(q^{-2} z)}$
**Baxter’s transformation:**

It is equivalent to the following **Dynamical Gauge Equivalence**:

\[
R_{12}^{SOS}(z_1/z_2; w) = M_1(z_1; w q^{h_2}) M_2(z_2; w) R_{12}^{8V}(z_1/z_2) M_1(z_1; w)^{-1} M_2(z_2; w q^{h_1})^{-1}
\]

with

\[
M(z, w, p)^{-1} = \begin{pmatrix}
\vartheta_p^4 (-p^3 w^{-2} z) & pz^{-1} \vartheta_p^4 (-p^{-1} w^2 z) \\
w \vartheta_p^4 (-pw^{-2} z) & \vartheta_p^4 (-pw^2 z)
\end{pmatrix} \Lambda(z; w, p)
\]

where \( \Lambda(z; w, p) = \text{diagonal matrix} \).

→ Eigenvalues and eigenvectors of the 8-Vertex Transfer Matrix (Baxter, 1973)

→ Correlation Functions of the 8-Vertex Model from the SOS ones (Lashkevich and Pugai, 1997)
Gervais-Neveu-Felder $R-$matrix:

In their study of $\mathfrak{sl}_{r+1}-$Toda Field theory, Gervais-Neveu ($r = 1$ case, 1984), and then Cremmer-Gervais (1989), introduced the "Face-type" standard trigonometric solution

$$R^{GN}(x) = q^{-\frac{1}{r+1}} \left\{ q \sum_i E_{i,i} \otimes E_{i,i} + \sum_{i \neq j} \left( q - q^{-1} \frac{\nu_i}{\nu_j} \right) \left( 1 - \frac{\nu_i}{\nu_j} \right)^{-1} E_{i,i} \otimes E_{j,j} \right. $$

$$+ \left. (q - q^{-1}) \sum_{i \neq j} \left( 1 - \frac{\nu_i}{\nu_j} \right)^{-1} E_{i,j} \otimes E_{j,i} \right\}$$

$$(\prod_{i=1}^{r+1} \nu_i = 1 \text{ and } x_i^2 = \frac{\nu_i}{\nu_{i+1}} ) \text{ of the QDYBE:}$$

$$R^{GN}_{12}(x)R^{GN}_{13}(xq^{h_2})R^{GN}_{23}(x) = R^{GN}_{23}(xq^{h_1})R^{GN}_{13}(x)R^{GN}_{12}(xq^{h_3})$$

where $xq^h = (x_1 q^{h\alpha_1}, \ldots, x_r q^{h\alpha_r})$. 
Cremmer-Gervais-Bilal $R$–matrix:

It exists, in the fundamental representation of $U_q(\mathfrak{sl}_{r+1})$,

- an invertible element $M(x)$
- a certain non-standard solution $R^J$ of the (non-dynamical) QYBE (Cremmer-Gervais's R-matrix)

such that

$$R^{GN}(x)M_1(xq^{h_2})M_2(x) = M_2(xq^{h_1})M_1(x)R^{CG}$$

Explicitly,

$$R^{CG} = q^{-\frac{1}{r+1}} \left\{ q \sum_{t,s} q^{-\frac{2(s-t)}{r+1}} E_{tt} \otimes E_{s,s} \right.$$

$$+ (q - q^{-1}) \sum_{i,j,k} \eta(i, j, k) q^{-\frac{2(i-k)}{r+1}} E_{i,j+i-k} \otimes E_{j,k} \right\},$$

(with $\eta(i, j, k) = 1$ if $i \leq k < j$, $-1$ if $j \leq k < i$, 0 otherwise) and $M(x)$ is given by a Van der Monde matrix (up to a diagonal matrix $U(x)$):

$$M(x)^{-1} = \left( \sum_j \nu_j^{i-1} E_{i,j} \right) U(x).$$
$g$ fin. dim. or affine Lie algebra, $\mathfrak{h}$ its Cartan subalg., let $I = \{\alpha_1, \cdots, \alpha_r\}$, and if $g = \mathfrak{sl}_{r+1}$, $I := I$, if $g = A_r^{(1)}$, $I = \{\alpha_0\} \cup I$

$(\zeta^i, h_{\alpha_j}) = \delta_{i,j}, \forall i, j \in I$, $(\zeta^d, h_{\alpha_j}) = 0, \forall j \in I$, $(\zeta^d, d) = 1.$

$U_q(g)$ is the algebra generated by $e_{\pm \alpha_i}, i \in I$ and $q^h, h \in \mathfrak{h}$ with relations:

$$q^h e_{\pm \alpha_i} q^{-h} = q^{\pm \alpha_i(h)} e_{\pm \alpha_i}, \quad q^h q^{h'} = q^{h+h'}, \quad [e_{\alpha_i}, e_{-\alpha_j}] = \delta_{ij} \frac{q^{h \alpha_i} - q^{-h \alpha_i}}{q - q^{-1}},$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \begin{array}{c} 1-a_{ij} \end{array} \right]_q e_{\pm \alpha_i}^{1-a_{ij}-k} e_{\pm \alpha_j} e_{\pm \alpha_i}^k = 0, \quad i \neq j$$

$U_q(b^{\pm})$ (resp. $U_q(n^{\pm})$): sub Hopf algebra generated by $q^h, e_{\pm \alpha_i}$ (resp. $e_{\pm \alpha_i}$; $i \in I$). $U_q^{\pm}(g) = \text{Ker}(\iota^{\pm})$ with $\iota^{\pm} : U_q(b^{\pm}) \to U_q(h)$ projections.

$U_q(g)$ is a quasitriangular Hopf algebra with $\Delta(e_{\alpha_i}) = e_{\alpha_i} \otimes q_i^{h_{\alpha_i}} + 1 \otimes e_{\alpha_i}$, $\Delta(e_{-\alpha_i}) = e_{-\alpha_i} \otimes 1 + q_i^{-h_{\alpha_i}} \otimes e_{-\alpha_i}$, and $\Delta(h) = h \otimes 1 + 1 \otimes h$ with universal R-matrix:

$$R = K \hat{R} \quad \text{with} \quad K = q^{\sum_i h_i \otimes \zeta^i} \in U_q(\mathfrak{h}) \otimes^2, \quad \hat{R} \in 1 \otimes^2 \oplus U_q^+(g) \otimes U_q^-(g)$$
Universal solutions $R(x)$ of the QDYBE can be build in terms of the standard solution $R$ of the QYBE and \textit{Quantum Dynamical Cocycles} as

$$R(x) = J_{21}(x)^{-1} R_{12} J_{12}(x),$$

\textbf{Definition : Quantum Dynamical Cocycle}

$$J : (\mathbb{C}^\times)^{\dim l} \to U_q(g) \otimes^2$$ where $J(x)$ is invertible, is of zero $l$-weight, and satisfies the Quantum Dynamical Cocycle Equation (QDCE):

$$(\Delta \otimes id)(J(x)) J_{12}(x q^h) = (id \otimes \Delta)(J(x)) J_{23}(x).$$

Quasi-Hopf interpretation, Dynamical Vertex-Operator approaches of 8–vertex correlation functions.

Explicit construction by means of an auxiliary linear equation:

$\rightarrow$ \textit{Standard solution} $R(x)$ in the finite dimensional case (Arnaudon, Buffenoir, Ragoucy, Roche, 1998)

$\rightarrow$ \textit{Standard IRF solution} and Belavin-Baxter’s \textit{Vertex solution} in the $A_r^{(1)}$ case (Jimbo, Konno, Odake, Shiraishi, 1999)

$\rightarrow$ Extension to other solutions (in the formalism of \textit{generalized Belavin-Drinfeld Triple}) (Etingof, Schedler, Schiffmann, 2000)
Verma modules

\[ \mathcal{V}_\eta = U_q(g) \cdot \varnothing > \eta \quad x \cdot \varnothing > \eta = 0, \quad \forall x \in U_q^+(g). \]
\[ u \cdot \varnothing > \eta = \mathcal{V}_\eta(u) \cdot \varnothing > \eta, \quad \forall u \in U_q(h). \]

Shapovalov's form

\[ S_\eta: \mathcal{V}_{-\eta^*} \times \mathcal{V}_\eta \to \mathbb{C} \]

unique hermitian form such that

\[ S_\eta(\varnothing > -\eta^*, \varnothing > \eta) = 1 \quad S_\eta(\nu, x \cdot \nu') = S_\eta(x^* \cdot \nu, \nu') \]

with \( q^* = q \quad e_{\alpha_i}^* = e_{-\alpha_i} q^{h_{\alpha_i}} \quad e_{-\alpha_i}^* = q^{-h_{\alpha_i}} e_{\alpha_i} \quad h_{\alpha_i}^* = h_{\alpha_i} \)

non-degenerate if and only if \( \mathcal{V}_\eta \) is irreducible.
### Fusion Operators of Verma Modules

Φ element of Hom\(_{U_q(g)}(V_{\eta'}, V \otimes V_{\eta})\) (where \(V\) is a finite dimensional \(U_q(g)\)–module). We define \(<\Phi>\) by

\[<\Phi> \in V[\eta' - \eta]/\Phi(|\emptyset>_{\eta'}) = <\Phi> \otimes |\emptyset>_{\eta} + \sum_{(\gamma) \neq \emptyset} a(\gamma) \otimes f(\gamma)|\emptyset>_{\eta}.\]

If \(V_{\eta'}\) is irreducible then \(<> : \text{Hom}_{U_q(g)}(V_{\eta'}, V \otimes V_{\eta}) \rightarrow V[\eta' - \eta]\) is an isomorphism. For any homogeneous element \(v_\mu\) of weight \(\mu\), we denote \(\Phi_{\eta \mu}^{\eta'} \in \text{Hom}_{U_q(g)}(V_{\eta + \mu}, V \otimes V_{\eta})\) the unique element such that \(<\Phi_{\eta \mu}^{\eta'}> = v_\mu\).

### Fusion Operators and “Face Type” Quantum Dynamical Cocycles

For any homogeneous vector \(v_{\mu_i} \in V\) of weight \(\mu_i\) and any \(\eta \in \mathfrak{h}^*\), we have

\[\Phi_{\eta \mu}^{\eta'} |\emptyset>_{\eta + \mu} = (\hat{\pi} \otimes \text{id})(J_F(\hat{\chi}))(v_\mu \otimes |\emptyset>_{\eta}) \quad \hat{\chi}(h_{\alpha_i}) = q, \forall i\]

\[\left((1 \otimes \Phi_{\eta \mu}^{\eta' \mu_2}) \Phi_{\eta + \mu_1 + \mu_2}^{\eta + \mu_1} \right) |\emptyset>_{\eta + \mu_1 + \mu_2} = \Phi_{\eta}(J_F(\hat{\chi}q)(v_{\mu_1} \otimes v_{\mu_2}) |\emptyset>_{\eta + \mu_1 + \mu_2}\]

\(J_F\) verifies quantum Dynamical Cocycle equation.

Relation of matrix elements of \(J_F\) and Shapovalov’s coefficients...
**Definition : Generalized Translation Quadruple \((\theta_+^{[x]}, \theta_-^{[x]}, \varphi, S)\)**

\(\theta_+^{[x]} \in \text{End}(U_q(b^\pm))\), \(\varphi\) and \(S\) are invertible elements of \(U_q(h)^{\otimes 2}\) such that

\[
\begin{align*}
\theta_+^{[x]} \in \text{End}(U_q(h^{\pm})), & \quad \varphi \text{ and } S \text{ are invertible elements of } U_q(h)^{\otimes 2} \\
\theta_+^{[x]}(\theta_-^{[x]})_1 &= Ad_{\varphi}^{[x]} \circ \theta_+^{[x]}_1 \\
\theta_+^{[x]}(\varphi_{12}) &= \theta_+^{[x]}(\varphi_{12}) = \varphi_{12} \\
\varphi_{12}K_{12}S_{21}^{-1}S_{12} &\quad \theta_-^{[x]}(K_{12}S_{21}S_{12}^{-1}) , \theta_-^{[x]}(v) = 0.
\end{align*}
\]

**Theorem : Auxilliary Linear Problem (ABRR equation)**

Let \((\theta_+^{[x]}, \theta_-^{[x]}, \varphi, S)\) be a generalized translation quadruple. The linear equation

\[
\hat{J}(x) = \theta_-^{[x]} \left( Ad_{S^{-1}}(\hat{R}) \hat{J}(x) \right)
\]

admits a unique solution \(\hat{J}(x) \in 1 \otimes 1 + U_q^+(g) \otimes U_q^-(g)\) and \(J(x) = S\hat{J}(x)\) satisfies the Quantum Dynamical Cocycle Equation.

\[\rightarrow\text{ explicit formula for quantum dynamical cocycles as infinite product:}\]

\[
J_{12}(x) = S_{12} \prod_{k=1}^{\infty} (\theta_-^{[x]}_2)^k \left( S_{12}^{-1} \hat{R} S_{12} \right)
\]
Standard IRF solutions

**Standard IRF Solution** $\mathfrak{g} = \mathfrak{sl}_{r+1}$ or $\mathfrak{g} = A_r^{(1)}$, $l = \mathfrak{h}$

\[
\theta_{[x]}^{\pm} = Ad_{B(x)}^{\pm 1}, \quad \varphi = K^{-2}, \quad S = 1,
\]

with $B(x) \in U_q(\mathfrak{h})$ such that $\Delta(B(x)) = B_1(x) B_2(x) K^2$, $B_1(x q^{h_2}) = B_1(x) K^2$. This solution leads to the fusion matrix $J_F$.

Arnaudon, Buffenoir, Ragoucy, Roche, 1998 for the finite case, Jimbo, Konno, Odake, Shiraishi, 1999, for the affine case

→ "Face-type" solution $R^{GN}(x)$ in the $\mathfrak{g} = A_n$ case in the fundamental evaluation representation

→ "Face-type" solution $R^{SOS}(z; w, p)$ in the $\mathfrak{g} = A_1^{(1)}$ case (and higher rank generalizations) in the fundamental evaluation representation
Extremal Vertex Solutions

### Cremmer-Gervais-Bilal’s Vertex Solutions \((g = A_r, l = 0)\)

Obtained with the following choice of Generalized Translation Quadruple:

\[
\begin{align*}
\theta_{\pm}(e_{\pm\alpha_i}) &= e_{\pm\alpha_i}, & \theta_{\pm}(e_{\alpha_1}) &= 0, & \theta_{\pm}(e_{-\alpha_r}) &= 0, & \varphi &= 1, \\
\theta_{\pm}(\zeta^{\alpha_i}) &= \zeta^{\alpha_i}, & \theta_{\pm}(\zeta^{\alpha_1}) &= 0, & \theta_{\pm}(\zeta^{\alpha_r}) &= 0, & S &= q^{\sum_{i=1}^{r-1} \zeta^{\alpha_i} \otimes \zeta^{\alpha_i-1}}.
\end{align*}
\]

→ Cremmer-Gervais’s vertex solution \(R^{CG}\) (and higher rank generalizations) in the fundamental representation

### Belavin-Baxter Elliptic Vertex solutions \((g = A_r^{(1)}, l = c\mathbb{C})\)

\[
\begin{align*}
\theta_{\pm} &= Ad_{D_{\pm}(p)} \circ \sigma_{\pm}, & D^+(p) &= p^{\frac{2}{r+1}} \varpi, & D^-(p) &= p^{-\frac{2}{r+1}} \varpi q^{-\frac{2}{r+1}} c \varpi, \\
\sigma_{\pm}(e_{\pm\alpha_i}) &= e_{\pm\alpha_{[i+1]}}, & \sigma_{\pm}(h_{\alpha_i}) &= h_{\alpha_{[i+1]}}, & \sigma_{\pm}(\varpi) &= \varpi,
\end{align*}
\]

where \(\varpi = \sum_{i=0}^{r} \zeta^{\alpha_i}, [i] = i \mod r + 1\), and \(p\) is the component of \(x\) along \(q^c\).

→ eight-vertex solution \(R^{8V}(z; p)\) (and higher rank generalizations)
In terms of Quantum Dynamical Cocycles, the Vertex-IRF is reformulated as

**Generalized Quantum Dynamical Coboundary Problem:**

Let $J_F(x)$ be the standard "Face-type" solution of the Quantum Dynamical Cocycle Equation, and $J_V(x)$ a "Vertex-type" solution.

Does it exists an invertible element $M(x) \in U_q(g)$ such that

$$J_F(x) = \Delta(M(x)) J_V(x) M_2(x)^{-1} M_1(xq^{h_2})^{-1} \ ?$$

$\rightarrow$ Such a generalized Quantum Dynamical Coboundary is a Vertex-IRF transformation:

$R_F(x) = J_F(x)^{-1} R J_F(x)_{12}$, and $R_V(x) = J_V(x)^{-1} R J_V(x)_{12}$ satisfy

$$R_F(x)_{12} = M(xq^{h_1})_2 M(x)_1 R_V(x)_{12} M(x)^{-1}_2 M(xq^{h_2})^{-1}_1.$$
Theorem : Auxiliary Linear Problem (Buffenoir, Roche, Terras)

If $M(x)$ is expressed as

$$
M(x) = M^{(0)}(x) M^{(-)}(x)^{-1} M^{(+)}(x), \quad M^{(\pm)}(x) = \prod_{k=1}^{+\infty} c^{[\pm k]}(x)^{\pm 1}
$$

with $c^{[+k]}(x) = \left(\theta_\times^+\right)^{k-1}(c^{[+]}(x)), \quad c^{[-k]}(x) = B(x)^{-k} c^{[-]}(x) B(x)^{k}. \ And$ if $M^{(0)}(x) \in U_q(\mathfrak{h}), \quad c^{[\pm]}(x) \in 1 \oplus U_q^\pm(\mathfrak{g})$ satisfy

**Fusion**

$$
\Delta(M^{(0)}(x)) = S_{12}^{-1} M_1^{(0)}(x q^{h_2}) M_2^{(0)}(x),
$$

$$
\Delta(c^{[\pm]}(x)) = K_{12} A_{[S_{21}]}(c_1^{[\pm]}(x)) A_{[K_{12}^{-1} S_{12}]}(c_2^{[\pm]}(x)) K_{12}^{-1}
$$

**Shift**

$$
\mathcal{C}_1^{[\pm]}(x q^{h_2}) = A_{[S_{12}^{-1} S_{21} K_{12}]}(\mathcal{C}_1^{[\pm]}(x)),
$$

**Hexagonal**

$$
\mathcal{C}_2^{[-]}(x) \mathcal{C}_1^{[+]}(x q^{h_2}) \left[A_{B_2(x) \circ \theta_{[2]}^-}(S_{12}^{-1} \mathcal{R}_{12} S_{12})
\right.

= (S_{12}^{-1} \mathcal{R}_{12} S_{12}) \mathcal{C}_1^{[+]}(x q^{h_2}) \mathcal{C}_2^{[-]}(x).
$$

then it satisfies quantum dynamical coboundary equation.
→ trivial to find \( M^{(0)}(x) \) solution of fusion + adequate shift in terms of \( S \)

→ easy to build explicit solutions \( C^{[\pm]}(x) \) of fusion and shift:
they are generically associated to couple \( \chi = (\chi^+, \chi^-) \) of some non singular characters \( \chi \) of the subalgebras \( U_q(\mathfrak{n}^\mp)\omega^\mp \) of \( U_q(\mathfrak{b}^\mp) \) generated respectively by

\[
\begin{align*}
\omega^+ & = q^{-(\alpha_i \otimes id)(\omega^+)} e_{\alpha_i} \\
\omega^- & = e^{-\alpha_i} q^{(\alpha_i \otimes id)(\omega^-)}
\end{align*}
\]

\[
C^{[\pm]}(x) = Ad_{M^{(0)}(x)^{-2}}(C^{[\pm]}_{\chi_{\mp}})
\]

\[
C^{[\pm]}_{\chi_{\mp}} = (id \otimes \chi_{\mp})(S_{21} R_{12}^{(\pm)} K_{\mp}^{\mp T} S_{21}^{-1})
\]

→ hexagonal relation is satisfied only for specific Lie algebras and only for Standard IRF-type dynamical cocycles and Extremal Vertex-type cocycles (verified for the Cremmer-Gervais and Belavin-Baxter case for any rank) as soon as

\[
-q^{-1}(q - q^{-1})^2 \chi^+(e_{\alpha_i}) \chi^-(e^{-\alpha_i}) = 1.
\]
Quantum Whittaker Vectors

Let $\mathcal{V}$ be a $U_q(g)-$module and let $\chi^+$ be a non-singular character of $U_q(n^+)-\omega^+$, a vector $\omega_{\chi^+} \in \mathcal{V}$ is said to be a $\chi^+$—quantum Whittaker vector if
\[
\forall x \in U_q(n^+)-\omega^+, \quad x.\omega_{\chi^+} = \chi^+(x)\omega_{\chi^+}.
\]

Quantum Whittaker Functions

A quantum Whittaker function denoted $\mathcal{W}_{\xi}^{\omega^-}_{\omega^+}$ associated to a central character $\xi$ and non-singular characters $\omega^\pm$ of $U_q(n^\pm)-\omega^\pm$, is an element of $(U_q(g))^*$ defined such that
\[
\forall x \in U_q(n^-)-\omega^- , z \in U_q(n^+)-\omega^+, y \in U_q(g), a \in \mathbb{Z}_q(g) \text{ we have }
\]
\[
\mathcal{W}_{\xi}^{\omega^-}_{\omega^+}(x.y.z) = \omega^- (x)\omega^+(z)\mathcal{W}_{\xi}^{\omega^-}_{\omega^+}(y).
\]
\[
\mathcal{W}_{\xi}^{\omega^-}_{\omega^+}(a.y) = \xi(a) \mathcal{W}_{\xi}^{\omega^-}_{\omega^+}(y).
\]
The basic motivation is the Toda Quantum Mechanics (ex. $q = 1, g = A_1$)

$$
\xi(c)\mathcal{W}_{\varrho^{-},\varrho^{+}}^{\xi}(e^{\phi h}) = \mathcal{W}_{\varrho^{-},\varrho^{+}}^{\xi}(e^{\phi h}(e^{-}e^{+} + \frac{1}{2}h^{2} + h))
$$

$$
= \left(\varrho^{-}(e^{-})\varrho^{+}(e^{+})e^{-2\phi} + \frac{1}{2}\partial_{\phi}^{2} + \partial_{\phi}\right)\mathcal{W}_{\varrho^{-},\varrho^{+}}^{\xi}(e^{\phi h})
$$

Then, $f(\phi) = e^{\phi}\mathcal{W}_{\varrho^{-},\varrho^{+}}^{\xi}(e^{\phi h})$ obeys

$$
\left(\partial_{\phi}^{2} + 2\varrho^{-}(e^{-})\varrho^{+}(e^{+})e^{-2\phi} - 2\xi(c) - 1\right) \cdot f = 0
$$

$(\phi^{i})_{i} \mapsto \mathcal{W}_{\varrho^{-},\varrho^{+}}^{\xi}(e^{\phi^{i}h_{\alpha_{i}}})$ is a wave function of $q$–deformed quantum Toda hamiltonian derived from the action of the casimir $(\varrho^{-1} := m(S)B(\widehat{\chi}))$

$$
c_{q}^{\omega} := (\text{tr}_{q}^{\omega} \otimes id)(Ad_{q^{-}\omega} - (\widehat{R}_{12}^{-1})K_{12}^{-2}Ad_{q^{-}\omega} + (\widehat{R}_{21}^{-1})) \rightarrow
$$

$$
0 = (\text{tr}_{q}^{\omega} \left(Ad_{\varrho^{i}e^{\phi^{i}h_{\alpha_{i}}}}(C_{\varrho^{-}}^{[-]} - 1)q^{-2\xi^{i}\partial_{\phi}^{i}C_{\varrho^{+}}^{[-]}) - \xi(c_{q}^{\omega})\right)) \mathcal{W}_{\varrho^{-},\varrho^{+}}^{\xi}(e^{\phi^{i}h_{\alpha_{i}}})
$$
Quantum Whittaker vectors can be identified as elements of a certain completion of a Verma module:

**Explicit realization of quantum Whittaker vectors in Verma modules**

An explicit realization of quantum Whittaker vectors is given by

\[ W^{\xi,\eta}_{\chi^+} = M(\tilde{x})^{-1} \emptyset > \eta \]

The quantum Whittaker function is given by

\[ \mathcal{W}^{\xi,\eta}_{\chi^+,\chi^-}(u) = S_{\eta}(W^{\xi,\eta}_{\tilde{x}^+}, \emptyset u W^{\xi,\eta}_{\chi^+}) \]

\[ \tilde{\varpi}_1^\pm = (\varpi_2^\pm)^* \quad \tilde{\chi}^+(x) = \chi^-(Ad_{\varphi}(x^*))^* \]
Using the dynamical Coboundary equation as well as expression of Fusion operators, we deduce

$$\Phi_{\eta}^{\nu \mu} \hat{W}_{\chi^+}^{\xi \eta + \mu} = \Phi_{\eta}^{\nu \mu} M(\hat{x})^{-1} |\emptyset >_{\eta + \mu}$$

$$= (\frac{\hbar}{\pi} \otimes id) (\Delta(M(\hat{x})^{-1})) \Phi_{\eta}^{\nu \mu} |\emptyset >_{\eta + \mu}$$

$$= (\frac{\hbar}{\pi} \otimes id) (\Delta(M(\hat{x})^{-1})) (\frac{\hbar}{\pi} \otimes id) (J_F(\hat{x})) . (v_{\mu} \otimes |\emptyset >_{\eta})$$

$$= (\frac{\hbar}{\pi} \otimes id) (J_V(\hat{x})M(\hat{x})^{-1} M(\hat{x}q^{h2})^{-1}) . (v_{\mu} \otimes |\emptyset >_{\eta})$$

$$= (\frac{\hbar}{\pi} \otimes id) (J_V(\hat{x})M(\hat{x}q^{\eta(h)})^{-1}) . (v_{\mu} \otimes \hat{W}_{\chi^+}^{\omega \xi \eta}) .$$

These fusion formulas implies the basic finite difference equations describing the shift on $\eta$ for Whittaker functions in terms of finite difference operators acting on the $\phi_i$ variables.
Explicit universal solutions of quantum dynamical cocycle and coboundary equation in the finite dimensional as well as affine Lie algebra case: → universality is fundamental to build correlation functions of the model as matrix elements in highest weight reps. of quantum algebras.

Relation between “Face type” quantum dynamical cocycles and the Fusion theory of Verma modules.

Relation between quantum dynamical coboundaries and Whittaker vectors.

Relation between “Extremal Vertex type” quantum dynamical cocycles and the Fusion theory of Whittaker modules.

Main Open Problems

The explicit expression of quantum dynamical coboundary (in the quantum affine case) in the highest weight modules gives the “tail operator” used in Lashkevich/Pugai and Konno/Kojima/Weston’s works → New approach to compute correlation functions of the 8–vertex model.

Analysis of the quantum dynamical coboundary equation → new results on Whittaker functions.