PHYSICS AND COMBINATORICS: THE MIRACLES OF INTEGRABILITY

2D integrable lattice models

- ice model
- fully packed loops

Combinatorics

- alternating sign matrices

Algebraic Geometry

- orbital varieties (multidegree)

1, 2, 7, 42, 429, ...

1, 3, 23, 441, 21009, ...

$$\begin{tikzcd}
\text{JB} \arrow[r, "\text{virus}"] \arrow[dr, "\text{virus}"] \arrow[d, "\text{conjectures}"'] & \text{P.D.F.} \\
\text{C. Krattenthaler} & \text{P. Zinn-Justin}
\end{tikzcd}$$

13 papers to this day and conjecture still open

QUANTUM SPIN CHAINS / LOOP MODELS AND INTEGER NUMBERS:

O(1) LOOP MODEL ON A CYLINDER AND THE RAZUMOV-STROGANOV CONJECTURE

Bijections between configurations

- 6 VERTEX MODEL WITH DOMAIN-WALL BOUNDARIES (DWBC) on a square $n \times n$ grid

$$\Updownarrow$$

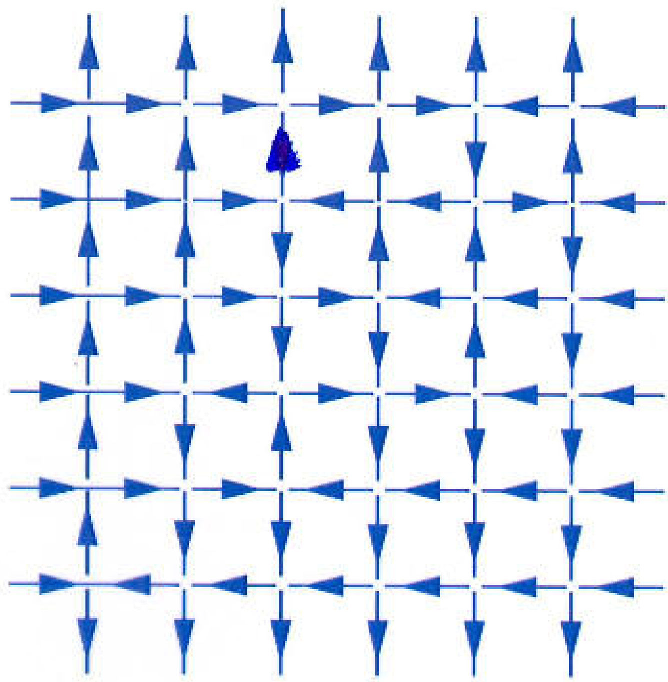
- FULLY-PACKED LOOP (FPL) model on a square $n \times n$ grid

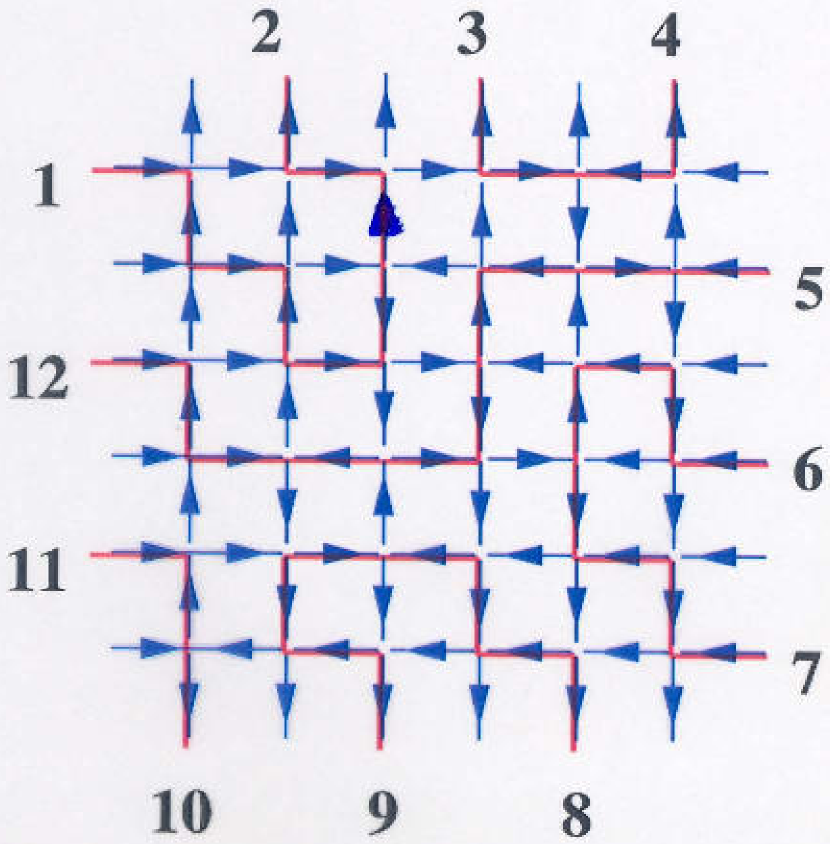
$$\Updownarrow$$

- ALTERNATING SIGN MATRICES $n \times n$ with elements $0, \pm 1$, each column and row reading $\{\underbrace{0..0}_{\text{arbitrary}}, +1, \underbrace{0..0}, -1, \underbrace{0..0}, +1 \ldots, \underbrace{0..0}, +1, \underbrace{0 \ldots 0}\}$

$A_n = 1, 2, 7, 42, 429, \ldots$

$n = 1, 2, 3, 4, 5, \ldots$

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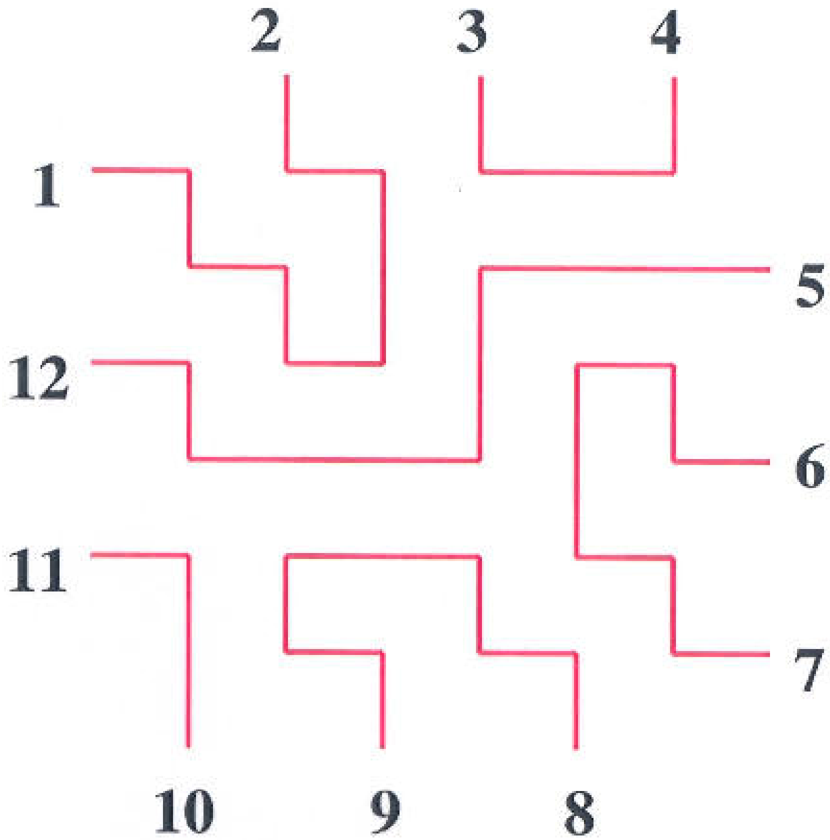
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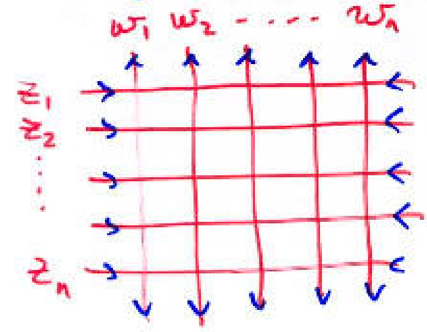
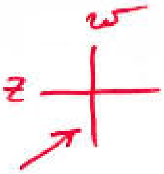
$$
\begin{array}{ccc}
 & \begin{array}{cccccc} & 2 & & 3 & & 4 \end{array} & \\
\begin{array}{c} 1 \\ \\ 12 \\ \\ 11 \\ \end{array} &
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & -1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} &
\begin{array}{c} \\ 5 \\ \\ 6 \\ \\ 7 \end{array} \\
 & \begin{array}{cccccc} 10 & & 9 & & 8 & \end{array} &
\end{array}
$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Izergin-Korepin determinant

• inhomogeneous 6V model w/DWBC



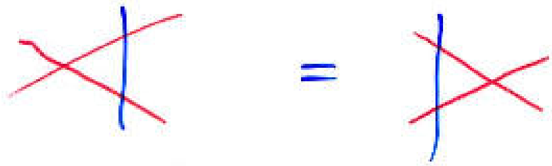


= "R-matrix"

$$\begin{pmatrix} a(z,w) & 0 & 0 & 0 \\ 0 & b(z,w) & c(z,w) & 0 \\ 0 & c(z,w) & b(z,w) & 0 \\ 0 & 0 & 0 & a(z,w) \end{pmatrix}$$

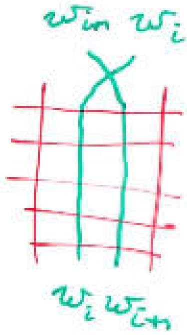
$Z(z_1 \dots z_n; w_1 \dots w_n) = $ p.f.

a, b, c such that R obey Yang-Baxter equation



Then (1) Z symmetric in the z's and w's

(zipper proof)



((2) $a, b, c \to$ parameter q if $q^3 = 1$
then symmetric in all z's and w's together)

(3) recursion relation $w_1 = q^2 z_1 \to$ size 1 less

$$\boxed{Z(z_1, z_2, \dots w_n) = (\dots) \det\left(\frac{c(z_i, w_j)}{a(z_i, w_j)\, b(z_i, w_j)}\right)_{1 \le i,j \le n}}$$

IK determinant

at $q^3 = 1$, $z_i = w_j = 1$ $\quad Z \to 3^{\frac{n(n-1)}{2}} A_n$

used by Kuperberg.

(ASMs)

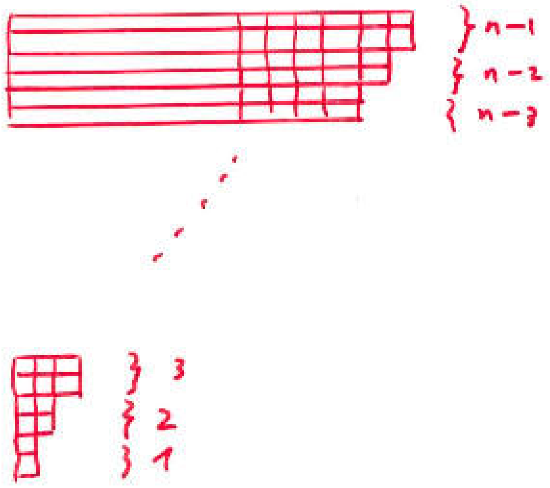
IK DETERMINANT at $q = e^{2i\pi/3}$

OKADA FORMULA ('04)

$$Z_n(z_1 \ldots z_{2n}) = S_{Y_n}(z_1 \ldots z_{2n})$$

schur fctn

where $Y_n =$



New formula

$$Z(z_1 \ldots z_{2n})^2 = \prod_{1 \le i < j \le 2n} \frac{z_i^3 - z_j^3}{(z_i - z_j)^2} \times$$

$$\times \ \mathrm{Pfaffian}\left(\frac{(z_i - z_j)^2}{z_i^3 - z_j^3} \right)_{1 \le i, j \le 2n}$$

Homogeneous limit

$$Z_n(1,1,1 \ldots 1) = 3^{\frac{n(n-1)}{2}} \times A_n = 3^{\frac{n(n-1)}{2}} \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

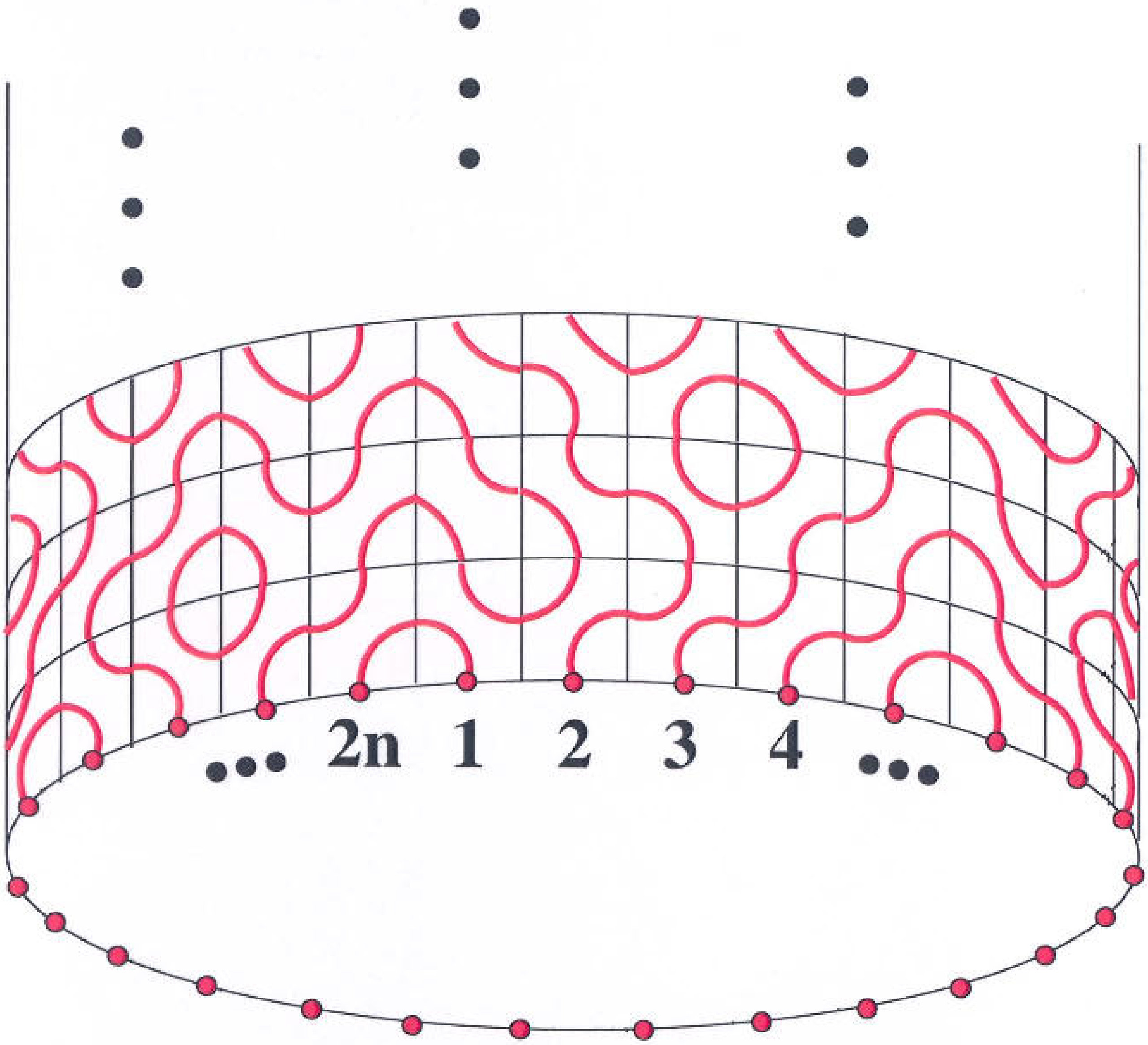
normalization of Ψ_{π_0} # ASMs

O(1) LOOP MODEL ON A SEMI-INFINITE CYLINDER

- semi-infinite cylinder of square lattice, perimeter $2n$
- on each square face draw the configs

w/probas $1-t$ t

- label $1,2,\dots,2n$ the centers of boundary edges



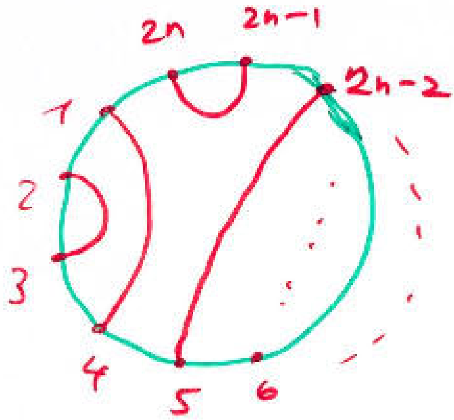
... 2n 1 2 3 4 ...

Ψ_π = Proba for a random config to be connecting the boundary points according to the link pattern π

Link pattern

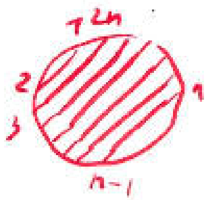
||

planar chord diagram with $2n$ external points on a circle



$\{\pi\} = LP_n$ $\qquad$ $\mathrm{card}(LP_n) = \dfrac{(2n)!}{n!(n+1)!}$ (Catalan)

CONJECTURES: let $\pi_0 =$



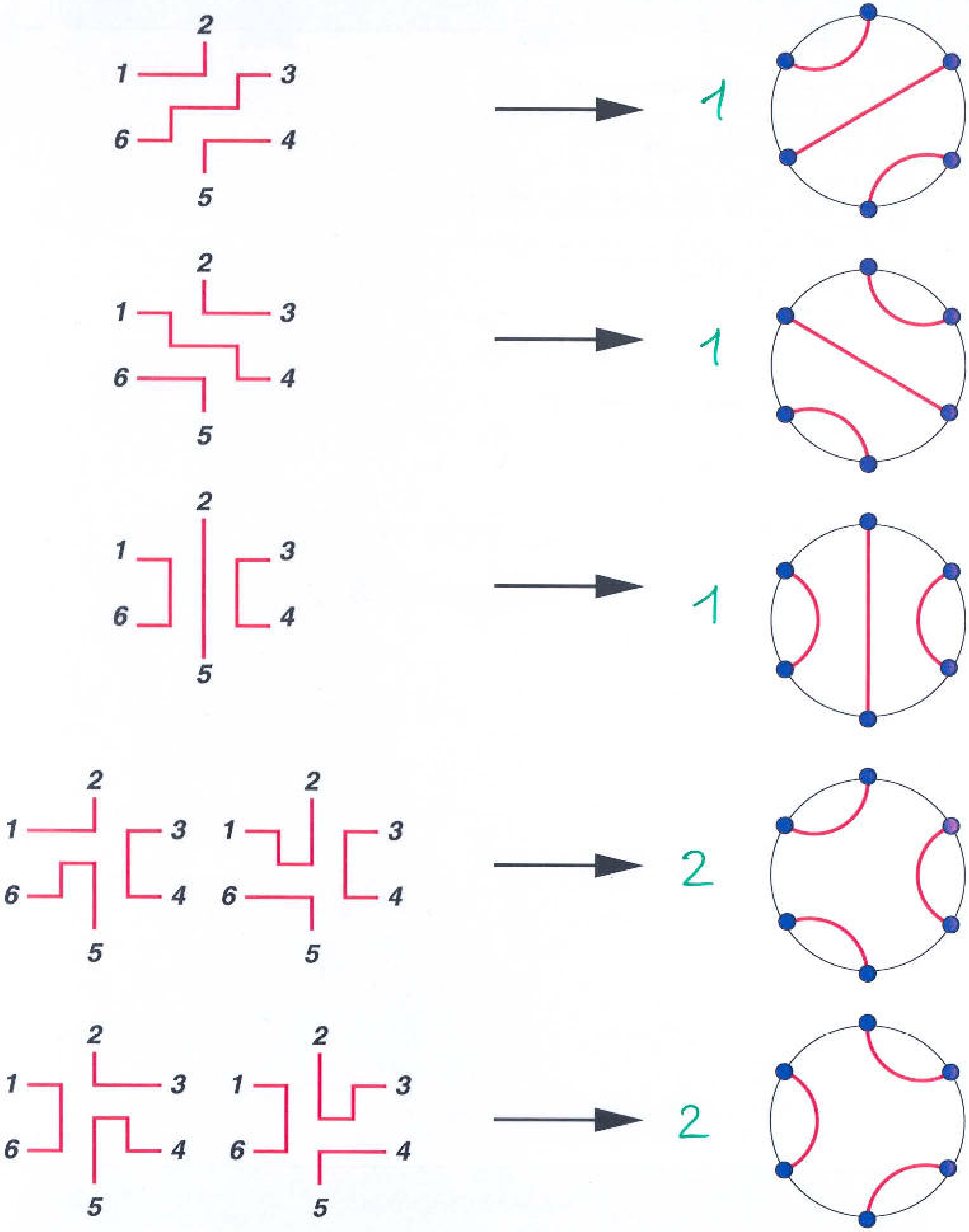
- Normalize Ψ so that $\boxed{\Psi_{\pi_0} = 1}$

C1 (Batchelor-DeGier-Nienhuis)(01) $\boxed{\sum_{\pi \in LP_n} \Psi_\pi = A_n}$
(SUM RULE)

A_n = # Alternating sign matrices $n \times n$.
= # Configs of 6V-DWBC on $n \times n$ grid = 1, 2, 7, 42, 429...

C2 (Razumov-Stroganov)(01)

$\Psi_\pi = A_n(\pi)$ = # Configs of FPL on $n \times n$ grid with connectivities given by π.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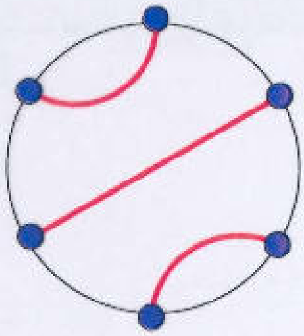
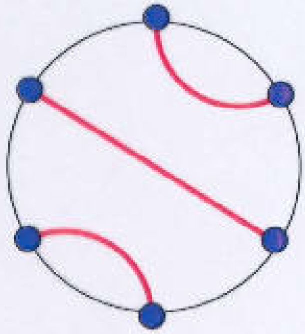
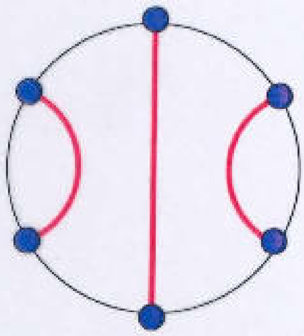
1 

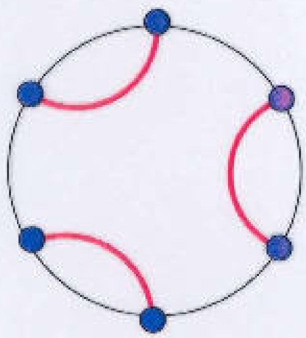
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1 

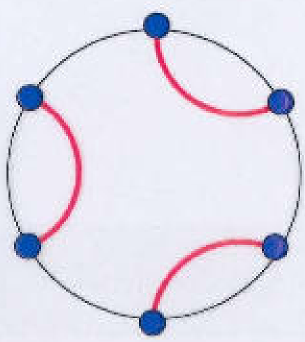
$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

1 

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

2 

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

2 

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

PROOF OF THE RAZUMOV-STROGANOV SUM RULE

$$\boxed{\Psi_{\pi_0} = 1 \quad ; \quad \sum_{\pi} \Psi_{\pi} = A_n}$$

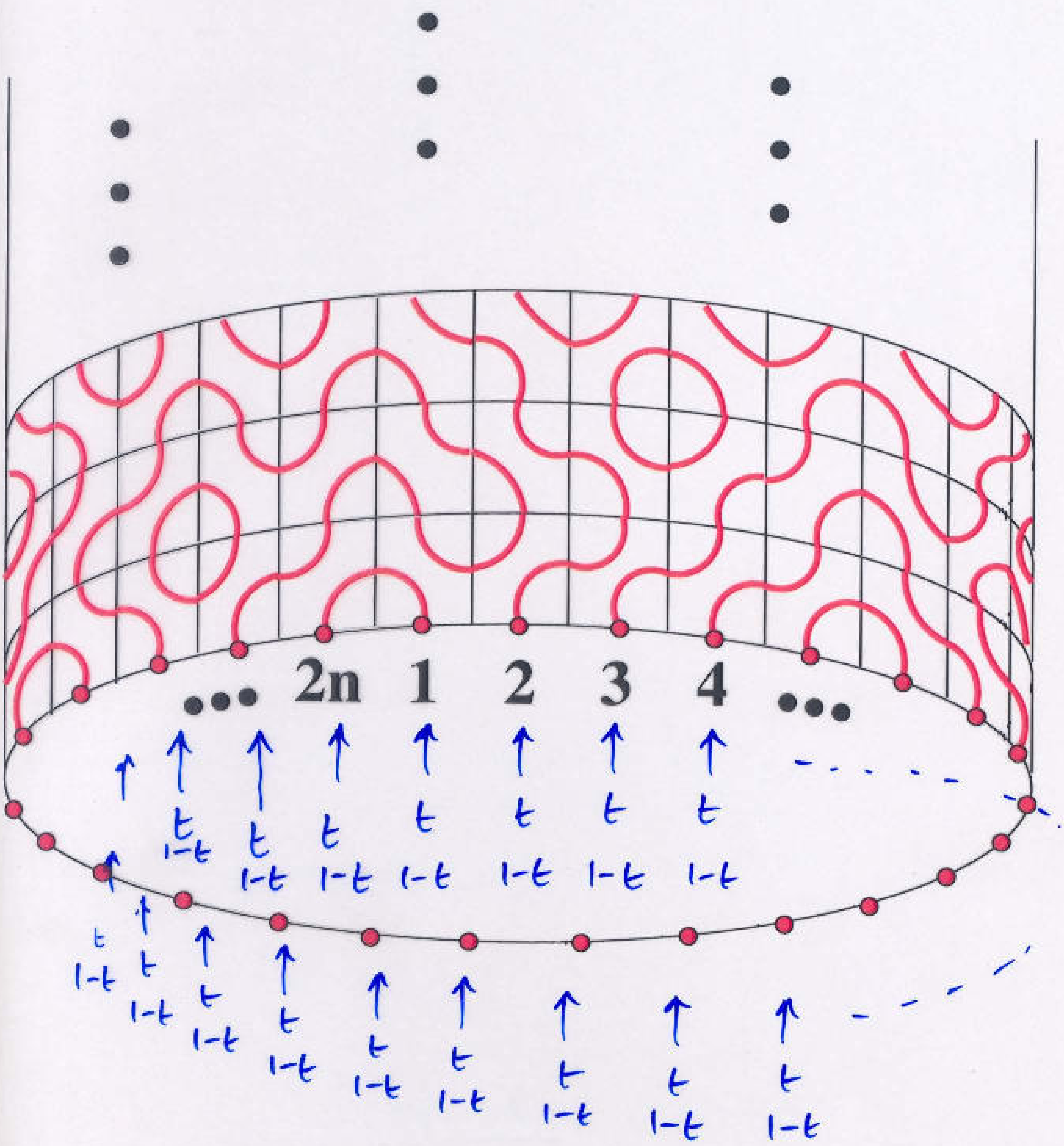
- make the problem inhomogeneous
  ($\Psi_\pi \to$ polynomials of $\{z_1, \ldots, z_{2n}\}$)
  while keeping it integrable i.e. face Boltzman weights satisfy Yang-Baxter eqn.

- prove that

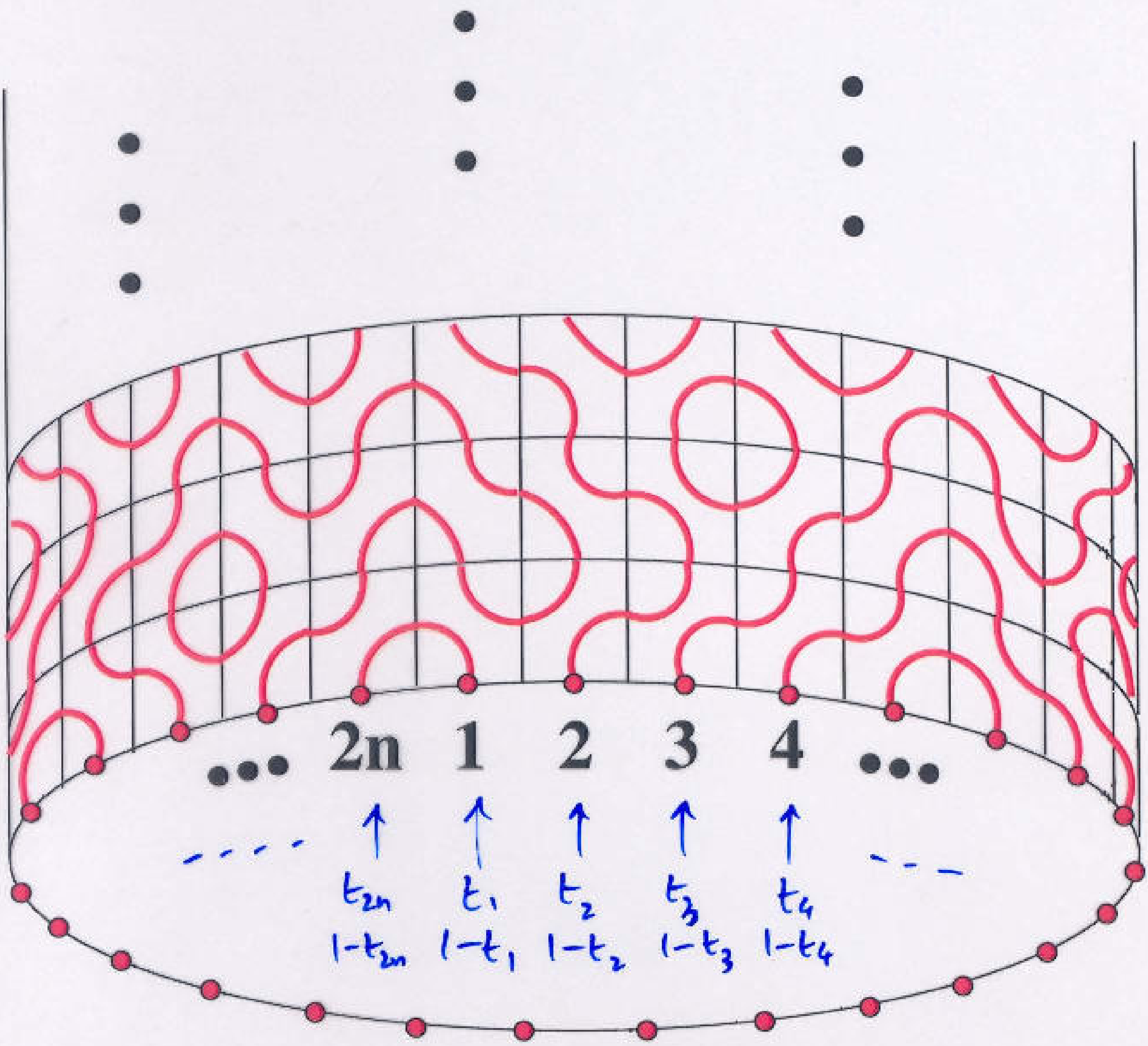
  $\sum_{\pi} \Psi_\pi(z_1, \ldots, z_{2n}) =$ partition function of the 6V model with DWBC and with inhomogeneities $\underbrace{z_1 \ldots z_n}_{\text{rows}}$; $\underbrace{z_{n+1} \cdots z_{2n}}_{\text{columns}}$, and $q^3 = 1$

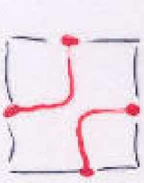
  = « Izergin-Korepin determinant »

HOMOGENEOUS (UNIFORM) PROBABILITIES



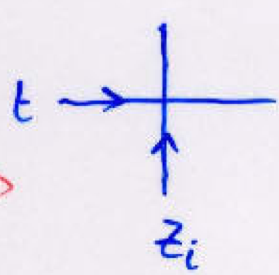
INHOMOGENEOUS PROBABILITIES

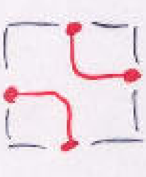




$$t_i = \frac{qt - z_i}{q z_i - t}$$

$\Longleftrightarrow$

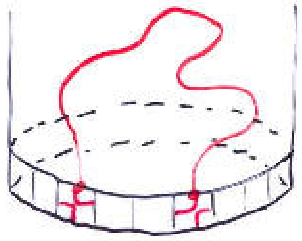


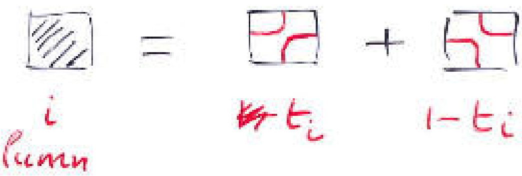


$$1 - t_i = \frac{q^2(t - z_i)}{q z_i - t}$$

• TRANSFER MATRIX play movie

add one raw to the cylinder : acts on link patterns

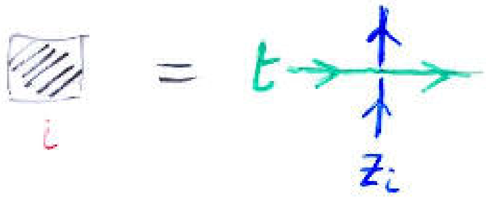




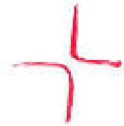
• INTEGRABILITY

if we pick $t_i = \dfrac{q z_i - t}{q t - z_i}$, $q = e^{\frac{2i\pi}{3}}$, then

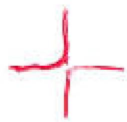
the "R-matix"



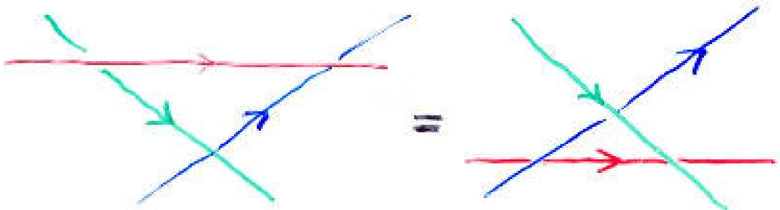
$$= \frac{q z_i - t}{q t - z_i}$$

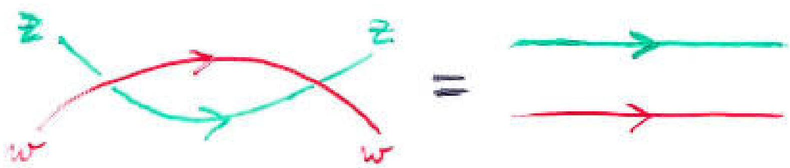


$$+ \frac{q^2 (z_i - t)}{q t - z_i}$$



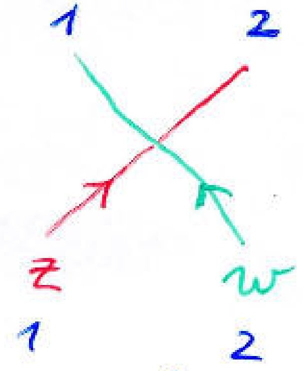
Satisfies the YANG-BAXTER and UNITARITY relations



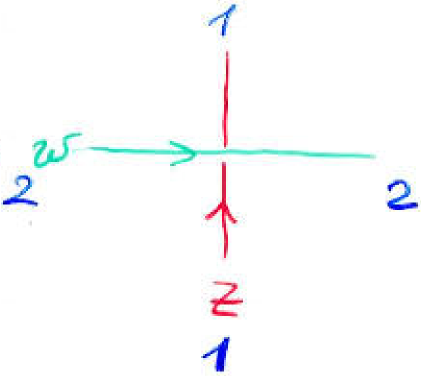


$\check{R}(z,w)\,\check{R}(w,z) = \mathbb{1}$

$\check{R}_{1,2}(z,w) =$

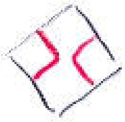


$R_{1,2}(z,w) =$



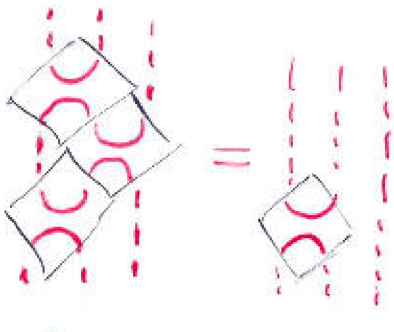
related via space permutation operator P: 12 → 21

NB

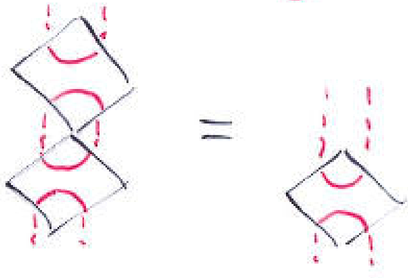


$$\check{R}_{i,i+1}(z,w) = \frac{q z_i - t}{q t - z_i}\, I + \frac{q^2(z_i - t)}{q t - z_i}\, e_i$$

generators of the Temperley-Lieb algebra TL(1) acting on link patterns



$e_i e_{i+1} e_i = e_i$



$e_i^2 = e_i$

$e_i e_j = e_j e_i \quad |i-j|>1$

Defs ① Let $\{\Psi_{n,\pi}(z_1 \cdots z_{2n})\}_{\pi \in LP_n} \equiv \Psi_n$ = vector of probabilities that a random configuration of the $O(1)$ loop model be connected according to $\pi \in LP_n$ (with proba t_i, $1-t_i$ in column i)

② The transfer matrix reads:

$$T(t \,|\, z_1 z_2 \ldots z_{2n}) =$$

$$= \mathrm{Tr}_0\Big(R_{2n,0}(z_{2n},t)\, R_{2n-1,0}(z_{2n-1},t) \cdots R_{1,0}(z_1,t)\Big)$$

Then: probas are invariant under addition of one row $\Longleftrightarrow$

$$\boxed{T(t \,|\, z_1 \cdots z_{2n})\, \Psi_n(z_1 \cdots z_{2n}) = \Psi_n(z_1 \cdots z_{2n})}$$

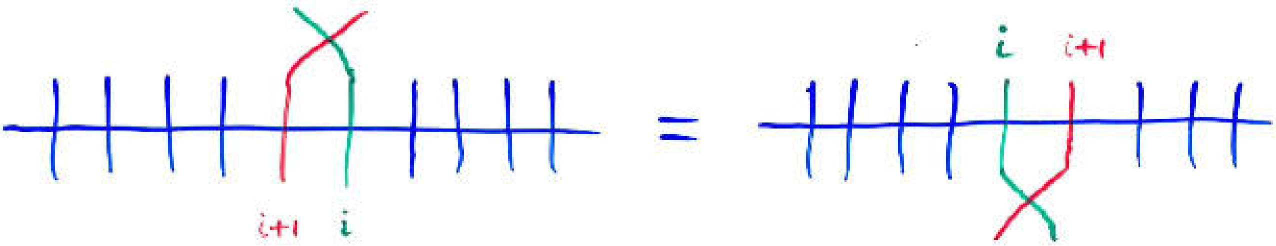
NB: • Ψ is Perron-Frobenius eigenvector when $t_i \in [0,1]$, i.e. $z_j \in U(1) \Rightarrow$ generically non-degenerate
• well-defined up to global normalization

Pick Ψ to be polynomial of the z's, with coprime components (minimal degree).

THE BASIC RELATION

$$\boxed{\begin{aligned}&\Psi(z_1, z_2 \ldots, \overset{\curvearrowleft\curvearrowright}{z_{i+1}, z_i}, z_{i+2} \ldots z_{2n}) \\ &\qquad = \check{R}_{i,i+1}(z_{i+1}, z_i)\, \Psi(z_1, z_2 \ldots z_{2n})\end{aligned}}$$

- a consequence of Yang-Baxter



Allows to compute Ψ entirely in terms of Ψ_{π_0}, itself fixed to be:

$$\Psi_{\pi_0} = \prod_{1 \le i < j \le n} (q z_i - q^{-1} z_j) \prod_{n+1 \le i < j \le 2n} (q z_i - q^{-1} z_j)$$

THE SUM RULE

let $v = (1, 1 \ldots 1)$ then $v \check{R}_{i,i+1} = v$

hence $Z = v \cdot \Psi = \sum_{\pi} \Psi_{\pi}$ symmetric

Allows to compute Z exactly (+ recursion relations, also consequences of YBE)

$$\tau_i \Psi = \check{R}_i \Psi = \left(\frac{q^{-1} z_i - q z_{i+1}}{q^{-1} z_{i+1} - q z_i} I + \frac{(z_i - z_{i+1})}{q^{-1} z_{i+1} - q z_i} e_i \right) \Psi$$

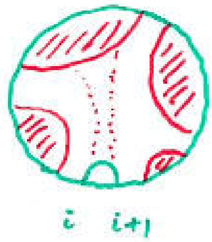
- write it in components
- use the divided difference operator $\partial_i = \frac{1}{z_i - z_{i+1}} (\tau_i - 1)$

$$\partial_i f(z_1, z_2, \ldots, z_i, z_{i+1}, \ldots z_N) = \frac{f(\cdots z_{i+1}, z_i \cdots) - f(\cdots z_i, z_{i+1} \cdots)}{z_i - z_{i+1}}$$

$$(q^{-1} z_{i+1} - q z_i)\, \partial_i \Psi = (e_i + q + q^{-1})\, \Psi$$

$$\boxed{(q^{-1} z_{i+1} - q z_i)\, \partial_i \Psi_\pi = \sum_{\substack{\pi' \neq \pi \\ e_i \pi' = \pi}} \Psi_{\pi'}}$$

antecedents under e_i

$\pi =$ 

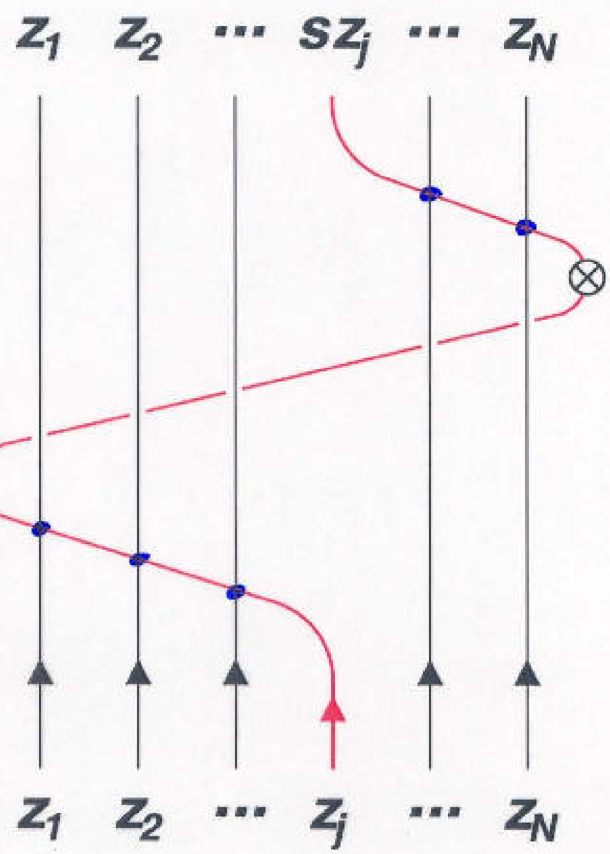
The system is solvable triangularly $\equiv$ express all Ψ_π in terms of Ψ_{π_0}

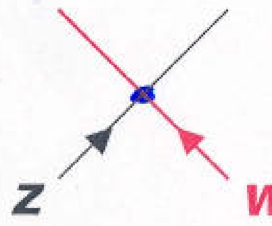
$\pi_0 =$ 

THE QUANTUM KNIZHNIK-ZAMOLODCHIKOV EQUATION ($\mathcal{U}_q(sl(2))$, level 1)

$$\boxed{\Psi(z_1,\dots,z_{j-1}, s z_j, z_{j+1},\dots z_N) = Q_j \, \Psi(z_1,\dots,z_j,\dots,z_N)}$$

(Frenkel-Reshetikhin)





$= \check{R}(z,w)$

$$s = q^{2(k+2)} \qquad k = \text{level}$$

$$k=1 \qquad s = q^6 = 1 \ \text{ at } q^3 = 1.$$

<u>Thm</u> our Ψ solves the qKZ equation of level 1 at $q^3 = 1$.

$\downarrow$

look at q generic?

AN EQUIVALENT SYSTEM:

- fundamental relations

$$\boxed{\tau_i \Psi = \check{R}_i(z_{i+1}, z_i)\,\Psi \qquad i = 1 \dots N-1}$$

- cyclic/shifted BC

$$\boxed{c \cdot \Psi(z_2, z_3, \dots, z_N, s z_1) = \Psi^{\sigma}(z_1 \dots z_N)}$$

rotation of link patterns by 1 unit.

level 1 $\Rightarrow$ $s = q^6$; $N = 2n$

compatibility $\Rightarrow$ $c = q^{3(1-n)}$

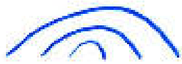
Solve? (Ψ polynomial of the z's, minimal degree)

$$\Psi_{\pi_0} = \prod_{1 \le i < j \le n} (q z_i - q^{-1} z_j) \prod_{n+1 \le i < j \le 2n} (q z_i - q^{-1} z_j)$$

all other Ψ_π's determined by the fundamental relations.

L = 6 qKZ$_1$ solution

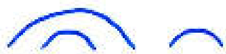
```
In[41]:= a[i_, j_] := q z[i] - q^-1 z[j]
         b[i_, j_] := q^-2 z[i] - q^2 z[j]
```



```
g[1] := a[1, 2] a[1, 3] a[2, 3] a[4, 5] a[4, 6] a[5, 6]
```

```
g[2] := a[1, 2] a[3, 4] a[5, 6]
   (-q^4 z[1] z[2] z[3] - q^4 z[1] z[2] z[4] + q^2 z[1] z[3] z[4] +
    q^2 z[2] z[3] z[4] + q^2 z[1] z[2] z[5] - q^-2 z[3] z[4] z[5] +
    q^2 z[1] z[2] z[6] - q^-2 z[3] z[4] z[6] - q^-2 z[1] z[5] z[6] -
    q^-2 z[2] z[5] z[6] + q^-4 z[3] z[5] z[6] + q^-4 z[4] z[5] z[6])
```

```
g[3] := a[2, 3] a[2, 4] a[3, 4] a[5, 6] b[6, 1] b[5, 1]
```



```
g[4] := a[1, 2] a[3, 4] a[3, 5] a[4, 5] b[6, 1] b[6, 2]
```

```
g[5] := a[2, 3] a[4, 5] b[6, 1]
   (q^5 z[1] z[2] z[3] - q z[2] z[3] z[4] - q z[2] z[3] z[5] -
    q z[1] z[4] z[5] + q^-1 z[2] z[4] z[5] + q^-1 z[3] z[4] z[5] -
    q z[1] z[2] z[6] - q z[1] z[3] z[6] + q^-1 z[2] z[3] z[6] +
    q^-1 z[1] z[4] z[6] + q^-1 z[1] z[5] z[6] - q^-5 z[4] z[5] z[6])
FullSimplify[Table[Factor[g[j]/g[1] /. z[k_] -> 1], {j, 1, 5}] /.
   {{q -> -Exp[I π / 3]}, {q -> -1}}] // TableForm
```

Out[48]//TableForm=

1	2	1	1	2
1	4	4	4	10

COMBINATORIAL POINTS $(n = -(q+q^{-1}))$

$q = -e^{i\pi/3}$ → RAZUMOV-STROGANOV $(n=1)$

$q = -1$ → "rational limit" $(n=2)$

again, the entries of $\Psi(z_1=1, z_2=1, \ldots, z_{2n}=1)$ (homogeneous case) reduce to integers !!!

example size $2n = 6$

$q = -e^{i\pi/3}$ $\Psi = (1, 1, 1, 2, 2)$ $\Sigma = 7$

$q = -1$ $\Psi = (1, 4, 4, 4, 10)$ $\Sigma = 23$

what are these?

Degree of the variety M.M = 0

M upper triangular complex n × n

In[1]:=
```
Clear[p, m, M, f, n]
p[i_, j_] := Block[{val}, val = 0; If[i < j, val = m[i, j]]; val]
M[n_] := Array[p, {n, n}]
n = 6; M[n] // MatrixForm
f = Flatten[Table[Table[(M[n].M[n])[[i, j]], {j, i + 2, n}], {i, :
f // TableForm
```

Out[4]//MatrixForm=

$$\begin{pmatrix} 0 & m[1,2] & m[1,3] & m[1,4] & m[1,5] & m[1,6] \\ 0 & 0 & m[2,3] & m[2,4] & m[2,5] & m[2,6] \\ 0 & 0 & 0 & m[3,4] & m[3,5] & m[3,6] \\ 0 & 0 & 0 & 0 & m[4,5] & m[4,6] \\ 0 & 0 & 0 & 0 & 0 & m[5,6] \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[6]//TableForm=
```
m[1, 2] m[2, 3]
m[1, 2] m[2, 4] + m[1, 3] m[3, 4]
m[1, 2] m[2, 5] + m[1, 3] m[3, 5] + m[1, 4] m[4, 5]
m[1, 2] m[2, 6] + m[1, 3] m[3, 6] + m[1, 4] m[4, 6] + m[1, 5] m[5, 6]
m[2, 3] m[3, 4]
m[2, 3] m[3, 5] + m[2, 4] m[4, 5]
m[2, 3] m[3, 6] + m[2, 4] m[4, 6] + m[2, 5] m[5, 6]
m[3, 4] m[4, 5]
m[3, 4] m[4, 6] + m[3, 5] m[5, 6]
m[4, 5] m[5, 6]
```

In[7]:=
```
Clear[f0, g, g0, i, j, l, L, num]
f0 = Flatten[Table[Table[M[n][[i, j]], {j, i + 1, n}], {i, 1, n - 1
L = Length[f]; num = Floor[n/2] Floor[(n + 1)/2];
For[i = 1, i ≤ num, i++, l[i] = Random[Complex];
 For[j = 1, j ≤ Length[f0], j++, l[i] = l[i] + Random[Complex] * f0
Clear[i, j]
NSolve[Join[f, Table[l[i], {i, 1, num}]] == 0, f0];
Length[%]
```

Out[13]= 23

In[14]:=
```
y = f0 /. %12; l = Length[y]; r = Length[y[[1]]]; mes = {};
For[i = 1, i ≤ l, i++, tes = {};
 For[j = 1, j ≤ r, j++, If[Abs[y[[i, j]]] ≤ .0001, tes = Append[te
 mes = Append[mes, tes]]
a[i_] := Length[Position[mes, Sort[mes][[i]]]]
Print["M2,3=M4,5=0                         " → a[1]]
Print["M1,2=M3,4=M5,6=0                    " → a[11]]
Print["M1,2=M3,4=M3,5=M4,5=0               " → a[16]]
Print["M2,3=M2,4=M3,4=M5,6=0               " → a[22]]
Print["M1,2=M1,3=M2,3=M4,5=M4,6=M5,6=0     " → a[23]]
```

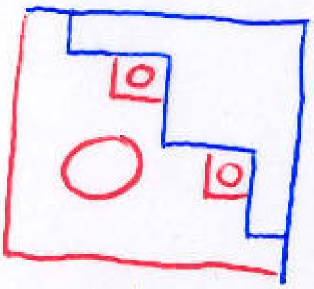
$M_{2,3}=M_{4,5}=0$ → 10

$M_{1,2}=M_{3,4}=M_{5,6}=0$ → 4

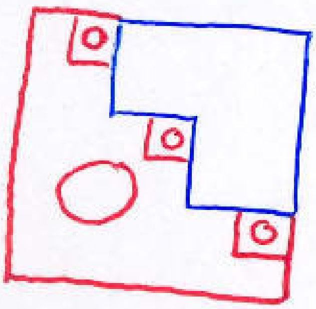
$M_{1,2}=M_{3,4}=M_{3,5}=M_{4,5}=0$ → 4

$M_{2,3}=M_{2,4}=M_{3,4}=M_{5,6}=0$ → 4

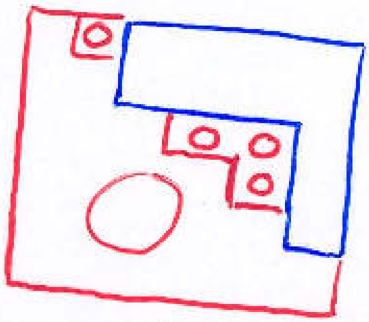
$M_{1,2}=M_{1,3}=M_{2,3}=M_{4,5}=M_{4,6}=M_{5,6}=0$ → 1



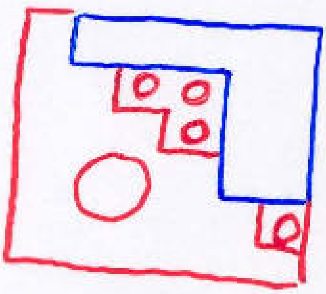
10



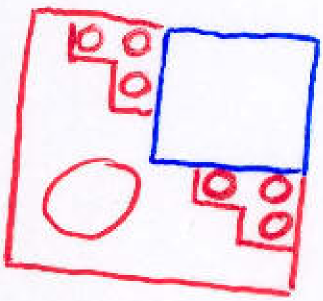
4



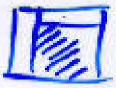
4

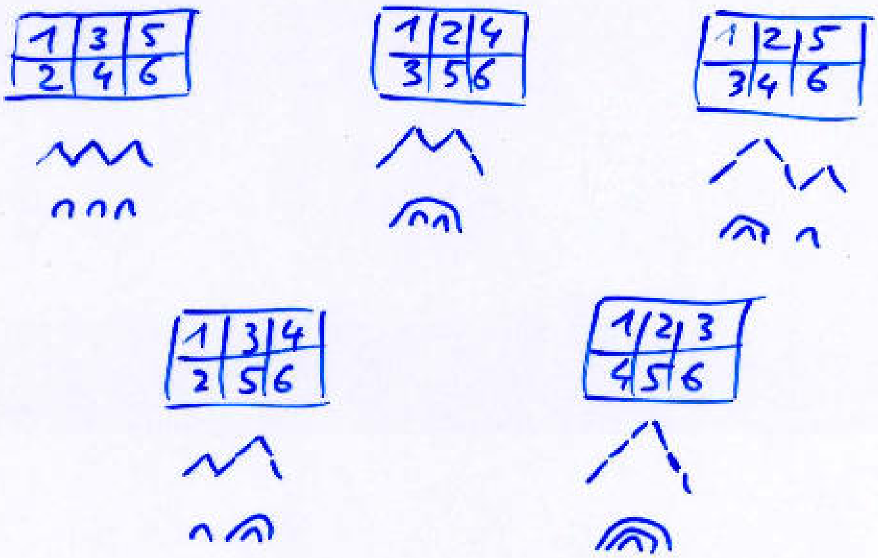


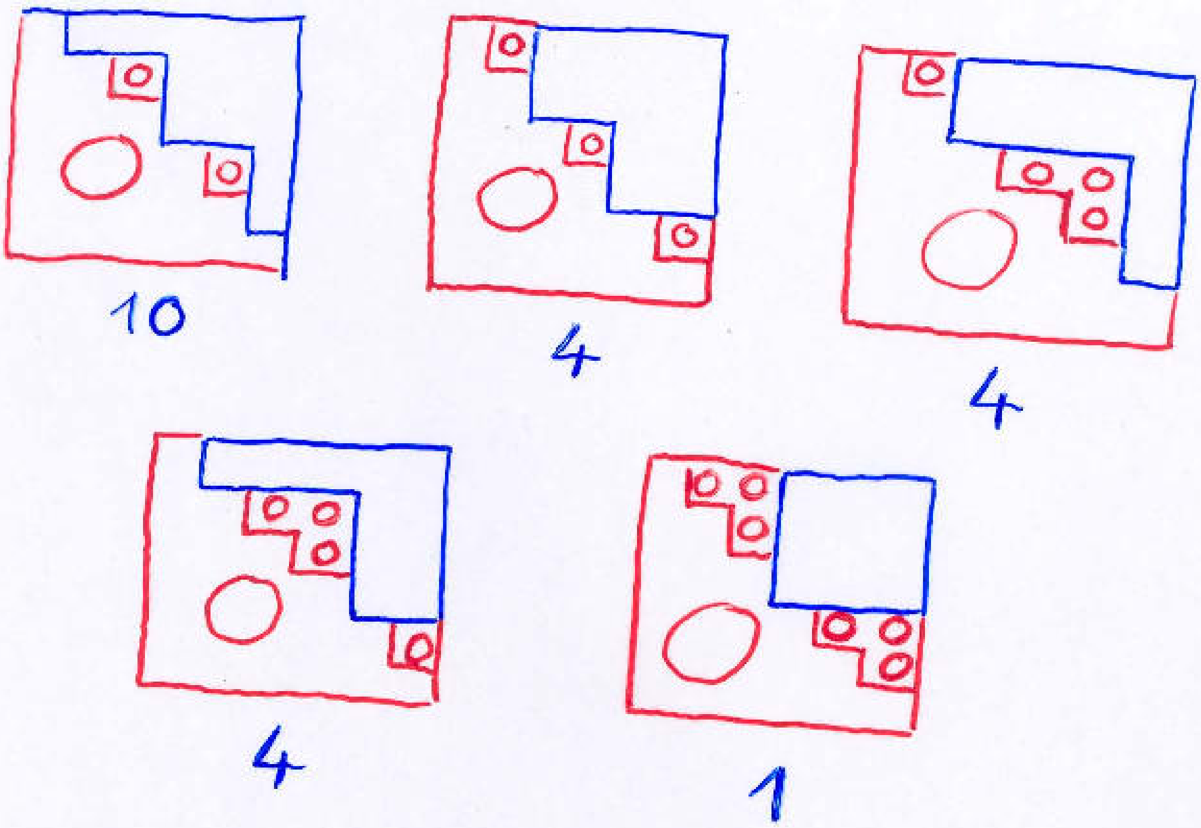
4



1

Succession of Jordan Blocks (1/2) in peelings of M: 





ORBITAL VARIETY

G Lie group $\quad \mathfrak{b} = \underset{\text{Cartan}}{\mathfrak{t}} \oplus \underset{\text{nilpotent}}{\mathfrak{n}}$ a Borel subalgebra of $\mathfrak{g}$

orbital varieties = irred. comp. of $V = \overline{\mathfrak{b} \cap G\cdot x}$ $\quad \begin{cases} x \in \mathfrak{n} \\ G\cdot x = \{g x g^{-1}\} \end{cases}$

Here: $M \in M_n(\mathbb{C})$, $\quad V = \{ \underset{\text{upper triang.}}{M} \mid M^2 = 0 \} = \bigcup_\pi V_\pi$ ← indexed by link patterns ≡ SYT (Jordan)

$V_\pi = \overline{B_\pi c}$

DEGREE / MULTIDEGREE

Formally: T-equivariant cohomology $\quad H^*_T(M_N(\mathbb{C})) \subset \mathbb{C}[a, z_1 \cdots z_N]$
for action of the torus T: $(a, z_1 \cdots z_N)$: $\quad M_{ij} \longmapsto e^{a + z_i - z_j} M_{ij}$

Concretely multidegree axioms

$\text{mdeg}_W X$ = multidegree of T-invariant $X \subset W$

TRIV. (1) if $X = W = \{0\}$ $\text{mdeg}_W X = 1$

ADDIT. (2) if X has top-dim^{al} components X_i, w/multiplicity m_i, then $\text{mdeg}_W(X) = \sum_i m_i \,\text{mdeg}_W X_i$

INT. (3) if X is a variety, H T-invariant hyperplane in W

(a) if $X \not\subset H$, then $\text{mdeg}_W(X) = \text{mdeg}_H(X \cap H)$

(b) if $X \subset H$, then $\text{mdeg}_W(X) = [W/H]_T \times \text{mdeg}_H(X)$

here $W = M_n(\mathbb{C})$ $\quad$ weight wrt torus action here $[M_{ij}] = a + z_i - z_j$ for the hyperplane $\{M_{ij} = 0\}$.

Thm $\quad \boxed{\text{mdeg}_W(V_\pi) = \Psi_\pi(a, z_1, \dots, z_N)}$

↓

rational limit of qKZ sol'n $\quad \begin{cases} q = -e^{-\varepsilon a/2} \\ z_i = e^{-\varepsilon z_i} \end{cases} \quad \varepsilon \to 0$

WHY? Hotta Construction

Pick V_π, intersect it with $H_i = \{ M_{i,i+1} = 0 \}$

2 cases:

(a) $V_\pi \subset H_i$ set $\boxed{s_i \pi = \pi}$

$\rightarrow \times (a + z_i - z_{i+1})$ on multideg

(b) intersection "transverse" $\dim (V_\pi \cap H_i) = \dim V_\pi - 1$

sweep with levi subgroup L_i → upper triangular matrices + element $(i+1, i)$

$\rightarrow \partial_i$ on multideg

$$\{ PMP^{-1}, P \in L_i, M \in V_\pi \cap H_i \} = \bigcup_{\pi' \neq \pi} \underbrace{C_{i\pi}^{\pi'}}_{\text{multiplicities}} V_{\pi'}$$

set $\boxed{s_i \pi = -\pi - \sum_{\pi'} C_{i\pi}^{\pi'} . \pi'}$

Then $\boxed{s_i \Psi_\pi = \left[-\tau_i + a \partial_i \right] \Psi_\pi}$

$$\begin{cases} \partial_i = \dfrac{\tau_i - 1}{z_{i+1} - z_i} & \text{(divided difference)} \\ \tau_i f(z_i, z_{i+1}) = f(z_{i+1}, z_i) \end{cases}$$

write $e_i = +1 - s_i \rightarrow TL(2)$

then $\boxed{\tau_i \Psi = R_i \Psi}$ $R_i \equiv R_i(z_i - z_{i+1})$

$$R_i(u) = \frac{a - u + u e_i}{a + u}$$ qed.

SUM RULES

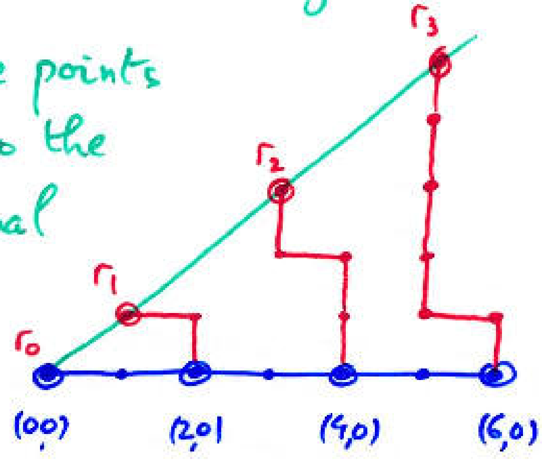
q \ size L	2	4	6	8	10	
$-e^{i\pi/3}$	1	2	7	42	429	← ASM
-1	1	3	23	441	21009	← $\deg\{M^2=0\}$

$q=-1$ The degrees d_{2n} of $\{M^2=0\}$ are the sums of minors of size $n\times n$ in the matrix $A_{i,r}$ $n\times 2n$:

$$A_{i,r} = \binom{2i}{r} \qquad \begin{array}{l} i=0,1,\dots,n-1 \\ r=0,1,\dots,2n-2 \end{array}$$

$$\boxed{d_{2n} = \sum_{\substack{0\le r_0<r_1\dots<r_{n-1} \\ r_i\le 2i}} \det\binom{2i}{r_j}_{0\le i,j\le n-1}}$$

also count the configs of n lattice paths going left/up from the points $(2i,0)$ to the first diagonal (LGV)



also expression via matrix integral

$$d_{2n} \propto \prod_1^n \int_0^{+\infty} dx_i\, e^{-x_i}\, |\Delta(x^2)|$$

↑ Vandermonde determinant ($\beta=1$, orthogonal case)

GENERALIZATIONS

- other boundary conditions

  as many as classical Lie groups (root systems)

→ $q = -e^{i\pi/3}$ other symmetry classes of ASMs / PPs
(Z = character of classical Lie group)

→ $q = -1$ $M^2 = 0$, M with additional symmetries
($\in$ Borel of other classical Lie groups)

- qKZ_1 with other q-groups $\mathcal{U}_q(sl_k)$

→ $q = -e^{i\pi/k+1}$ generalized RS point → numbers that generalize ASMs

→ $q = -1$ $M^k = 0$

- Both ? in progress

- other solutions of YBE : e.g. Brauer crossing loops
  $M \cdot M = 0$, deformed product
  degree of the commuting variety $\{(X,Y) \in M_n(\mathbb{C})^2,\ XY = YX\}$

CONCLUSION

Integrable loop models

$\updownarrow$

qKZ equation

q = root of unity $= -e^{i\pi/k+2}$ $(sl(k))$

Combinatorics : ASM + generalizations

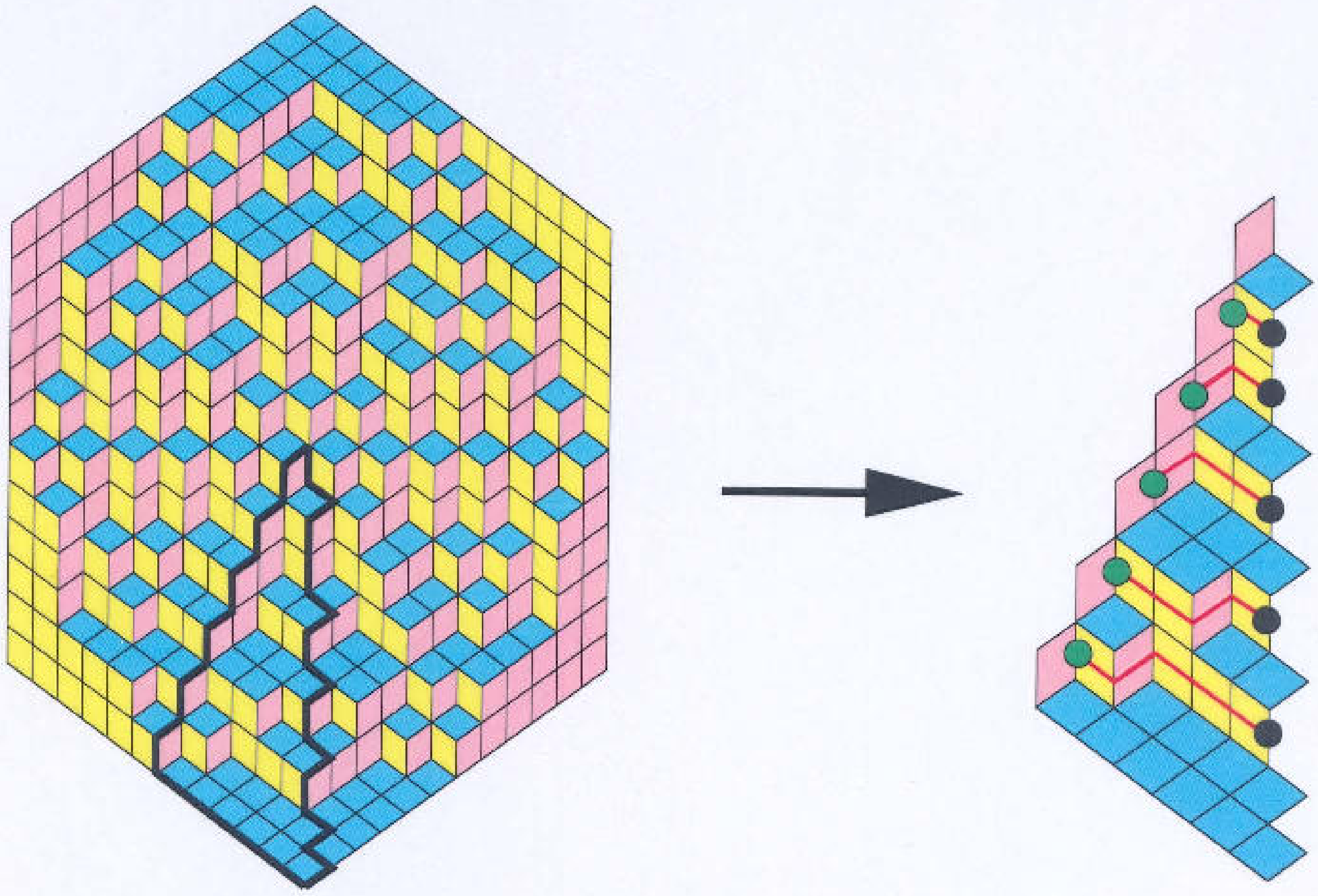
$q = -1$ "rational limit"

Algebraic geometry : multidegrees of nilpotent varieties $(M^k = 0)$

q generic ? K-theory ?

All the integers popping out count families of non-intersecting lattice paths = "fermion lines" (Lindström-Gessel-Viennot formulas)

$q = -e^{i\pi/3}$ The ASM_n numbers also count the $TSSCPP_n$ (Totally symmetric self-complementary Plane Partitions) in a hexagon $2n \times 2n \times 2n$.



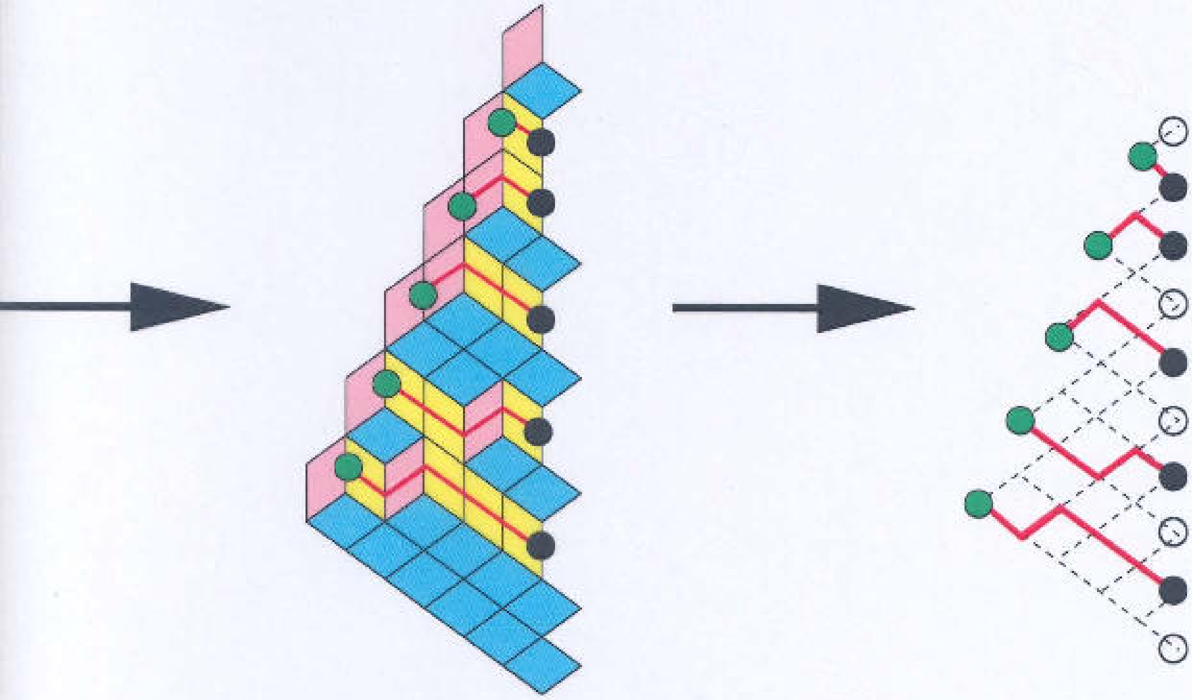
TSSCPP

$$\boxed{A_n = \sum_{r_0 < r_1 \cdots < r_{n-1}} \det \binom{i}{r_j - i}_{0 \le i,j \le n-1}}$$

= sum of minors$_{n \times n}$ of the matrix $n \times 2n$

$$A_{i,r} = \binom{i}{r-i} = \#\text{paths}_{i \to r}$$

$q = -e^{i\pi/3}$ The ASM_n numbers also count the $TSSCPP_n$ (Totally symmetric self-complementary Plane Partitions) in a hexagon $2n \times 2n \times 2n$.



NILP

$$\boxed{A_n = \sum_{r_0 < r_1 \cdots < r_{n-1}} \det \binom{i}{r_j - i}_{0 \le i,j \le n-1}}$$

= sum of minors $_{n\times n}$ of the matrix $n \times 2n$

$A_{i,r} = \binom{i}{r-i} = \#\text{paths}_{i \to r}$