S-Matrix of AdS/CFT

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Gauge Theory Setup / Coordinate Bethe Ansatz

Take a scalar $Z = \phi_5 + i\phi_6$

The state $|0\rangle = |\ldots ZZZ \ldots\rangle$ is ½ BPS and receives no anomalous dimension. $\overset{D-J}{H}|0\rangle = 0$. Ferrom. Vac. ~~State~~ solved.

Flip one spin → one magnon state (momentum eigenstate)

$$|p^A\rangle = \sum_k e^{ipk} |\ldots Z \overset{k}{A} Z \ldots\rangle$$

Is automatically eigenstate of Hamiltonian if A can only be shifted around. Guaranteed if $D_0 - J = 1$ ← lowest possible value (besides 0 for Z)

Possible spin flips (single magnons)

ϕ_j	$j = 1 \ldots 4$	$D_0 = 1$	$J = 0$	$D_0 - J = 1$
$D_\mu Z$	$\mu = 0 \ldots 3$	$D_0 = 2$	$J = 1$	$D_0 - J = 1$
$\psi, \bar{\psi}$	8 fermions	$D_0 = 3/2$	$J = 1/2$	$D_0 - J = 1$

Unsuitable spin flips (multiple magnons) possible "decays"

$\bar{Z}$	$D_0 = 1$	$J = -1$	$D_0 - J = 2$	$Z\bar{Z} \to \phi_j \phi_j$
$F_{\mu\nu}$	$D_0 = 2$	$J = 0$	$D_0 - J = 2$	$ZF_{\mu\nu} \to \psi\psi$
$D_\mu \phi_j$	$D_0 = 2$	$J = 0$	$D_0 - J = 2$	$Z D_\mu \phi_j \to D_\mu Z \; \phi_j$
$D_\mu D_\nu Z$	$D_0 = 3$	$J = 1$	$D_0 - J = 2$	$Z D_\mu D_\nu Z \to D_\mu Z \, D_\nu Z$

etc.

Multiple spin flips, Asymptotic states

$$|p^A q^B r^C\rangle = \sum_{k \ll l \ll m} e^{ipk + iql + irm} |\ldots \overset{k}{A} \ldots \overset{l}{B} \ldots \overset{m}{C} \ldots\rangle$$

- magnons well separated
- ~~local~~ interactions act on one magnon at a time.

<u>Group Theory</u> $PSU(2,2|4) \to \cancel{\mathbb{R}}\,(PSU(2|2) \times PSU(2|2)) \ltimes \mathbb{R}$

$PSU(2|4)$ has 30/32 generators. Matrix form

$$\left(\begin{array}{c|c|c} SU(2) & Q & P \\ \hline S & SU(4) & \bar{Q} \\ \hline K & \bar{S} & \overline{SU(2)} \end{array}\right)$$

<u>Residual algebra</u> that preserves magnon number $D_0 - J$

$$\left(\begin{array}{c|c} PSU(2|2) & \text{magnon creation} \\ \hline \text{magnon annihilation} & PSU(2|2)' \end{array}\right)$$

magnon creation ← 8/8 generators

8/8 gen. → magnon annihilation

<u>Grouping of Magnons</u>

8/8 flavours of magnons

$$\left(\begin{array}{cc|cc} \phi & \phi & \bar{\psi} & \bar{\psi} \\ \phi & \phi & \bar{\psi} & \bar{\psi} \\ \hline \psi & \psi & DZ & DZ \\ \psi & \psi & DZ & DZ \end{array}\right)$$

The two $PSU(2|2)$'s act on rows/colums respectively.

scattering of rows and colums independent $S^{N=4} = S^{PSU(2|2)} \cdot S^{PSU(2|2)'}$

Simplify by considering just one row $(\phi^1, \phi^2 | \psi^1, \psi^2)$

→ closed $SU(2|3)$ sector made from $(\phi^1\, \phi^2\, \phi^3 \,|\, \psi^1\, \psi^2)$.

PSU(2|2) algebra & action

Generators

$R^a{}_b$ SU(2) internal symmetry

$L^\alpha{}_\beta$ SU(2) spacetime symmetry (from conformal sym)

$Q^\alpha{}_b$ 4 Poincaré supercharges

$S^a{}_\beta$ 4 special conformal supercharges

C central charge → Hamiltonian. SU(2|2)

Lie brackets

$$[R^a{}_b, R^c{}_d] = \delta^c_b R^a{}_d - \delta^a_d R^c{}_b \qquad \text{su(2) algebra}$$

$$[R^a{}_b, Q^\gamma{}_d] = -\delta^a_d\, Q^\gamma{}_b + \tfrac{1}{2}\delta^a_b Q^\gamma{}_d \qquad \text{fundamental } Q$$

$$[R^a{}_b, C] = 0 \qquad \text{central}$$

similar for $L^\alpha{}_\beta$, $S^a{}_\beta$ etc.

$$\{Q^\alpha{}_b, S^c{}_\delta\} = \delta^c_b L^\alpha{}_\delta + \delta^\alpha_\delta R^c{}_b + \delta^c_b \delta^\alpha_\delta C$$

$$\{Q^\alpha{}_b, Q^\gamma{}_d\} = 0 \qquad \{S^a{}_\beta, S^c{}_\gamma\} = 0$$

Action ~~fundamental representation~~

$$R^a{}_b |\phi^c\rangle = \delta^c_b |\phi^a\rangle - \tfrac{1}{2}\delta^a_b |\phi^c\rangle \qquad \text{can only be this : SU(2) spin } 1/2$$

$$R^a{}_b |\psi^\gamma\rangle = 0 \qquad \text{spin } 0$$

$L^\alpha{}_\beta$ ... etc.

don't know these due to perturbative effects

$$Q^\alpha{}_b |\phi^c\rangle = a\,\delta^c_b |\psi^\alpha\rangle \qquad \delta_\epsilon \Phi_A = \bar\epsilon \gamma_A \Psi$$

$$S^a{}_\beta |\psi^\gamma\rangle = d\,\delta^\gamma_\beta |\phi^a\rangle \qquad \text{conjugate action}$$

$$Q^\alpha{}_b |\psi^\gamma\rangle = b\,\varepsilon^{\alpha\gamma}\varepsilon_{bd} |Z\phi^d\rangle \qquad \delta_\epsilon \Psi = \gamma^{AB}[\Phi_A, \Phi_B]\epsilon + \gamma^{A\mu} D_\mu \Phi_A \epsilon + \gamma^{\mu\nu} F_{\mu\nu}\epsilon$$

$$S^a{}_\beta |\phi^c\rangle = c\,\varepsilon^{ac}\varepsilon_{\beta\delta} |Z^{-1}\psi^\delta\rangle$$

conjugate action

$$C|\phi^a\rangle = C|\phi^a\rangle$$

$$C|\psi^\alpha\rangle = C|\psi^\alpha\rangle$$

op num Energy with full quantum effects

Fundamental Representation & Central Extension

Is it a representation of PSU(2|2)? okay for $SU(2)\times SU(2)$

$$\{Q^\alpha{}_b, S^c{}_\delta\}|\phi^e\rangle$$

$$= bc\,\varepsilon^{\alpha k}\varepsilon_{bg}\,\varepsilon^{ce}\,\varepsilon_{\delta k}|\phi^g\rangle + ad\,\delta^e_b\,\delta^\alpha_\delta|\phi^c\rangle$$

$$= bc\,\delta^\alpha_\delta\,\delta^c_b|\phi^e\rangle - bc\,\delta^\alpha_\delta\,\delta^e_b|\phi^c\rangle + ad\,\delta^e_b\,\delta^\alpha_\delta|\phi^c\rangle$$

$$= \underbrace{(ad-bc)}_{=1}\,\delta^\alpha_\delta\,R^c{}_b|\phi^e\rangle + \tfrac{1}{2}\underbrace{(ad+bc)}_{C}\,\delta^\alpha_\delta\,\delta^c_b|\phi^e\rangle + \ldots L^\alpha{}_\delta|\phi^e\rangle$$

constraint $ad-bc=1$ $\quad\boxed{C=\tfrac{1}{2}(ad+bc)}$

$$\{Q^\alpha{}_b, Q^\gamma{}_d\}|\phi^e\rangle = ab\,\varepsilon^{\alpha\gamma}\,\varepsilon_{bf}\,\delta^e_d|Z\psi^f\rangle + ab\,\varepsilon^{\alpha\gamma}\,\varepsilon_{df}\,\delta^e_b|Z\psi^f\rangle$$

$$= ab\,\varepsilon^{\alpha\gamma}\,\varepsilon_{bd}|Z\psi^e\rangle$$

$$\{Q^\alpha{}_b, Q^\gamma{}_d\}|\psi^\varepsilon\rangle = ab\,\varepsilon^{\gamma\varepsilon}\,\varepsilon_{df}\,\delta^f_b|Z\phi^\alpha\rangle + ab\,\varepsilon^{\alpha\varepsilon}\,\varepsilon_{bf}\,\delta^f_d|Z\phi^\gamma\rangle$$

$$= ab\,\varepsilon^{\alpha\gamma}\,\varepsilon_{bd}|Z\phi^\varepsilon\rangle$$

constraint $ab=0$

similarly $cd=0$ from $\{S,S\}$

$\Rightarrow C=\pm 1/2$. but not quite right b/c C is Hamiltonian and spectrum is not integer-spaced!

Gauge Transformations

Recall that

$$\cancel{[\delta_{\varepsilon_1},\delta_{\varepsilon_2}]\phi_A = \ldots = \bar\varepsilon\gamma^\mu\varepsilon D_\mu\phi_A + \bar\varepsilon\gamma^B\varepsilon[\phi_B,\phi_A]}$$

$$[\delta_{\varepsilon_1},\delta_{\varepsilon_2}]X = \bar\varepsilon\gamma^\mu\varepsilon\,\underset{\substack{\uparrow\\ \text{mom gen.}\\ \text{not here}}}{D_\mu X} + \bar\varepsilon\gamma^A\varepsilon\,\underset{\substack{\uparrow\\ \text{gauge trans.}}}{[\phi_A, X]}$$

So $\{Q^\alpha{}_b, Q^\gamma{}_d\} = \varepsilon^{\alpha\gamma}\varepsilon_{bd}\,P \leftarrow$ gauge trans $X\to[Z,X]$

$\{S^a{}_\beta, S^c{}_\delta\} = \varepsilon^{ac}\varepsilon_{\beta d}\,K \leftarrow$ quantum gauge trans $X\to[Z^{-1},X]$

$\boxed{P=ab}$ $\boxed{k=cd}$ central extension $PSU(2|2)\ltimes\mathbb{R}^3$

Length Fluctuations & Dispersion

Find that $P|X\rangle = ab\,|ZX\rangle$ parametrise ignorance

Compare with $X \xrightarrow{P} g\alpha[Z, X]$

single particle state $|X\rangle$ represents $\sum_k e^{ipk}|Z\ldots \overset{k}{X}\ldots Z\rangle$

$|ZX\rangle$ represents $\sum_k e^{ipk}|Z\ldots \overset{k}{Z}\overset{k+1}{X}\ldots Z\rangle$

$|XZ\rangle$ represents $\sum_k e^{ipk}|Z\ldots \overset{k}{X}\overset{k+1}{Z}\ldots Z\rangle$

$$= \sum_k e^{ipk+ip}|Z\ldots \overset{k+1}{X}\ldots Z\rangle = e^{ip}|ZX\rangle$$

$$P|X\rangle = g\alpha|ZX\rangle - g\alpha|XZ\rangle = g\alpha(1-e^{ip})|ZX\rangle$$

constraint: $ab = g\alpha(1-e^{ip})$

similarly $cd = g\alpha^{-1}(1-e^{-ip})$

Put together with $C = \frac{1}{2}(ad+bc)$ $\quad C^2 - abcd = \frac{1}{4}(ad-bc)^2 = \frac{1}{4}$

$$C = \sqrt{\tfrac{1}{4}+abcd} = \tfrac{1}{2}\sqrt{[illegible]^2 \sin^2(\tfrac{p}{2})}$$

Dispersion relation $C = C(p)$ from algebra. Exact up to def. of g^2.

Note: two solutions for $C = \pm\frac{1}{2}\sqrt{1+16g^2\sin^2\frac{p}{2}}$

Particle & Antiparticle

Fundamental & Anti-fundamental representation

Smoothly connected in complex plane

Length Fluctuations & Coproduct

Know that $P|X\rangle = \overset{\text{num}}{P_x}|ZX\rangle$

furthermore $P|XY\rangle = P_x|ZXY\rangle + P_Y|XZY\rangle$

$$= (P_x + e^{ip_x}P_Y)|ZXY\rangle$$

How to formulate this algebraically? coproduct — Gomez Hernandez, Plefka Spill Torrielli

$$\Delta(P) = P\otimes 1 + U^2\otimes P$$
$$\Delta(U) = U\otimes U$$
$$U|X\rangle = e^{ip/2}|X\rangle$$

Can we also use this for Q?

$$Q|\phi\rangle \sim |\psi\rangle \qquad Q|\psi\rangle \sim |Z\phi\rangle$$

different powers of Z

Redefine $|\psi\rangle \to |Z^{1/2}\psi\rangle$ leads to

$$Q|\phi\rangle \sim |Z^{1/2}\psi\rangle \qquad Q|Z^{1/2}\psi\rangle \sim |Z\phi\rangle$$

or $Q|\psi\rangle \sim |Z^{1/2}\phi\rangle$

Now Q has same ~~charge~~ power of $Z^{1/2}$ for all states

$$\Delta(Q) = Q\otimes 1 + U\otimes Q$$

similarly $\Delta(S) = S\otimes 1 + U^{-1}\otimes S$

other coproducts undeformed.

compatible with the algebra

$$\Delta(X) = X\otimes 1 + U^{[X]}\otimes X$$

where $[X]$ is grading of X

$[C]=[R]=[L]=0 \quad [P]=+2 \quad [Q]=+1 \quad [S]=-1 \quad [K]=-2$

This coproduct respects the algebra $\Delta[X,Y]=[\Delta X,\Delta Y]$.

Furthermore $P=K=0 \Leftrightarrow \prod_k e^{ip_k}=1 \Leftrightarrow \sum_k p_k = 2\pi m$

Coproduct & light cone gauge for strings

Frolov Plefka Zamaklar

Coproduct deformed for PSU(2|2) algebra, eg:

$$\Delta^3(Q) = Q\otimes 1\otimes 1\otimes 1 + U\otimes Q\otimes 1\otimes 1 + U\otimes U\otimes Q\otimes 1 + U\otimes U\otimes U\otimes Q$$

Action of generators becomes non-local on ~~chain~~ spin chain.

In gauge theory: Length fluctuations

In string theory: Light-cone gauge

In light cone gauge use $X_\pm = \overset{\text{AdS}_5 \text{ time}}{t} \pm \overset{S^5 \text{ great circle}}{\varphi}$.

Use Virasoro constraints to solve for $X_-' = \frac{\partial}{\partial\sigma} X_-$.

Then $x^-(\sigma)$ is non-local: $x^-(\sigma) = \int_0^\sigma x_-'(\sigma')\, d\sigma'$.

Now supercharges Q, S contain x^- and are thus non-local.

Moreover the non-locality is towards the left of Q, S and homogeneous.

$\Rightarrow$ Construction applies to strings on $AdS_5 \times S^5$ as well.

The S-Matrix of AdS/CFT.

Scattering matrix for two magnons $|\phi^a\rangle, |\psi^\alpha\rangle$.
Statistics: $S: |\phi\phi\rangle \leftrightarrow |\psi\psi\rangle$ or $|\phi\psi\rangle \leftrightarrow |\psi\phi\rangle$
First: bosonic symmetries $SO(2)\times SO(2)$. Coproduct undeformed:

$$S|\phi^a\phi^b\rangle = A|\phi^{(a}\phi^{b)}\rangle + B|\phi^{[a}\phi^{b]}\rangle + C\varepsilon^{ab}\varepsilon_{\gamma\delta}|\psi^\gamma\psi^\delta\rangle$$
$$S|\psi^\alpha\psi^\beta\rangle = D|\psi^{(\alpha}\psi^{\beta)}\rangle + E|\psi^{[\alpha}\psi^{\beta]}\rangle + F\varepsilon^{\alpha\beta}\varepsilon_{cd}|\phi^c\phi^d\rangle$$
$$S|\phi^a\psi^\beta\rangle = G|\psi^\beta\phi^a\rangle + H|\phi^a\psi^\beta\rangle$$
$$S|\psi^\alpha\phi^b\rangle = K|\psi^\alpha\phi^b\rangle + L|\phi^b\psi^\alpha\rangle$$

Second: fermion symmetries Q, S. Coproduct deformed:
consider state $|\phi^1\phi^1\rangle$, act with ΔQ^1_1

$$S|\phi^1\phi^1\rangle = A|\phi^1\phi^1\rangle$$
$$\Delta(Q^1_1)|\phi^1\phi^1\rangle = a_1|\psi^1\phi^1\rangle + e^{ip_1/2}a_2|\phi^1\psi^1\rangle$$
$$S|\psi^1\phi^1\rangle = K|\psi^1\phi^1\rangle + L|\phi^1\psi^1\rangle$$
$$S|\phi^1\psi^1\rangle = G|\psi^1\phi^1\rangle + H|\phi^1\psi^1\rangle$$

$$\begin{aligned}0 = [\Delta(Q^1_1), S]|\phi^1\phi^1\rangle = {} & a_2 A|\psi^1\phi^1\rangle + e^{ip_2/2}a_1 A|\phi^1\psi^1\rangle \\ & - a_1 K|\psi^1\phi^1\rangle - a_1 L|\phi^1\psi^1\rangle \\ & - e^{ip_1/2}a_2 G|\psi^1\phi^1\rangle - e^{ip_1/2}a_2 H|\phi^1\psi^1\rangle\end{aligned}$$

$$\Rightarrow \quad a_2 A = a_1 K + e^{ip_1/2}a_2 G$$
$$\Rightarrow \quad e^{ip_2/2}a_1 A = a_1 L + e^{ip_1/2}a_2 H$$

~~All ... at~~ Demanding $[\Delta(Q,S), S] = 0$ for all states relates all $A, \ldots, L$ to each other. Only one undet. function remains: ~~eg D~~ R^0_{12}