I Introduction ~~to~~ to AdS/CFT correspondence

Anti-de-Sitter space-time AdS_d is a

- $(d-1,1)$-dimensional space-time
- with constant negative curvature
- with a boundary $\mathbb{R} \times S_{d-2} \simeq \mathbb{R}^{d-2,1}$ (conformal)
- which has $SO(d-1,2)$ group of isometries
- is the space-time analog of hyperbolic space $\mathbb{H}^d$

The AdS/CFT correspondence ~~relates~~ claims the (exact) equivalence of

- string theories on AdS_d backgrounds ~~to~~ and
- conformal quantum field theories on $\partial AdS_d \simeq \mathbb{R}^{d-2,1}$.

key example of AdS/CFT

- Superstring theory (IIB) on $AdS_5 \times S^5$
- Maximally ($N=4$) extended supersymmetric gauge theory in $D=4$

Benefits / shortcomings of these models

+ Well tractable due to nice features (susy, surprises ...)
− unrealistic model (too much susy, conformal symmetry)

Outline of this lecture

- $N=4$ super Yang-Mills theory with $U(N)$ gauge group
- IIB string theory on $AdS_5 \times S^5$
- AdS/CFT holographic duality

$\mathcal{N}=4$ Super Yang-Mills theory

- 4D quantum field theory
- much like standard model of particle physics (technically)

Field content

- gauge field / connection A_μ ← 4D vector index
- 4 flavours of fermions Ψ^{a}_{α} — a ← 4D flavour index, α ← 4D spinor index
- 6 flavours of scalars Φ^m ← 6D flavour index
- all fields in adjoint of $U(N)$: $N\times N$ (hermitian) matrices

Gauge symmetry

Local $U(N)$ symmetry $U(x)$ $\qquad \frac{\partial}{\partial x_\mu} U \neq 0$

Covariant transformation $X \to U X U^{-1}$ $\qquad X = \Psi, \Phi$

derivatives $\partial_\mu X \to U\left(\partial_\mu X + [U^{-1}\partial_\mu U, X]\right) U^{-1}$

~~gauge connection~~ covariant derivative $D_\mu X \to U D_\mu X U^{-1}$ st. $D_\mu X = \partial_\mu X + i\overset{\text{coupling const}}{g}[A_\mu, X]$

gauge connection $A_\mu \to U\left(A_\mu + \frac{i}{g} U^{-1}\partial_\mu U\right)$ to compensate

covariant fieldstrength $F_{\mu\nu} = -\frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$

Action (up to numerical factors)

$$S = N \int \frac{d^4x}{(2\pi)^2} \operatorname{tr}\Big(F_{\mu\nu}F^{\mu\nu} + D_\mu \Phi D^\mu \Phi + \bar{\Psi}\gamma^\mu D_\mu \Psi$$
$$+ g\Psi\gamma^m \Phi_m \Psi + g\bar{\Psi}\gamma^m \Phi_m \bar{\Psi} + g^2 [\Phi^m, \Phi^n][\Phi_m, \Phi_n]\Big)$$

Symmetries of N=4 SYM

Gauge Symmetry see above

Flavour Symmetry: $SU(4) \sim SO(6)$ global — R flavour rotation

- fermions Ψ^a_α ← fundamental 4-dim of $SU(4)$, spinor 4-dim of $SO(6)$
- scalars Φ^m ← antisymmetric bifundamental 6-dim of $SU(4)$, vector of $SO(6)$
- gluons A_μ invariant / singlet

Lorentz $SO(3,1) \sim SL(2,\mathbb{C})$ — $L, \dot{L}$ Lorentz rotation

- rotates spacetime x_μ & spinor/gluon indices

Poincaré $SO(3,1) \ltimes \mathbb{R}^{3,1}$ — P momentum

- coordinate shifts

Supersymmetry (Poincaré susy) $SO(3,1) \ltimes \mathbb{R}^{2N=4} \ltimes \mathbb{R}^{3,1}$ — Q supercharge

- gluons $\delta A_\mu \sim \varepsilon \gamma_\mu \Psi$ (numerical factors)
- scalars $\delta \Phi^m \sim \varepsilon \gamma^m \Psi$
- fermions $\delta \Psi \sim \gamma^{\mu\nu} \varepsilon F_{\mu\nu} + \gamma^\mu \gamma^n \varepsilon D_\mu \Phi_n + \gamma^{mn} \varepsilon g [\Phi_m, \Phi_n]$

SUSY algebra $[Q, Q] \sim P$ + gauge transformation

Conformal scaling — D Dilatation generator

- $D = 3/2$: Ψ
- $D = 1$: P, Φ_m, D_μ, A_μ
- $D = 1/2$: Q
- $D = 0$: $D, R, L, \dot{L}$
- $D = -1/2$: S ← special conformal supercharge
- $D = -1$: K ← special conformal boost

Conformal / superconformal symmetry $PSU(2,2|4)$

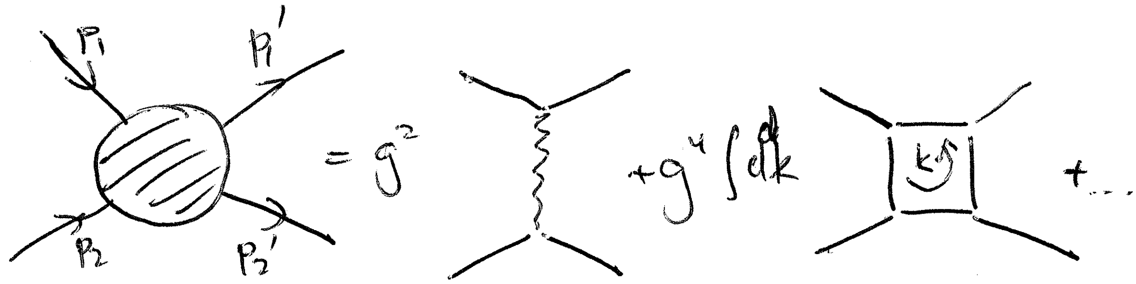
- Conformal symmetry $SO(4,2) \sim SU(2,2)$ exact in QFT! $\beta_g = 0$
- Conformal symmetry joins with susy: superconformal $PSU(2,2|4)$

Scattering Amplitudes

Typically in 4D QFT's one computes scattering amplitudes
$\rightsquigarrow$ collider experiments
Go to momentum space. Particle propagators

$$\Delta_m(p) = \frac{1}{-p^2+m^2}$$

Feynman rules / diagrams, perturbation series

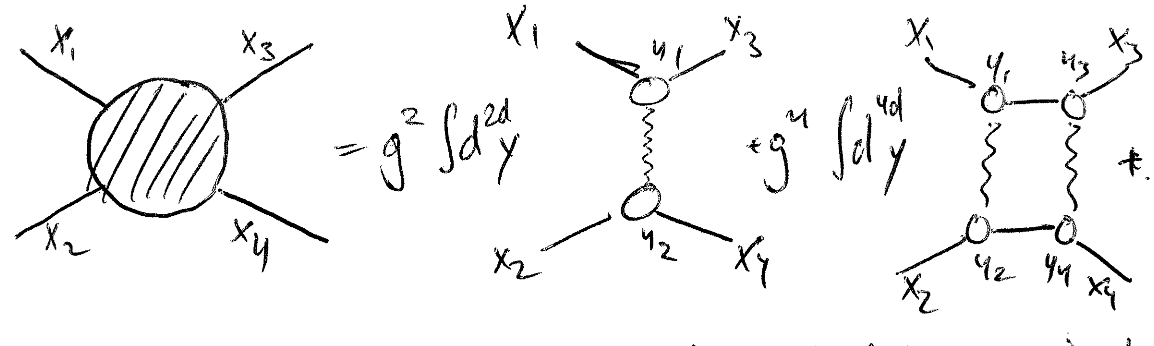


Hard calculations, but well-known rules & elaborate technology

Here: Position space different. Propagators

$$\Delta_m(x,y) = \frac{K_{d/2-1}(m|x-y|)}{2\pi\,(2\pi|x-y|/m)^{d/2-1}} \xrightarrow{\text{massless}} \Delta(x,y) = \frac{\Gamma(d/2-1)}{4\pi^{d/2}\,|x-y|^{d-2}}$$

Feynman rules. n-point correlation functions



Only practical in massless case, more integrations to be done, similar.

Local operators & Correlators

In gauge theory only gauge-invariant quantities are observable:
- local gauge-invariant combinations of the fields (local operators)
- Wilson loops (with field insertions): Holonomy of gauge field
- on-shell fields

Non gauge-invariant quantities average out over gauge orbits in PI.

Local Operators

Recall: All fields $X \in \{\Phi, \Psi, F_{\mu\nu}, D_\mu \dots\}$ are $N\times N$ matrices
and transform like $X \to U X U^{-1}$

Take trace of a product, eg. $\mathcal{O}_1 = \mathrm{Tr}\, \phi_m \phi_n$, $\mathcal{O}_2 = \mathrm{Tr}\, F_{\mu\nu} D_\nu \Psi D_\rho \Phi$
Also linear combination and multiple traces are permitted

$$\mathcal{O} = 16\, \mathcal{O}_1 + 3\, \mathcal{O}_2 \cdot \mathcal{O}_3 + \dots$$

Correlators in CFT

- one-point functions $\langle \mathcal{O}_\alpha(x) \rangle = 0$ typically in CFT
- two-point functions $\langle \mathcal{O}_\alpha(x)\, \mathcal{O}_\beta(y) \rangle = \dfrac{\delta_{\alpha\beta} \cdot C_\alpha}{|x-y|^{2D_\alpha}}$ in CFT

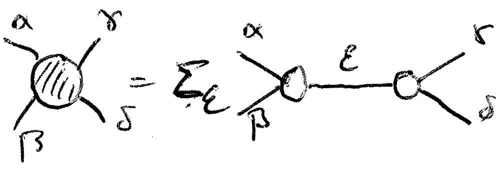
  C_α (unphysical) normalisation; D_α scaling dimension of $\mathcal{O}_\alpha$
  spectrum of CFT
- three-point function $\langle \mathcal{O}_\alpha(x)\, \mathcal{O}_\beta(y)\, \mathcal{O}_\gamma(z) \rangle$

$$= \frac{C_{\alpha\beta\gamma}}{|x-y|^{D_\alpha+D_\beta-D_\gamma}\, |y-z|^{D_\beta+D_\gamma-D_\alpha}\, |z-x|^{D_\gamma+D_\alpha-D_\beta}}$$

$C_{\alpha\beta\gamma} \leftarrow$ structure constants of CFT

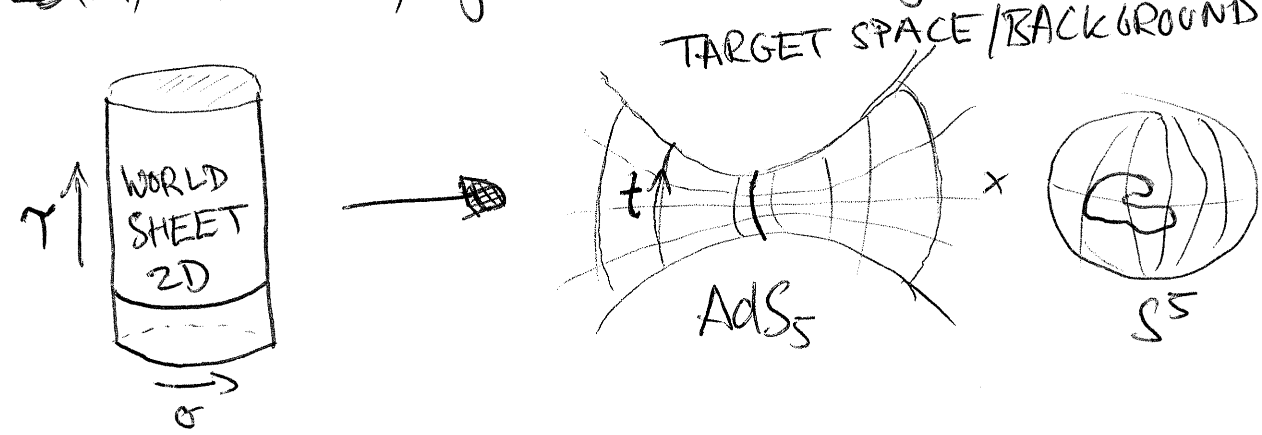
$\leftarrow$ determined by conf. sym.

- four-point functions, n-point functions
- structure can now depend on conformal invariants / cross ratios $\frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$ etc.
- can be inferred from OPE



IIB superstrings on $AdS_5\times S^5$ background

String theory:
(1+1) D (non-linear) sigma model coupled to gravity



Fields

$X_m(\sigma,\tau)$: Embedding coordinates of the string

$\gamma_{ab}(\sigma,\tau)$: World-sheet metric

$\psi(\sigma,\tau)$: Fermionic embedding coordinates (target superspace)
Worldsheet scalars

Action
up to prefactors

target space metric

$$S = \sqrt{\lambda}\int \frac{d\sigma d\tau}{2\pi}\sqrt{\gamma}\left(\gamma^{ab}\partial_a X_m \partial_b X_n g^{mn} + \dots + R g_s\right)$$

~ "Area" of worldsheet embedding

Equations of motion

- "$\Box X_m = \dots$" wave equation, periodicity → string modes
- "$\Box \psi = \dots$" similar
- $(\partial_\pm X)^2 = 0$ (from metric) Virasoro constraints.
  → WS metric is induced metric from background

Symmetries

- Worldsheet diffeomorphisms
- kappa symmetry: local supersymmetry on world sheet

  } needs to be gauge fixed.
- Isometries of target space (global sym)

Quantum Strings and Backgrounds

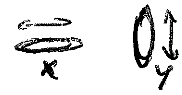
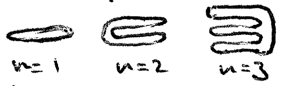
Action

$$S = \sqrt{\lambda} \int \frac{d^2\sigma}{2\pi} \left(\tfrac{1}{2}\sqrt{\gamma}\,\gamma^{ab}\partial_a X^m \partial_b X^n \underset{\uparrow}{g_{mn}(X)} + \tfrac{1}{2}\varepsilon^{ab}\partial_a X^m \partial_b X^n \underset{\uparrow}{B_{mn}(X)} + g_s R \ldots \right)$$

bg metric ← string couples to bg geometry → bg 2-form potential

Quantisation

Vacuum: • string shrunk to a point $\frac{\partial}{\partial\sigma}X = 0$ • like a point particle in bg.
- bosonic strings: tachyonic particle, instability
- superstrings: absent

Excitations: Fluctuations of vacuum configuration
- one per transverse direction  etc.
- one per fourier mode of worldsheet  etc
- combine to get infinite collection of spinning particles.

Lowest excited modes: ~~[illegible]~~
- massless
- closed strings: δg_{mn}, δB_{mn}, etc. (fluctuations of bg geometry)
- open strings: δA_m (gauge fields)

higher excited modes: • massive • stringy (non-geom) modes

Quantum Consistency

Anomalies present. Cancel if g_{mn} satisfies Einstein Eq, B_{mn} satisfies 2-form field eq. } (super) gravity background

Second Quantisation

- infinite collection of particles → infinite collection of ~~the~~ quantum fields
- contains gravitons → Theory of quantum gravity

The $AdS_5 \times S^5$ coset model

The sphere S^{N-1} is the coset $\frac{SO(N)}{SO(N-1)}$. $S^5 = \frac{SO(6)}{SO(5)} = \frac{SU(4)}{Sp(2)}$

Anti de Sitter AdS_{N-1} is the coset $\frac{\widetilde{SO(N-2,2)}}{SO(N-2,1)}$. $AdS_5 = \frac{\widetilde{SO(4,2)}}{SO(4,1)} = \frac{\widetilde{SU(2,2)}}{Sp(1,1)}$

$AdS_5 \times S^5$ superspace is the coset $\frac{\widetilde{PSU(2,2|4)}}{Sp(1,1) \times Sp(2)}$.

String theory on this coset has the fields

- $g(\sigma,\tau)$ element of $\widetilde{SU(2,2|4)}$.
- $\gamma_{ab}(\sigma,\tau)$ world sheet metric. Defines Hodge dual $*dx^a = \frac{\gamma^{ab}}{\gamma} \epsilon_{bc} dx^c$.
- $\Lambda(\sigma,\tau)$ Lagrange multiplier for $su(2,2|4)$

Local symmetries

- diffeomorphisms, kappa symmetry
- $g(\sigma,\tau) \to \xi(\sigma,\tau)\, g(\sigma,\tau)$, $\xi(\sigma,\tau)$ factor, reduces to $\widetilde{PSU(2,2|4)}$
- $g(\sigma,\tau) \to g(\sigma,\tau) h(\sigma,\tau)$, $h \in Sp(1,1) \times Sp(2)$, reduces to coset

Algebra of currents

Current $J = -g^{-1} dg \in su(2,2|4)$, $dJ = J \wedge J$

Z_4 grading (supersymmetric coset)

- 0 elements of $sp(1,1) \times sp(2)$
- 2 remaining bosonic elements of $psu(2,2|4)$
- 1,3 16+16 fermionic elements of $psu(2,2|4)$

$$\left.\right\} \; J = \underset{0}{H} + \underset{1}{Q_1} + \underset{2}{P} + \underset{3}{Q_2}$$

Action

reduces to $su(2,2|4)$

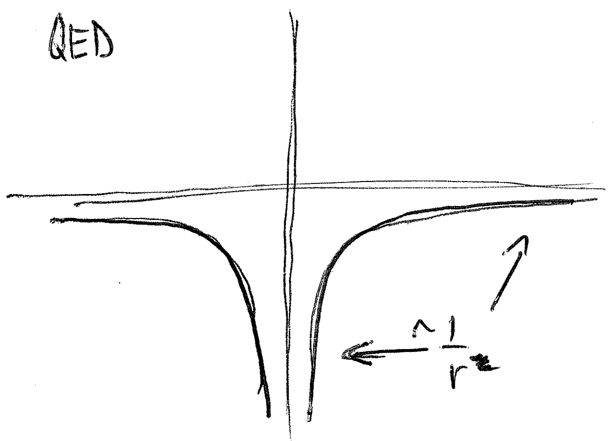
$$S = \frac{\sqrt{\lambda}}{2\pi} \int \left(\frac{1}{2} \operatorname{str} P \wedge *P - \frac{1}{2} \operatorname{str} Q_1 \wedge Q_2 + \Lambda \wedge \operatorname{str} P \right)$$

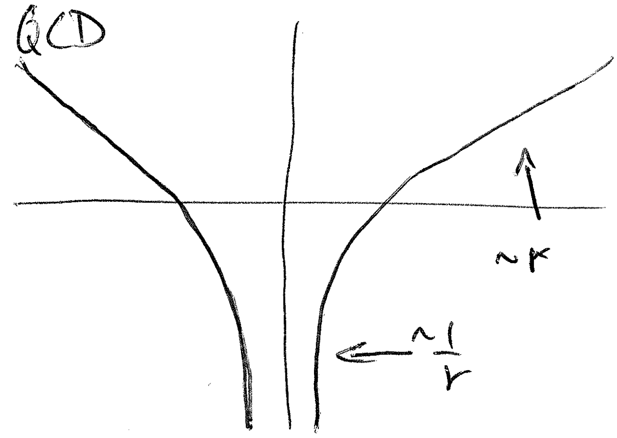
$$Q = \frac{\sqrt{\lambda}}{2\pi} \oint g\left(*P + \tfrac{1}{2} Q_1 - \tfrac{1}{2} Q_2\right) g^{-1}$$

Noether charges for $PSU(2,2|4)$ global symmetry

QCD String

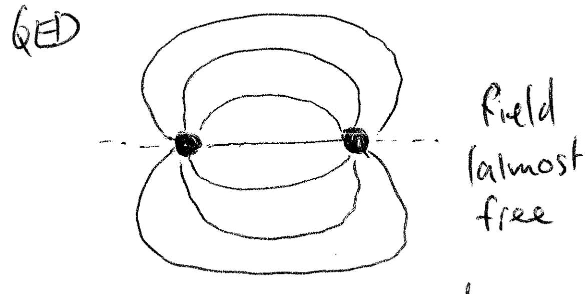
Unlike the electromagnetic forces (QED), the strong nuclear forces (QCD) are confining. Compare electron / quark potential:



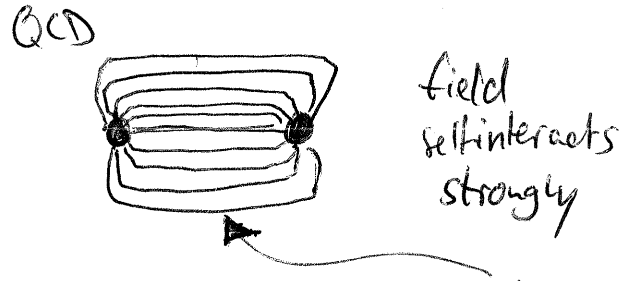


Electron / positron pairs can be separated with finite ~~energy~~ amount of energy. Quark pairs cannot (confinement).

Field lines of electromagnetic / gluon field:

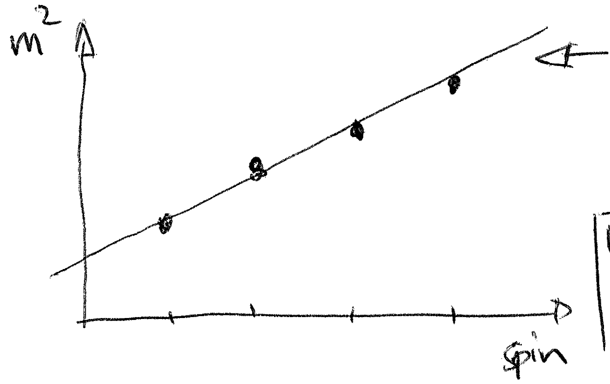


Flux spreads throughout space



Flux condensed into flux tube.
String-like 1D object with tension.
Linear potential with slope ~ tension.

Spectrum of hadronic excitations



Linear relation between m^2 and S.
Regge trajectories.
Spectrum similar to string theory.

$\boxed{\text{4D gauge theories behave stringy in strong coupling regime}}$

Ij

The Planar limit 't Hooft

Consider a $U(N)$ gauge theory with large N.

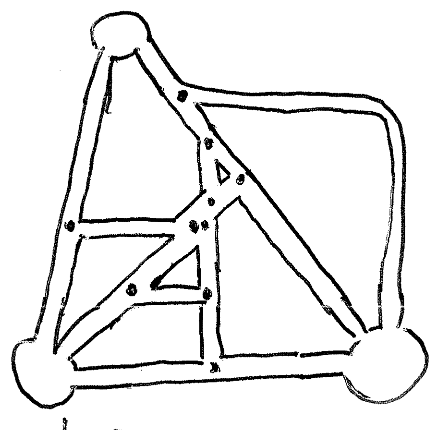
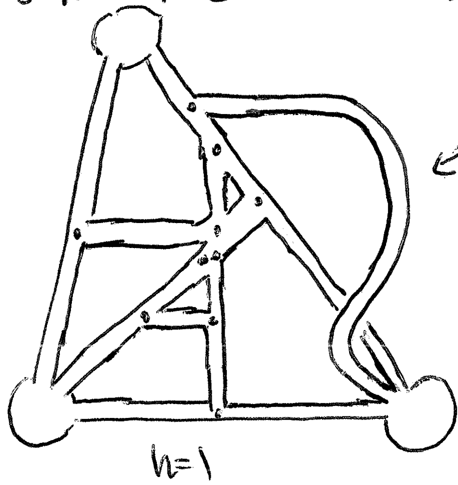
In Feynman graphs draw $U(N)$ representations as

- fundamental $\uparrow$
- conjugate fundamental $\downarrow$
- adjoint/gauge field $\uparrow\downarrow$

Vertices take the form

A^3  , A^4  , $\psi A \bar{\psi}$  etc.

Correlator can be drawn on genus h surface without crossing lines

handle of surface

$h=0$				$h=1$	
$\#V = 10$	vertices	$\sim N g$		$\#V = 10$	$\sim N g$
$\#L = 19$	lines	$\sim N^{-1}$		$\#L = 19$	$\sim N^{-1}$
$\#F = 8$	faces	$\sim N$	$\mathrm{tr}\,1 = N$	$\#F = 6$	$\sim N$
		$\sim N^5 g^{10}$			$\sim N^3 g^{10}$

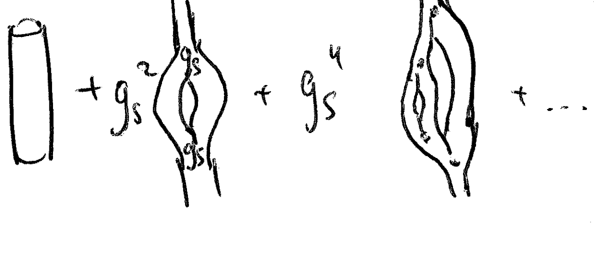
Order of general graph $N^{2-2h+\#E}$ ← external punctures.

't Hooft coupling $\boxed{\lambda \sim g^2}$

$\Rightarrow$ large-N expansion is genus expansion

~~genus counting~~ $\boxed{g_s \sim \frac{1}{N}}$

String perturbation theory



AdS/CFT Correspondence

Exact <u>equivalence</u> of

- String theory on AdS$_d$ bg. (IIB strings on AdS$_5 \times S^5$)
- Conformal quantum field theory on boundary ($N=4$ SYM)

Symmetries match

- Isometry group $\widetilde{SO(d-1,2)}$ ($\widetilde{PSU(2,2|4)}$)
- Conformal spacetime symmetry group $SO(d-1,2)$ ($PSU(2,2|4)$)

Strong/Weak duality

- Classical physics of both models very different
- classical regimes of coupling constants do not overlap
- hard to test in practice,
- makes predictions about **strong**-coupling behaviour of other model.

Coupling constants

coupling of CFT ↓ ; string coupling

- 't Hooft coupling $\lambda \sim g^2 \sim g_{YM}^2 N \sim \frac{R^4}{l_s^4} \begin{cases} \lambda \ll 1 & \text{"weak" coupling (strong)} \\ \lambda \gg 1 & \text{"strong" coupling (weak)} \end{cases}$
- genus counting $\frac{1}{N} \sim \frac{l_s^4}{R^4} g_s \sim \frac{g_s}{\lambda}$

Usually deduced via stack of N D3-branes

Effective ~~lower~~ Low-energy theory: $N=4$ SYM with $U(N)$ gauge

Throat near the D3-branes has asymptotic AdS$_5 \times S^5$ geometry.

Correspondence becomes exact when zooming in on throat (supposedly).

Holography

Statement of duality

$$Z_{CFT}[\phi_0(x)] = Z_{string}[\phi(x,w)|_{w=0} = \phi_0(x)]$$

source of local operator; bulk field; at boundary of AdS; value at boundary

On shell string fields (2nd quantisation) are specified through values at boundary (Cauchy).

Some useful coordinate (patches) for AdS_5

Embedding in $\mathbb{R}^{4,2}$	metric	name	$\partial AdS_5 = \mathbb{R}\times S^3$	metric
$X \in \mathbb{R}^{4,2}$ with $X^2=-1$	induced	embed.	$X\in\mathbb{R}^{4,2}$ with $X^2=0$, $X \simeq \lambda\cdot X$	induced conformal
$X=\begin{pmatrix}\sec r\cos t\\ \sec r\sin t\\ \tan r\times S^3\in\mathbb{R}^4\end{pmatrix}$	$ds^2=\sec^2 r(-dt^2+dr^2)+\tan^2 r\, dS_3^2$	global coords.	$X=\begin{pmatrix}\cos t\\ \sin t\\ S^3\in\mathbb{R}^4\end{pmatrix}$	$ds^2\simeq -dt^2+dS_3^2$
$X=\frac{1}{w}\begin{pmatrix}\frac{1+x^2+w^2}{2}\\ x_t\\ x_x\\ x_y\\ x_z\\ \frac{1-x^2-w^2}{2}\end{pmatrix}$	$ds^2=\frac{1}{w^2}(dx^2+dw^2)$	half-space	$X=\begin{pmatrix}\frac{1+x^2}{2}\\ x_t\\ x_x\\ x_y\\ x_z\\ \frac{1-x^2}{2}\end{pmatrix}$	$ds^2\simeq dx^2$

- global coordinates best for global structure. see that light-ray interacts with boundary in finite time $N=4$ SYM on $\mathbb{R}\times S^3$ (complicated due to curved space)
- half-space model useful for gauge theory (CFT on $\mathbb{R}^{3,1}$ flat) only a patch of coordinates of global AdS_5

What about S^5?

- $SO(6)$ charges of local operators in gauge theory correspond to spherical harmonics on S^5.

$$\partial(AdS_5\times S^5) = CFT_4 \text{ with } S^5 \text{ decomposed into harmonics}$$