Parton Dynamics, 30 years later

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A review of the basics of the QCD parton evolution picture is given with an emphasis on recent findings that reveal intrinsic beauty, and hint at hidden potential simplicity, of the perturbative quark–gluon dynamics.

I. INTRODUCTION

We are witnessing an explosive progress in analytical and numerical methods and techniques for deriving sophisticated pQCD results, prompted to a large extent by the LHC needs. The permanent fight for increasing the accuracy of pQCD predictions is being fought on two fronts: on the one hand, increasing the α_s-order of the exact matrix element calculations (hard parton cross sections) and, on the other hand, improving perturbative description of space-like (parton distribution functions) and time-like (fragmentation functions) quark–gluon cascades. The first battleground is process specific; the second one is universal and is usually referred to as *parton dynamics*.

The universal nature of the parton dynamics goes under the name of *factorisation* of collinear ("mass") singularities. Physically, it is due to the fact that quark–gluon multiplication processes happen at *much larger* space–time distances than the hard interaction itself. It is this separation that makes it possible to describe quark–gluon cascades in terms of independent *parton splitting processes*. They success one another in a cleverly chosen *evolution time*, $t \sim \ln Q^2$, whose flow "counts" basic parton splittings that occur at well separated, strongly ordered, space–time scales. Perturbative structure of the cross section of a given process p characterised by the hardness scale Q^2 can be cast, symbolically, as a product (convolution) of three factors (for a review see [1]):

$$\sigma_h^{(p)}(\ln Q^2) \;\propto\; C^{(p)}[\alpha_s(t)] \otimes \exp\left(\int_{t_0}^{t} d\tau\, P[\alpha_s(\tau)]\right) \otimes w_h(t_0), \qquad t \sim \ln Q^2. \tag{1}$$

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Here the functions $C[\alpha_s]$ (hard cross section; coefficient function) and $P[\alpha_s]$ (parton evolution; anomalous dimension matrix) are perturbative objects analysed in terms of the α_s-expansion. The last factor w_h embeds non-perturbative information about parton structure of the participating hadron(s) h, be it a target hadron in the initial state (parton distribution) or a hadron triggered in the final state (fragmentation function).

A borderline between perturbative and non-perturbative ingredients in (1) is fictitious; it is set arbitrarily by choosing the launching hardness scale $t_0 \sim \ln Q_0^2$. This is however not the only arbitrariness present in the representation (1). Namely, beyond the leading approximation (one loop; $P \propto \alpha_s$), the separation between the C and $\exp(P)$ factors becomes scheme dependent. Here one talks about *factorisation scheme* dependence. Another negotiable object is the expansion parameter α_s itself whose definition depends on the ultraviolet renormalisation procedure (*renormalisation scheme* dependence). The so-called MS-bar scheme — a precisely prescribed procedure for eliminating ultraviolet divergences, based on the dimensional regularisation — won the market as the best suited scheme for carrying out laborious high order calculations. It is this scheme in which the parton "Hamiltonian" P was recently calculated up to next-to-next-to-leading accuracy, α_s^3, by Moch, Vermaseren and Vogt in a series of papers [2, 3].

Formally speaking, the physical answer does not depend on a scheme (either factorisation or renormalisation) one chooses to construct the expansion. There is a *big if* however which renders this motto meaningless. It would have been the case, and consolation, if we had hold of the *full* perturbative expansion for the answer. But this goal is not only technically unachievable. More importantly, it is actually useless. Perturbative expansions in QFT are *asymptotic series*. This means that starting from some order, $n > n_{\rm crit} = {\rm const}_{(p)}/\alpha_s$, a series for any observable (p) inevitably goes haywire and ceases to represent the answer. For QED where $n_{\rm crit} \sim 100$ this is an academic problem. In QCD on the contrary the best hope the perturbative expansion may offer is a reasonable numerical *estimate* based on the first few orders of the perturbation theory (whose intrinsic uncertainty can often be linked with genuine non-perturbative effects). This being understood, it becomes legitimate, and mandatory, to play with perturbative series and try to recast a formal α_s expansion in the most relevant way, the closest to the physics of the problem.

In the beginning of the lecture I will remind you of the basics of parton dynamics, of the origin of *logarithmically enhanced contributions* that lie in the core of the QCD parton picture. The basic one-loop parton Hamiltonian was long known to possess a number of symmetry properties which made the problem look over-restricted and hinted at possible hidden simplicity of the parton dynamics.

We shall discuss in detail the notion of the parton *evolution time* and how it gets modified due to coherence effects in space-like (DIS) and time-like (e^+e^- annihilation) processes.

Then, we will have a closer look at an intimate relation between evolution of space- and time-like parton cascades which I believe has not been properly explored. I will argue that paying a deeper respect to this inter-relation should help to better grasp the complicated structure of two and three loop anomalous dimensions as they are known today.

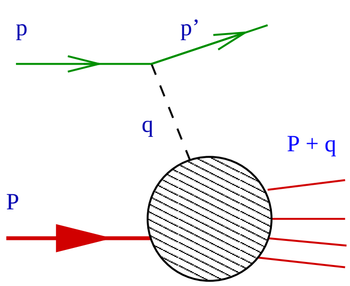
II. PARTON DYNAMICS

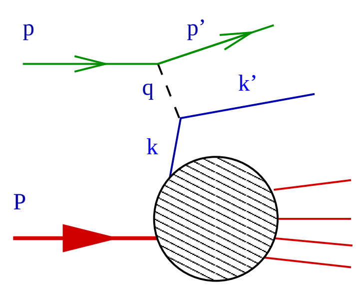
A. Hard QCD processes and partons

Hard processes answered the quest for finding out what hadrons are made of. The answer was rather childish but productive: take a hammer and *hit hard* to see what is it there inside your favourite toy.

We may hit (or heat, if you please) the *vacuum* as it happens in $e^+e^- \to q\bar{q} \to$ hadrons. Then, one may hit a *proton* with a sterile (electroweak) probe giving rise to the famous Deep Inelastic lepton-hadron Scattering (DIS) : $e^-p \to e^- + X$. Finally, make *two hadrons* hit each other hard to produce either a sterile massive object like a $\mu^+\mu^-$ pair (the Drell-Yan process), an electroweak vector boson (Z^0, $W^\pm$), or a Higgs, or a direct photon or a hadron with large transverse momentum with respect to the collision axis. Importantly, in all cases it is *large momentum transfer* which is a measure of the hardness of the process.

Let us turn to DIS as a classical example of a hard process.





Here the momentum q with a large space-like virtuality $Q^2 = |q^2|$ is transferred from an incident electron (muon, neutrino) to the target proton, which then breaks up into the final multihadron system. Introducing an invariant energy $s = 2(Pq)$ between the exchange photon (Z^0, $W^\pm$) with 4-momentum q and the proton with momentum P, one writes the invariant mass of the produced hadron system which measures *inelasticity* of the process as

$$W^2 \equiv (q+P)^2 - M_p^2 = q^2 + 2(Pq) = s(1-x)\,, \qquad x \equiv \frac{-q^2}{2(Pq)} \le 1\,.$$

The cross section of the process depends on two variables: the *hardness* q^2 and Bjorken x. For the case of *elastic* lepton-proton scattering one has $x \equiv 1$ and it is natural to write the cross section as

$$\frac{d\sigma_{el}}{dq^2\,[dx]} = \frac{d\sigma_{\mathrm{Ruth}}}{dq^2} \cdot F_{el}^2(q^2) \cdot [\delta(1-x)]\,. \tag{2a}$$

Here $\sigma_{\mathrm{Ruth}} \propto \alpha^2/q^4$ is the standard Rutherford cross section for e.m. scattering off a point charge and F_{el} stands for elastic proton form factor. For inclusive *inelastic* cross section one can write an analogous expression by introducing "inelastic proton form factor" which now depends on both the momentum transfer q^2 and the inelasticity parameter x:

$$\frac{d\sigma_{in}}{dq^2\,dx} = \frac{d\sigma_{\mathrm{Ruth}}}{dq^2} \cdot F_{in}^2(x, q^2)\,. \tag{2b}$$

What kind of behavior of the form factors (2) could one expect in the Bjorken limit $Q^2 \to \infty$? Quantum mechanics tells us how the Q^2-behavior of the electromagnetic form factor is related to the *charge distribution* inside a proton:

$$F_{el}(\mathbf{Q}^2) = \int d^3r\, \rho(\mathbf{r})\, \exp\{i\mathbf{Q}\cdot\mathbf{r}\}\,.$$

For a point charge $\rho(\mathbf{r}) = \delta^3(\mathbf{r})$, it is obvious that $F \equiv 1$. On the contrary, for a smooth charge distribution $F(Q^2)$ falls with increasing Q^2, the faster the smoother ρ is. Experimentally, the elastic *e-p* cross section does decrease with q^2 *much faster* that the Rutherford one ($F_{el}(q^2)$ decays as a large power of q^2). Does this imply that $\rho(\mathbf{r})$ is indeed regular so that there is no well-localized — point-charge inside a proton? If it were the case, the *inelastic* form factor would decay as well in the Bjorken limit: a tiny photon with the characteristic size $\sim 1/Q \to 0$ would penetrate through a "smooth" proton like a knife through butter, inducing neither elastic nor inelastic interactions.

However, as was first observed at SLAC in the late sixties, for a fixed x, F_{in}^2 stays practically constant with q^2, that is, the *inelastic cross section* (with a given inelasticity) is similar to the Rutherford cross section (Bjorken scaling). It looks *as if* there was a point-like scattering in the guts of it, but in a rather strange way: it results in inelastic break-up dominating over the elastic channel. Quite a paradoxical picture emerged; Feynman–Bjorken partons came to the rescue.

Imagine that it is not the proton itself that is a point-charge-bearer, but some other guys (quark–partons) inside it. If those constituents were *tightly* bound to each other, the elastic channel would be bigger than, or comparable with, the inelastic one: an excitation of the parton that takes an impact would be transferred, with the help of rigid links between partons, to the proton as a whole, leading to elastic scattering or to formation of a quasi-elastic finite-mass system ($N\pi$, $\Delta\pi$ or so), $1-x \ll 1$.

To match the experimental pattern $F^2_{el}(q^2) \ll F^2_{in}(q^2) = \mathcal{O}(1)$ one has instead to view the parton ensemble as a *loosely* bound system of quasi-free particles. Only under these circumstances does knocking off one of the partons inevitably lead to deep inelastic breakup, with a negligible chance of reshuffling the excitation among partons.

The parton model, forged to explain the DIS phenomenon, was intrinsically paradoxical by itself. In sixties and seventies, there was no other way of discussing particle interactions but in the field-theoretical framework, where it remains nowadays. But all reliable (renormalisable, 4-dimensional) quantum field theories (QFTs) known by then had one feature in common: an effective interaction strength $g^2(Q^2)$ — the running coupling — *increasing* with the scale of the hard process Q^2. Actually, this feature was widely believed to be a general law of nature. At the same time, it would be preferable to have it the other way around so as to be in accord with the parton model, which needs parton–parton interaction to *weaken* at small distances (large Q^2).

Only with the advent of non-Abelian QFTs (and QCD among them) exhibiting an anti-intuitive asymptotic-freedom behavior of the coupling, the concept of partons was to become more than a mere phenomenological model.

B. Partons and Quantum Field Theory

Thus, the existence of the limiting distribution

$$F^2_{\text{inelastic}}(q^2, x) \implies D^q_P(x)\,; \qquad |q^2| \to \infty,\ x = \text{const},$$

constituted the Bjorken scaling hypothesis. It became immediately clear however that the Bjorken scaling regime is unattainable in the QFT framework. Indeed, in QFT particle virtualities (transverse momenta) are *not limited* as the parton model suggested. In particular, in a DIS process, "partons" (quarks and gluons) may have transverse momenta $k_\perp^2$ up to $Q^2 = |q^2|$. As a result, the number of particles turns out to be large in spite of small coupling:

$$\int dw \;\propto\; \int^{Q^2} \frac{\alpha_s}{\pi}\frac{dk_\perp^2}{k_\perp^2} \;\sim\; \frac{\alpha_s}{\pi}\ln Q^2 = \mathcal{O}(1)\,.$$

Such — "*collinear*" — enhancement is typical for QFTs with dimensionless coupling, known as "*logarithmic*" Field Theories, and makes the probability of finding a QCD parton, q, inside the target, h, depend on the "resolution", q^2,

$$D_h^q = D_h^q(x, \ln Q^2)\,.$$

Physically, a particle is surrounded by a *virtual coat*; its visible content depends on the *resolution power* of the probe $\lambda = 1/Q = 1/\sqrt{-q^2}$. So, a QCD parton is *not* a point-like particle as the orthodox parton model implied.

Large probability of quark–gluon multiplication processes posed another serious problem. The Feynman–Bjorken picture of partons employed the classical probabilistic language, expressing the hadron interaction cross section as the product of the corresponding parton cross section and the *probability* to find a proper parton inside the hadron target: $\sigma_h = \sigma_q \otimes D_h^q$.

However, as we see, quarks and gluons multiply willingly, $w = \mathcal{O}(1)$. Is there any chance in these circumstances to speak of "QCD partons", to use the language of probabilities? The question may sound silly, since in QFT the number of Feynman graphs grows with the number n of participating particles very fast, roughly as $(n!)^2$, so that the quest of rescuing probabilistic interpretation of quark–gluon cascades looks hopeless.

However, let us ask ourselves, which are the graphs that contribute most? In other words, which are the *most probable* parton fluctuations? Selecting in the n-th order of the perturbative expansion the *maximally enhanced* contributions,

$$(\alpha_s)^n \implies (\alpha_s \cdot \ln Q^2)^n = \mathcal{O}(1),$$

constitutes the logic of the so-called Leading Log Approximation (LLA) [4].

In the DIS environment, the initial parton A with a negative (space-like) virtuality decays into B with the large space-like virtual momentum $|k_B^2| \gg |k_A^2|$ and a positive virtuality (time-like) C. The parton C generates a subjet of secondary partons ($\to$ hadrons) in the final state. As long as the process is inclusive, that is that no details of the final state structure are measured, integration over the subjet mass is due, dominated in LLA by the region $k_C^2 \ll |k_B^2|$. The latter condition makes C look *quasi-real* as compared with the hard scale of $|k_B^2|$. The same is true for the initial parton A.

Given this ordering of virtualities of participating partons, the splitting can be viewed as a large momentum transfer process of scattering (turnover) of a "real" target parton A into a "real" C in the external field mediated by high-virtuality B. At the next step of evolution it is B's turn to play a rôle of a next target $B \equiv A'$, "real" with respect to yet deeper probe $|k_B'^2| \gg |k_B^2|$, and so on.

Successive parton decays with step-by-step *increasing* space-like virtualities (transverse momenta) constitute the picture of parton wave-function fluctuations inside the proton. The sequence proceeds until the overall hardness scale Q^2 is reached.

C. Apparent and hidden beauty of Parton Dynamics

Dependence of the parton decay probability $A \to B[z] + C[1-z]$ on the momentum fraction variable z is given by the "splitting function" $\Phi_A^{BC}(z)$. When studying inclusive characteristics of parton cascades, one traces a single route of successive parton splittings. Having this in mind, we can drop the label that marks partons C whose fate does not concern us, $\Phi_A^{BC}(z) \Longrightarrow P_A^B(z)$ or, in the standard Altarelli–Parisi [5] notation, $\equiv P_{BA}(z)$.

We cannot *predict*, from the first principles, parton content (B) of a hadron (h). However, perturbative QCD tells us how it *changes* with resolution of the DIS process — momentum transfer Q^2. It is driven by the parton Evolution Equation whose structure reminds that of the Schrödinger equation. In the leading order (one loop, LLA) it reads

$$\frac{d}{d\ln Q^2} D_h^B(x,Q^2) \;=\; \frac{\alpha_s(Q^2)}{2\pi} \sum_{A=q,\bar{q},g} \int_x^1 \frac{dz}{z} P_A^B(z) \cdot D_h^A(\frac{x}{z},Q^2). \tag{3}$$

To fully appreciate the power of the probabilistic interpretation of parton cascades have a deeper look at parton splitting probabilities — our *evolution Hamiltonian.*

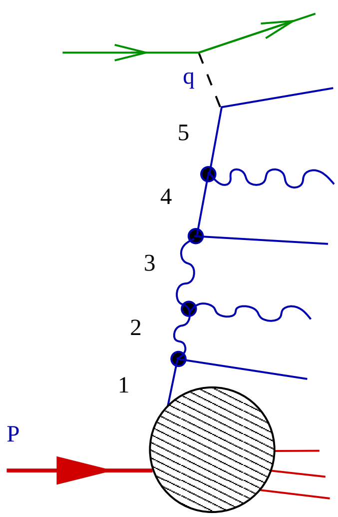
For discussion of the relations between the LLA splitting functions it is convenient to strip off colour factors and introduce

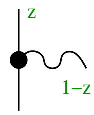
$$\begin{aligned} P_q^q(z) &= C_F\, V_q^q(z)\,, \qquad & P_q^g(z) &= C_F\, V_q^g(z)\,, \\ P_g^q(z) &= T_R\, V_g^q(z)\,, \qquad & P_g^g(z) &= C_A\, V_g^g(z)\,. \end{aligned} \tag{4}$$

Here C_F and $C_A = N_c$ are the familiar quark and gluon "colour charges", while T_R is a scientific name for one half:

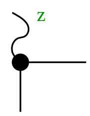
$$\mathrm{Tr}(t^a t^b) = \sum_{i,k=1}^{N_c} t^a_{ik}\, t^b_{ki} \;\equiv\; T_R\, \delta^{ab} = \frac{1}{2}\delta^{ab}\,.$$

In this notation the LLA splitting probabilities read

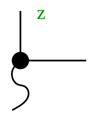




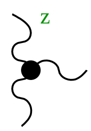
$$V_q^q(z) = \frac{1+z^2}{1-z} \tag{5a}$$



$$V_q^g(z) = \frac{1+(1-z)^2}{z} \tag{5b}$$



$$V_g^q(z) = \left[\, z^2 + (1-z)^2 \,\right] \tag{5c}$$



$$V_g^g(z) = \frac{1+z^4+(1-z)^4}{z(1-z)} \tag{5d}$$

They have the following remarkable symmetry properties.

Parton Exchange results in an obvious relation between probabilities to find decay products with complementary momentum fractions:

$$V_A^{B(C)}(x) \;=\; V_A^{C(B)}(1-x)\,. \tag{6a}$$

Another relation between the elements of the LLA parton Hamiltonian is the

Gribov-Lipatov Reciprocity

$$V_B^A\,(x) \;=\; (-1)^{2s_A+2s_B-1}\, x\, V_A^B\left(x^{-1}\right). \tag{6b}$$

It is a marriage between the

Drell-Levy-Yan Crossing Relation that links two splitting processes corresponding to opposite "evolution time" sequences [6],

$$\overline{V}_B^A\,(x) \;=\; (-1)^{2s_A+2s_B-1}\, x\, V_A^B\left(x^{-1}\right), \tag{6c}$$

(with s_A the spin of particle A) and the

Gribov-Lipatov relation

$$\overline{V}_A^B(x) \;=\; V_A^B(x), \tag{6d}$$

stating that "Hamiltonians" that govern the LLA dynamics of space- (V) and time-like parton cascades ($\overline{V}$) are simply identical [4].

The LLA relations (6a)–(6d) are of general nature as they hold not only in QCD but for any logarithmic QFT. In fact they were first established and discussed in the QFT models of fermions–"quarks" interacting with pseudoscalar and Abelian vector "gluons" [4, 7, 8].

The relations (6a) and (6b) do not leave much freedom for splitting functions. One could borrow V_q^q from QED textbooks, reconstruct V_q^g by exchanging the decay products (6a), and then obtain V_g^q using the crossing (6b). This is the way to generate all three splitting functions (5a)–(5c) relevant for the Abelian model.

The last gluon-gluon splitting function (5d) transforms into itself under both (6a) and (6b). The more surprising is the fact that the gluon self-interaction kernel could actually have been also obtained "from QED" using the

Super-Symmetry Relation [9]

$$V_q^q(x) + V_q^g(x) = V_g^q(x) + V_g^g(x)\,, \tag{6e}$$

which reflects the existence of the supersymmetric QFT closely related to real QCD. But even this is not the end of the story.

Conformal Invariance leads to a number of relations (involving derivatives) between splitting functions, the simplest of which reads [8]

$$\left(x\frac{d}{dx} - 2\right)\, V_g^q(x) \;=\; \left(x\frac{d}{dx} + 1\right)\, V_q^g(x)\,. \tag{6f}$$

Generality of the symmetry properties makes them practically useful when studying sub-leading effects in parton dynamics where one faces technically difficult calculations.

To illustrate the idea have a look at one of the most advanced results — the next-to-next-to-leading prediction for the ratio of mean parton multiplicities in gluon and quark jets derived by Gaffney and Mueller [10],

$$\frac{\mathcal{N}_g}{\mathcal{N}_q} \simeq \frac{N_c}{C_F} - \left(\frac{N_c}{C_F} + \frac{T_f}{C_F} - 2\frac{T_f}{N_c}\right)\left[\sqrt{\frac{\alpha_s N_c}{18\pi}} + \frac{\alpha_s N_c}{18\pi}\left(\frac{25}{8} - \frac{3T_f}{4N_c} - \frac{T_f C_F}{N_c^2}\right)\right].$$

Here $T_f \equiv 2n_f T_R$, with $2n_f$ the number of fermions (quarks and antiquarks of n_f flavours).

Symmetry between quarks (fermions) and gluons (bosons) is hidden in QCD. It becomes manifest in QCD's *supersymmetric partner* QFT in which "quark" and "gluon" belong to the same (adjoint) representation of the colour group. Given identity of all colour factors in (4), $C_F = C_A = T_R$, the relation (6e) can be spelled out as an equality of the *total* probabilities of "quark" and "gluon" decays. (By the way, the fact that it holds identically in x means that there is an infinite number of non-trivial hidden conservation laws in this theory!)

Equating the colour factors and bearing in mind another subtlety, $n_f = \frac{1}{2}$ (since the "quark" is a Majorana fermion there), it is easy to see that the ratio of multiplicities in "quark" and "gluon" jets indeed turns into unity, in all known (as well as in all unknown) orders. The SUSY-QCD [11] (see also [12] and references therein) had been also used to judge two contradictory calculations of the next-to-LLA (two-loop) anomalous dimensions in the early 80s [13, 14]. Let me mention that at the *three loop* level the fate of the SUSY relation for the anomalous dimension matrix remains unknown since the necessary translation of the MS-bar results [2, 3] to a SUSY-respecting renormalisation scheme (based on "dimensional reduction" rather than "regularisation" [15]) has not been established yet.

III. EVOLUTION TIME AND COHERENCE

A. Relating DIS and e^+e^-

As we have seen above, space- and time-like parton cascades are intimately related. No surprise, this. In the DIS case a large virtual momentum q transferred from an incident lepton to a target nucleon with momentum P is *space-like*, $q^2 < 0$. Remind you, inelasticity of the process is conveniently characterised by the *Bjorken* variable $x_B = -q^2/2(Pq)$. On the other side, inclusive fragmentation of an e^+e^- pair with total momentum q (large *positive* invariant mass squared q^2) into a final state hadron with momentum P is characterised by the *Feynman* variable $x_F = 2(Pq)/q^2$ (hadron energy fraction in the e^+e^- cms.). The fact that Bjorken and Feynman variables are indicated by the same letter is certainly not accidental. In both channels $0 \le x \le 1$ though these variables are actually reciprocal, $x_F \Longleftrightarrow 1/x_B$, rather than identical:

$$x_B = \frac{-q^2}{2(Pq)}, \qquad x_F = \frac{2(Pq)}{q^2}. \tag{7}$$

One x becomes the inverse of the other after the crossing operation $P_\mu \to -P_\mu$. Apart from the difference in the hadron momentum P belonging to the initial state in DIS and final state in e^+e^- case, Feynman diagrams for the two processes are just the same. In particular, "mass singularities" that emerge when some parton momentum become collinear to P are therefore also the same. That is why in the two processes similar parton interpretation emerges in terms of QCD evolution equations, and space- and time-like evolution anomalous dimensions turn out to be related.

In fact, relations between the two objects are many and this may cause confusion. Let us recall and discuss three important ones.

a. Drell-Levy-Yan relation. The DLY relation (6c) has a "kinematical" origin in a manner of speaking, as it follows directly from the comparison of the structure of Feynman diagrams in space- and time-like channels. As we have seen above it states that the e^+e^- splitting function can be obtained from that of DIS by replacing $x_B \to 1/x_F$ (modulo a kinematical factor). So, the DLY relation addresses *functional dependence* on x of two different functions, the time-like anomalous dimension $\gamma_+(\alpha, x)$ and the space-like one, $\gamma_-(\alpha, x)$, irrespectively to the value of the argument, x. In higher loops, (6c) was being used to *determine* time-like splitting functions from their space-like counterparts.

b. Analytic continuation. This is a different story. It is about deriving, say, $\gamma_+(x)$ by analytic continuation of the function $\gamma_-(x)$ into the unphysical region $x > 1$ (and then replacing $x \to 1/x < 1$). The continuation path crosses a singular point $x=1$. This calles for special care to be taken of defining certain complex logarithms in "arithmetic" sense, $\ln(1-x) \implies |\ln(1-x)|$, see [8, 9]; beyond the first loop, see [12] and references therein.

c. Gribov-Lipatov relation. Finally, the GL relation (6d) states simply $\gamma_+(\alpha, x) = \gamma_-(\alpha, x)$ and applies in the physical regions of both channels, $x \leq 1$, though the variables are actually given by *different* expressions (7). True in the leading order (LLA), this relation is known to break beyond the first loop. But why?...

B. Long live parton fluctuation time!

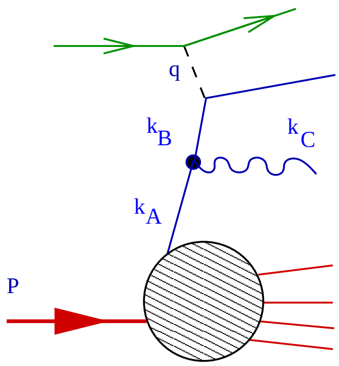
It is instructive to look more carefully into the origin of logarithmically enhanced contributions to the DIS cross section. Introducing two light-like vectors p_1^μ and p_2^μ one can write down Sudakov (light-cone) decomposition of momenta:

$$k^\mu = \beta p_1^\mu + \alpha p_2^\mu + k_\perp^\mu\,, \qquad k^2 = \alpha\beta s - \mathbf{k}_\perp^2 \qquad \big(s = 2Pq\,,\ (k_\perp^\mu)^2 = -\mathbf{k}_\perp^2\big). \tag{8}$$

Then, for $k_1^\mu + k_2^\mu + k_3^\mu = 0$ it is straightforward to derive the identity

$$\frac{k_1^2}{\beta_1} + \frac{k_2^2}{\beta_2} + \frac{k_3^2}{\beta_3} = \frac{\beta_1\beta_2}{\beta_3}\left(\frac{\mathbf{k}_{\perp 1}}{\beta_1} - \frac{\mathbf{k}_{\perp 2}}{\beta_2}\right)^2 . \tag{9}$$

Let us now apply this general relation to the parton splitting that involves a space-like parton A decaying into $B{+}C$. Choosing for the sake of simplicity the direction of p_1 so that $\mathbf{k}_{\perp A} = 0$ (so that $\mathbf{k}_{\perp B} = -\mathbf{k}_{\perp C} \equiv \mathbf{k}_\perp$ is *relative transverse momentum* in the splitting) the relation (9) applied to our basic space-like splitting $A \to B[z] + C[1-z]$ gives



$$\frac{-k_B^2}{z} = \frac{-k_A^2}{1} + \frac{k_C^2}{1-z} + \frac{k_\perp^2}{z(1-z)}\,, \tag{10}$$

where z is the longitudinal momentum fraction — the ratio of the Sudakov light-cone variables β. Since the 4-momenta of A and B are space-like, all terms in (10) are positive.

B being an intermediate virtual state, k_B^2 enters Feynman denominators in the matrix element. The collinear-log contribution arises upon integration over $k_\perp^2$, over the region where the last term dominates in the r.h.s. of (10), that is from the region

$$\frac{|k_B^2|}{z} \simeq \frac{k_\perp^2}{z(1-z)} \gg \frac{|k_A^2|}{1},\ \frac{k_C^2}{1-z}\,. \tag{11}$$

The physical origin of this strong inequality becomes transparent in terms of lifetimes of virtual states ($p_1^\mu \simeq P^\mu$, $p_2^\mu = q^\mu + xP^\mu$)

$$\frac{\beta_i P}{|k_i^2|} \simeq \frac{k_i^0}{|k_i^2|} = \tau_i, \qquad \tau_B \ll \tau_A\,,\ \tau_C. \tag{12}$$

This shows that LLA contributions originate from the sequence of branchings well separated in the *fluctuation time* (12). Invoking the local-scattering analogy (recall $A \to C$ on the "external field" B), we can say that the classical picture naturally implies "fast scattering": probing time τ_B much smaller than proper lifetimes of the "target" before (τ_A) and after the scattering occurs (τ_C).

In DIS kinematics, evolution goes from the proton side and, on the way towards the virtual probe Q^2, parton fluctuations become successively shorter-lived (the "probe" is faster than the fluctuation time of the "target"). Assembling a "ladder" of successive parton splittings we have the $n^{\rm th}$-order LLA contribution $(\alpha_s \ln Q^2)^n$ coming from time-ordered kinematics

$$\frac{P}{\mu^2} \gg \tau_1 \gg \tau_2 \gg \ \ldots \ \gg \tau_n \gg \frac{xP}{-q^2}\,; \qquad x = x_{\rm Bjorken} \equiv \frac{-q^2}{2Pq}. \tag{13a}$$

In the crossing channel, $e^+e^- \to q\bar{q} \to h(x)+X$, the process starts from a large scale q^2 (cms annihilation energy) and results in triggering a final particle h with momentum P. Here order of events is opposite: a parton of the generation $(i+1)$ lives *longer* than its parent (i):

$$\frac{P}{xq^2} \ll \tau_1 \ll \tau_2 \ll \ \ldots \ \ll \tau_n \ll \frac{P}{\mu^2}\,; \qquad x = x_{\mathrm{Feynman}} \equiv \frac{2Pq}{q^2}, \tag{13b}$$

where we have used that the energy of the initial quark stemming form the $\gamma^* \to q\bar{q}$ vertex is $q_0/2 = P/x_F$.

Comparing the two sequences (13) we see that the $x \to x^{-1}$ reciprocity is well present in the ordering of successive fluctuation times. So, why does the Gribov–Lipatov relation break up in higher orders? The answer is simple: it is because we *never* followed the fluctuation time ordering for constructing anomalous dimensions. And for a good reason it seemed.

C. Coherent effects in space- and time-like parton evolution

Beyond the 1st loop, it starts to matter how does one order successive parton splittings. That is, what variable precisely one takes for *parton evolution time* $t \sim \ln Q^2$.

Within the LLA framework it does not make much sense to argue which of possible "evolution times" $\ln(k^2/\beta)$, or $\ln k^2$, $\ln k_\perp^2$ or alike, does a better job: various options differ by *subleading* terms $\mathcal{O}(\alpha_s)$, negligible compared with $\alpha_s \ln Q \sim 1$. However, when numerically small values of Bjorken x are concerned the *next-to*-LLA mismatch contributions amount to

$$\alpha_s \ln^2 \frac{\beta_{i+1}}{\beta_i} = \alpha_s \ln^2 z \quad \Longrightarrow \quad (\alpha_s \ln^2 x)^n. \tag{14}$$

They become significant and must be taken care of — "resummed" — in all orders when $\alpha_s \ln^2 x \sim 1$. In this situation *soft gluon emission* comes onto stage. Here we better be careful: the catch is, for a relatively soft gluon with $z \ll 1$ to be emitted *later* does not guarantee being emitted *independently*. Quantum mechanics, you know. Interference diagrams with gluon radiation off harder partons of different generations enter the game. Does this imply losing probabilistic picture? Not necessarily. It was realised quite some time ago that probabilistic interpretation could be rescued by simply cutting off definite part of the logarithmic phase space formally allowed by the "kinematical" fluctuation time ordering.

In the DIS environment, the *transverse momentum* ordering proved to be the one that took good care of potentially disturbing corrections (14) in all orders, and in this sense became a preferable choice for constructing the probabilistic scheme for space-like parton cascades (DIS structure functions). On the other hand, in the case of time-like cascades it turned out to be the *relative angle* between offspring partons (rather than transverse momentum) that had to be kept ordered, decreasing along the evolutionary decay chain away from the hard production vertex;

$$dt_- = d\ln k_\perp^2 \quad \text{(space-like)}, \tag{15a}$$

$$dt_+ = d\ln \frac{k_\perp^2}{\beta^2} \quad \text{(time-like)}. \tag{15b}$$

Observing that $k_\perp/\beta P = k_\perp/k_+ = 2\tan(\theta/2)$, we confirm that the variable (15b) corresponds indeed to *angular ordering.*

The choice of the variables (15) is a clever dynamical move which takes into consideration *soft gluon coherence* and prevents explosively large terms (14) from appearing in higher loop anomalous dimensions. What is the difference between the two prescriptions (15) and how do they relate to the fluctuation time ordering represented by (13)?

For $z \ll 1$ we have $|k^2| \simeq k_\perp^2$ and the comparison goes as follows

$$\text{DIS} \quad \begin{cases} \text{time ordering:} & \tau_i = \dfrac{\beta_i P}{k_{\perp i}^2} \;>\; \tau_{i+1} = \dfrac{\beta_{i+1} P}{k_{\perp i+1}^2}, \\ k_\perp \text{ ordering:} & k_{\perp i} \;<\; k_{\perp i+1}; \end{cases}$$
$$\text{mismatch} \Longrightarrow \quad z \cdot k_{\perp i}^2 < k_{\perp i+1}^2 < k_{\perp i}^2, \tag{16a}$$

while for the time-like cascades

$$e^+e^- \quad \begin{cases} \text{time ordering:} & \tau_i = \dfrac{\beta_i P}{k_{\perp i}^2} \;<\; \tau_{i+1} = \dfrac{\beta_{i+1} P}{k_{\perp i+1}^2}, \\ \text{angular ordering:} & \theta_i = \dfrac{k_{\perp i}}{\beta_i P} \;>\; \theta_{i+1} = \dfrac{k_{\perp i+1}}{\beta_{i+1} P}; \end{cases}$$
$$\text{mismatch} \Longrightarrow \quad \theta_i^2 < \theta_{i+1}^2 < \frac{\theta_i^2}{z}. \tag{16b}$$

We conclude that in both cases the fluctuation time ordering turns out to be *more liberal* than the corresponding “clever dynamical variable”. Let me briefly remind you underlying physics of QCD coherence that overrides the τ ordering in each of the two channels.

1. DIS: Vanishing of forward inelastic diffraction

Let us zoom onto a bit of the DIS ladder — a two-step space-like evolution process

$$k_{i-1} \to k_i + k', \quad k_i \to k_{i+1} + k''; \qquad \beta_{i-1} > \beta_i \gg \beta_{i+1}, \tag{17}$$

which is composed of two successive decays producing in the end of the day a soft gluon k_{i+1}. In the kinematical region (16a) the *time-ordering* is still intact, which means that the virtual space-like momentum k_{i+1} is transferred fast as compared with the lifetime of the first fluctuation $k_{i-1} \to k_i + k'$.

Since k_{i+1} is the softest, energy-wise, the process can be viewed as *inelastic diffraction* in the external gluon field (k_{i+1}). The transverse size of this field is $\rho_\perp \sim k_{\perp i+1}^{-1}$. The characteristic size of the fluctuation $k_{i-1} \to k_i + k'$, according to (16a), is smaller: $\Delta r_\perp \sim k_{\perp i}^{-1} < \rho_\perp$. We thus have a *compact* state propagating through the field which is smooth at distances of the order of the size of the system and cannot therefore resolve internal structure of the fluctuation. In these circumstances you have to consider also scattering of the *final state* parton k' off the field, in addition to that of the soft virtual offspring k_i. You will observe that the two components of the fluctuation scatter *coherently.* As a result the sum of the two amplitudes will turn out to be identical, and opposite in sign, to the interaction of the external field (k_{i+1}) with the *initial state* k_{i-1}. Inelastic breakup does not occur.

In QCD the cancellation of these three amplitudes in the region (16a), and thus the $k_\perp$ ordering, is a direct consequence of conservation of colour current. The underlying physics is however more general. Phenomenon of vanishing of inelastic transitions in the forward direction was demonstrated by V.N. Gribov in the late 60s on the example of diffractive dissociation of deuteron, which example he used in order to construct the so-called *weak reggeon coupling regime* in general context of high energy hadron interactions.

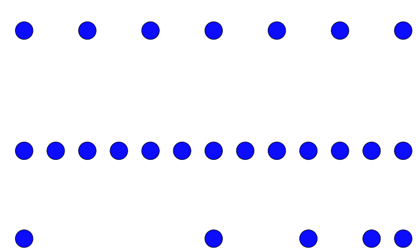
2. e^+e^-: No soft gluon multiplication at large angles

Now we have to examine the same two-step process (17) but with all the parton momenta being *time-like.* The way coherence works here turns out to be spectacularly different. Physics of QCD angular ordering in soft gluon multiplication has a good old QED predecessor — the Chudakov effect. In the 50s cosmic ray physics was a synonym of high energy physics.

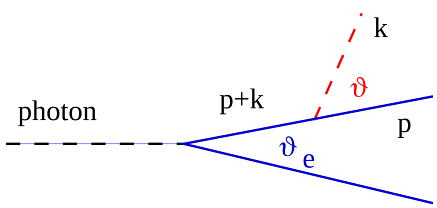
An electron traversing the cosmic ray detector was leaving a trace in photo-emulsion looking like this:

A pair of charges emerging from $\gamma \to e^+e^-$ was expected to show up as a double density track:

What was observed instead looked rather like →



Physical electron is a charge surrounded by proper Coulomb field. In quantum language the Lorentz-contracted Coulomb-disk attached to a relativistic particle may be treated as consisting of photons virtually emitted and, in due time, reabsorbed by the core charge. Such virtual emission and absorption processes form a coherent state which we call a "physical electron" (dressed particle). It is this Coulomb field which is responsible for electron interaction with the emulsion, ionising the atoms it passes by. But this field is not given for granted: the photon conversion produces not "physical" charges but "half-dressed" ones. Their proper field-coats lack components whose lifetime is larger than that of the production process itself, $\tau_0 \sim 1/\sqrt{(P_{e^+} + P_{e^-})^2}$. It takes time to *build up* Coulomb disks adjusted to the charges. In the course of this regeneration process, extra photon radiation takes place giving rise to two bremsstrahlung cones centered around P_{e^+} and P_{e^-}.



Let ϑ_e be an opening angle of the e^+e^- fork, and θ the angle of secondary photon radiation off one of the charges. To evaluate the formation time of the photon k we use the uncertainty relation to estimate the lifetime of the virtual electron state $p+k$ as follows:

$$t_{\rm form} \simeq \frac{(p+k)_0}{(p+k)^2} \simeq \frac{p_0}{2p_0k_0(1-\cos\vartheta)} \simeq \frac{1}{k_0\vartheta^2} \simeq \frac{1}{k_\perp}\cdot\frac{1}{\vartheta} = \lambda_\perp \cdot \frac{1}{\vartheta} \tag{18}$$

What will the photon "see" when its formation time (18) elapses and it has to decide whether to get radiated? Look at the distance between the charges $\Delta\mathbf{r}$ — the size of the e^+e^- dipole — and compare it with characteristic size of the photon-to-be, $\lambda_{||} \sim \omega^{-1}$, $\lambda_\perp \sim k_\perp^{-1} \simeq (\omega\vartheta)^{-1}$:

$$\Delta r_{||} \sim \left|v_{2||} - v_{1||}\right| \cdot c\, t_{\rm form} \sim \vartheta_s^2 \cdot \frac{1}{\omega\vartheta^2} = \left(\frac{\vartheta_s}{\vartheta}\right)^2 \lambda_{||} \;\Leftrightarrow\; \lambda_{||}\,; \tag{19a}$$

$$\Delta r_\perp \sim \vartheta_s \cdot c\, t_{\rm form} \sim \vartheta_s \cdot \frac{1}{\omega\vartheta^2} = \left(\frac{\vartheta_s}{\vartheta}\right) \lambda_\perp \;\Leftrightarrow\; \lambda_\perp\,. \tag{19b}$$

At angles *smaller* than the angle between the charges, $\vartheta < \vartheta_e$, the photon sees the two charges as independent classical emitters. On the contrary, when $\vartheta > \vartheta_e$ it cannot resolve the internal structure of the pair and interacts with the *total electric charge* of the system. In our QED example the latter is zero so that photons disappear altogether.

Returning to QCD cascades, see (17), large angle secondary gluon radiation (k_{i+1}) may be still present. However, coherent sum of emission amplitudes off the partons of the previous generation, k'' and k', will make its intensity proportional not to the sum of the squared colour charges of the "parents" but to the squared charge of the "granddad" k_{i-1}. Probabilistically, it is the granddad the emission of the soft gluon k_{i+1} has to be ascribed to. *Angular ordering*, that is. It does not matter whether the parton k_{i-1} actually split into two or whether there was a whole bunch of partons with small relative angles between them. Soft gluon radiation at large angles is sensitive only to the *total* colour charge of the final parton system, which equals the colour charge of the initial parton. This physically transparent statement holds for arbitrary processes involving quark and gluons (or any other colour objects for that matter).

IV. ISN'T QCD ACTUALLY SIMPLE?

So, fluctuation time ordering proves to be wrong both in space- and time-like kinematics. Interestingly, it also happens to be *equally*, symmetrically wrong: the τ-ordering positions itself just in the middle between the two "clever" ones:

$$k_\perp^2 \quad\Longrightarrow\quad \frac{k_\perp^2}{z} \quad\Longrightarrow\quad \frac{k_\perp^2}{z^2}.$$

What if we decided to play a fool and stubbornly stick to the "wrong" τ-ordering?

Combining (12) and (13) we get the upper limits of virtuality integrals to be

$$\left.\begin{array}{lll} \mathrm{DIS}: & |k_i^2| \ll \dfrac{\beta_i}{\beta_{i+1}}\,|k_{i+1}^2| = z^{-1}\cdot|k_{i+1}^2| \\ e^+e^-: & k_{i+1}^2 \ll \dfrac{\beta_{i+1}}{\beta_i}\;k_i^2 \;=\; z\,\cdot k_i^2 \end{array}\right\} \qquad z = \frac{\beta_{i+1}}{\beta_i} \le 1. \tag{20}$$

Different placing of the z factor causes, beyond the first loop, violation of the Gribov–Lipatov reciprocity (GLR). Moreover, it is likely to be the *one and only* source of this breaking!

A. Rescuing Gribov–Lipatov reciprocity

Let us probe this idea. Choosing $\kappa^2 = |k^2|$ as an integration variable and assembling parton evolution sequences, for the probability $D(x,Q^2)$ to find a parton with *virtuality* integrated up to a given Q^2 we obtain (omitting a trivial Born term)

$$D(x,Q^2) = \int_x^1 \frac{dz}{z}\int^{Q^2}\frac{\kappa^2}{\kappa^2}\,P[z,\alpha_s]\,D\left(\frac{x}{z}, z^\sigma\kappa^2\right); \qquad \sigma = \pm 1 \text{ for the T/S channel.} \tag{21}$$

The second argument of the D function on the r.h.s. of the equation follows from (20). The equation (21) looks more complicated than the standard integral equations that determined the anomalous dimensions of DIS structure functions (γ_-) and fragmentation functions (γ_+),

$$
\begin{aligned}
D^{(S)}(x, Q^2) \;&=\; \int_x^1 \frac{dz}{z} \int^{Q^2} \frac{\kappa^2}{\kappa^2}\, P^{(S)}[z, \alpha_s]\, D^{(S)}\left(\frac{x}{z}, \kappa^2\right), && \text{(22a)} \\
D^{(T)}(x, Q^2) \;&=\; \int_x^1 \frac{dz}{z} \int^{Q^2} \frac{\kappa^2}{\kappa^2}\, P^{(T)}[z, \alpha_s]\, D^{(T)}\left(\frac{x}{z}, \kappa^2\right). && \text{(22b)}
\end{aligned}
$$

In terms of *Mellin moments* of parton distributions and splitting functions,

$$
D_N(Q^2) \;=\; \int_0^1 \frac{dx}{x}\, x^N\, D(x, Q^2), \qquad \mathcal{P}(N, \alpha_s) \;=\; \int_0^1 dz\, z^{N-1}\, P[z, \alpha_s],
$$

one had

$$
\partial_{\ln Q^2} D_N(Q^2) \equiv \gamma(N, \alpha_s) D_N(Q^2) = \int_0^1 \frac{dz}{z}\, z^N\, P[z, \alpha_s]\, D_N\left(Q^2\right) \;=\; \mathcal{P}(N, \alpha_s) D_N(Q^2), \quad (23)
$$

which equated the anomalous dimensions with Mellin images of the corresponding splitting functions, $\gamma_- \equiv \mathcal{P}^{(S)}$ and $\gamma_+ \equiv \mathcal{P}^{(T)}$.

Non-locality of the new equation (21) in longitudinal (z) and transverse variables ($k^2 \propto k_t^2$) breaks identification of splitting functions with anomalous dimensions. What it offers instead is a *link* between the two channels by means of *universal* reciprocity respecting splitting function matrix $\mathcal{P}$, one and the same for T and S evolutions. In spite of the fact that the new "splitting functions" $\mathcal{P}$ in (21) do not correspond to any clever choice of the evolution variable, in either T- or S- channel (explosive $\alpha_s \ln^2 x$ terms being present in both cases), this universality can be exploited for relating DIS and e^+e^- anomalous dimensions.

One can expect that by separating the notions of splitting functions and anomalous dimensions by means of the Reciprocity Respecting Evolution equation (RRE) (21) the Gribov–Lipatov wisdom can be rescued in all orders. This guess ascends to an old remark made by Curci, Furmanski & Petronzio [13] who observed that the GLR violation in the second loop non-singlet quark anomalous dimension amounted to a "quasi-Abelian" term $\propto C_F^2$ with a suggestive structure

$$
\tfrac{1}{2}\left[P_{qq}^{(2,T)}(x) - P_{qq}^{(2,S)}(x)\right] = \int_0^1 \frac{dz}{z} \left\{P_{qq}^{(1)}\left(\frac{x}{z}\right)\right\}_+ \cdot P_{qq}^{(1)}(z) \ln z\,. \qquad (24)
$$

This observation hinted that the GLR violation was not a dynamical higher order effect but was *inherited* from the previous loop via a non-linear relation.

In the Mellin space the convolution (24) translates into $P_N \frac{d}{dN} P_N = P_N \dot{P}_N$. Let us check that it is this structure of the GLR breaking that emerges from (21). Differentiating (21),

$$\partial_{\ln Q^2} D^B(x, Q^2) = \int_x^1 \frac{dz}{z}\, P[z, \alpha_s]\, D^A\left(\frac{x}{z}, z^\sigma Q^2\right), \tag{25}$$

and taking Mellin moments of both sides of the equation we obtain

$$\gamma_\sigma(N)\, D_N(Q^2) = \int_0^1 \frac{dz}{z}\, z^N\, P[z, \alpha_s]\, z^{\sigma\partial_{\ln Q^2}} D_N(Q^2), \tag{26}$$

where we have used the Taylor expansion trick. The integral formally equals

$$\gamma_\sigma(N) \;=\; (D_N)^{-1}\, \mathcal{P}(N + \sigma\partial_{\ln Q^2})\, D_N, \tag{27}$$

expressing the anomalous dimension through the Mellin image of the splitting function with the *differential operator* for argument, $N \to N + \sigma\partial_{\ln Q^2}$. The derivative acts upon $D_N(Q^2)$ producing, by definition, $\gamma(N)D_N$. In high orders it will also act on the *running coupling* the anomalous dimension depends on, $\gamma = \gamma(N, \alpha_s)$. The latter action gives rise to terms proportional to the β-function. Such terms are *scheme dependent* as they can be reshuffled between the exponent and the coefficient function $C[\alpha_s]$ in (1). Neglecting for the time being such contributions by treating α_s as constant, (26) reduces to a functional equation

$$\gamma_\sigma(N) \;=\; \mathcal{P}\left(N + \sigma\gamma_\sigma(N)\right). \tag{28}$$

Since $\gamma = \mathcal{O}(\alpha_s)$, we can expand the argument of the splitting function perturbatively,

$$\gamma_\sigma \;=\; \mathcal{P} \;+\; \dot{\mathcal{P}}\cdot\sigma\gamma \;+\; \tfrac{1}{2}\ddot{\mathcal{P}}\cdot\gamma^2 \;+\; \mathcal{O}(\beta(\alpha)) \;+\; \mathcal{O}\big(\alpha^4\big). \tag{29a}$$

Solving (29a) iteratively we get

$$\gamma_\sigma \;= \mathcal{P} \;+\; \sigma\mathcal{P}\dot{\mathcal{P}} \;+\; \left[\mathcal{P}\dot{\mathcal{P}}^2 + \tfrac{1}{2}\mathcal{P}^2\ddot{\mathcal{P}}\right] +\; \ldots \tag{29b}$$

Restricting ourselves to the first loop, $\mathcal{P} = \alpha\, P^{(1)}$, with $P^{(1)}$ the (Mellin image of) good old LLA functions, gives

$$\gamma_\sigma \;=\; \alpha P^{(1)} \;+\; \alpha^2\, \sigma P^{(1)}\dot{P}^{(1)} \;+\; \ldots \tag{30}$$

The second term on the r.h.s. of (30) generates the two-loop Curci–Furmanski–Petronzio relation (24) all right.[1]

[1] A month after this lecture had been delivered, the structure of the GLR breaking predicted by (27), (28) was verified in [16] for non-singlet anomalous dimensions in three loops.

The same structure of the GLR violation holds for gluon→gluon evolution as well. Strictly speaking, this is true only for two colour structures that emerge in the second loop anomalous dimensions $\mathcal{P}_{gg}^{(2)}$ namely, C_A^2 and $C_A C_F$. The third one, $2n_f T_R C_F$, corresponds to the $g \to q(\bar{q}) \to g$ two-step transition that mixes gluon and quark states. The same colour factor is also present in quark evolution, $q \to g \to q$ described in two loops by the *singlet* anomalous dimension $\mathcal{P}_{qq,s}^{(2)}$. As a result, generalisation of (24) for this specific colour structure turned out to be more involved though natural [17]:

$$\tfrac{1}{2}[\mathcal{P}_{qq,s}^{(2,T)} - \mathcal{P}_{gg}^{(2,S)}] \Longrightarrow \mathcal{P}_{gq}^{(1)} \dot{\mathcal{P}}_{qg}^{(1)} , \qquad \tfrac{1}{2}[\mathcal{P}_{gg}^{(2,T)} - \mathcal{P}_{qq,s}^{(2,S)}] \Longrightarrow \mathcal{P}_{qg}^{(1)} \dot{\mathcal{P}}_{gq}^{(1)} . \tag{31}$$

The analysis of non-diagonal parton transitions is more difficult since here the scheme dependence is more pronounced. Stratmann and Vogelsang have addressed this issue in [12] where a detailed discussion was given of a possibility to rescue GLR in two loops in terms of factorisation scheme transformation. The problem remains open and should be further pursued.

The RRE framework allows us to relate various interesting phenomena that were discovered separately in DIS and e^+e^- context. The first example concerning large-x behaviour of anomalous dimensions was reported in [18].

B. Large x: classical radiation

The RRE (21) has an unexpectedly simple but powerful application to the large-x region, $(1–x) \ll 1$. Here non-diagonal $q \leftrightarrow g$ transitions do not matter and one can restrict oneself to non-singlet quark evolution. The large-N behaviour of corresponding anomalous dimensions can be parametrised as follows:

$$\gamma_\sigma(N) = -A(\psi(N+1) + \gamma_e) + B - C_\sigma \frac{\psi(N+1) + \gamma_e}{N} + \frac{D_\sigma}{N} + \mathcal{O}\left(\frac{\log^p N}{N^2}\right), \tag{32a}$$

where A, B, C, D are given in terms of series in α_s in a given renormalisation scheme. The coefficients A and B are the same in the two channels. In x space (32a) corresponds to

$$\gamma_\sigma(x) = \frac{A\,x}{(1-x)_+} + B\,\delta(1-x) + C_\sigma \ln(1-x) + D_\sigma + \mathcal{O}((1-x)\log^p(1-x)) . \tag{32b}$$

Specific structure of the first — the most singular — term $x/(1-x)$ is dictated by the Low–Burnett–Kroll theorem [19]. It is a consequence of the fact that soft radiation at the level of $d\sigma \propto d\omega(\omega^{-1} + \text{const})$ has *classical nature*. The coefficient A in front of this structure has a meaning of the "physical coupling" measured by the intensity of relatively soft gluon emission. This coefficient (calculated in three loops in the MS-bar scheme) is known to universally appear in all observables sensitive to soft gluon radiation: quark and gluon Sudakov form factors and Regge trajectories, threshold resummations, singular part of the Drell–Yan K-factor, distributions of jet event shapes in the near-to-two-jet kinematics, heavy quark fragmentation functions, etc. The structure (32) applies to large-x behaviour of the $g \to g$ anomalous dimension as well, with $A_{(g)}/A_{(q)} = C_A/C_F$, in all orders.

Quantum effects show up only at the level of $d\sigma \propto \omega\, d\omega$ that is at the level of contributions that were *neglected* in (32b). At one loop, subleading terms $C \ln(1-x)$ and the constant D in (32) are absent. This suggests that in higher loops they should emerge as "inherited" rather than non-trivial entries.

Indeed, to keep under control all the terms in (32) it suffices to use the following general expression for the large-N asymptote of the splitting function $\mathcal{P}^{(1)}$:

$$\mathcal{P}^{(1)} = -A(\psi(N+1) + \gamma_e) + B + \mathcal{O}\big(N^{-2}\big) \,, \tag{33a}$$

which gives

$$\dot{\mathcal{P}}^{(1)} = -\frac{A}{N} + \mathcal{O}\big(N^{-2}\big) \,. \tag{33b}$$

Substituting (33) into (29b) produces then the all-order relations [18]

$$C_\sigma \;=\; -\sigma\, A^2, \tag{34a}$$

$$D_\sigma \;=\; -\sigma\, A\, B. \tag{34b}$$

The relation (34a) is "conformal" while (34b) acquires corrections due to running of the coupling. In three loops (34a) was recently verified for the time-like channel in [16].

C. Small x: relating space- and time-like mysteries

As we have discussed above, in small-x kinematics the RR parton splitting functions $\mathcal{P}$ based on the fluctuation time ordering contain double logarithmic (DL) series, $(\alpha \log^2 x)^n$. In the anomalous dimensions γ_σ they combine with another series of DL enhanced terms that originate from *non-locality* of the RRE and are contained in the $\sigma \mathcal{P} \dot{\mathcal{P}}$ structure. As we already know, the resulting series can be absorbed into modifying the evolution time from fluctuation time to *transverse momentum* ordering in the case of DIS space-like parton cascades, and to the ordering of *angles* of successive parton splittings in e^+e^-. Such a clever transformation of the ordering variable is a result of cancellation, due to destructive soft gluon coherence, of a part of the logarithmic phase space allowed by the lifetime ordering. Thus, from the point of view of the RRE, the angular ordering in time-like evolution can be considered to be an image of the k_t ordering in DIS, and vice versa.

Even more, beyond the leading log approximation, the RRE should link together two puzzling small-x phenomena which, to the best of my knowledge, were never thought to be of a common origin. These are: absence of the α_s^2 and α_s^3 terms in the BFKL anomalous dimension in the DIS problem on one hand [20],

$$\gamma_-^{\mathrm{BFKL}}[N,\alpha_s] \;\sim\; \frac{\alpha_s}{N} + 0\cdot\left(\frac{\alpha_s}{N}\right)^2 + 0\cdot\left(\frac{\alpha_s}{N}\right)^3 + \left(\frac{\alpha_s}{N}\right)^4,$$

and, on the other hand, *exact angular ordering* [21] which seems to hold further than expected, down to the next-to-next-to-leading (N-MLLA) order (Malaza puzzle [22]).

Examine the system of equations

$$\gamma_-[N,\alpha_s] \;=\; \mathcal{P}\big(N - \gamma_-[N,\alpha_s],\alpha_s\big), \tag{35a}$$

$$\gamma_+[N,\alpha_s] \;=\; \mathcal{P}\big(N + \gamma_+[N,\alpha_s],\alpha_s\big). \tag{35b}$$

Knowing the DIS anomalous dimension γ_- we can plug it into (35a), find $\mathcal{P}$ and use this information in (35b) to determine γ_+.

Small-x limit corresponds to Mellin moments $N \to 0$. Perturbative series for the S- and T-case can be organised according to strength of the $N \to 0$ singularity as follows

$$\gamma_-[N,\alpha_s] = \sum_{p=1}^{\infty}\sum_{k=0}^{p} s_{p\,k}\,\frac{\bar{\alpha}_s^p}{N^k}\,, \qquad \gamma_+[N,\alpha_s] = \sum_{p=1}^{\infty}\sum_{k=0}^{2p-1} t_{p\,k}\,\frac{\bar{\alpha}_s^p}{N^k}\,. \tag{36}$$

For example, processing through (35) the leading term in the space-like function,

$$\gamma_-[N,\alpha_s] = \frac{\bar{\alpha}_s}{N}, \qquad \bar{\alpha}_s \equiv \frac{N_c\,\alpha_s}{\pi}, \tag{37a}$$

gives the time-like anomalous dimension in the DL approximation

$$\gamma_+^{\mathrm{DLA}}[N,\alpha_s] \;=\; \frac{1}{4}\left(-N + \sqrt{N^2 + 8\bar{\alpha}_s}\right). \tag{37b}$$

Incorporating the first subleading correction $\mathcal{O}(1)$ into (37a),

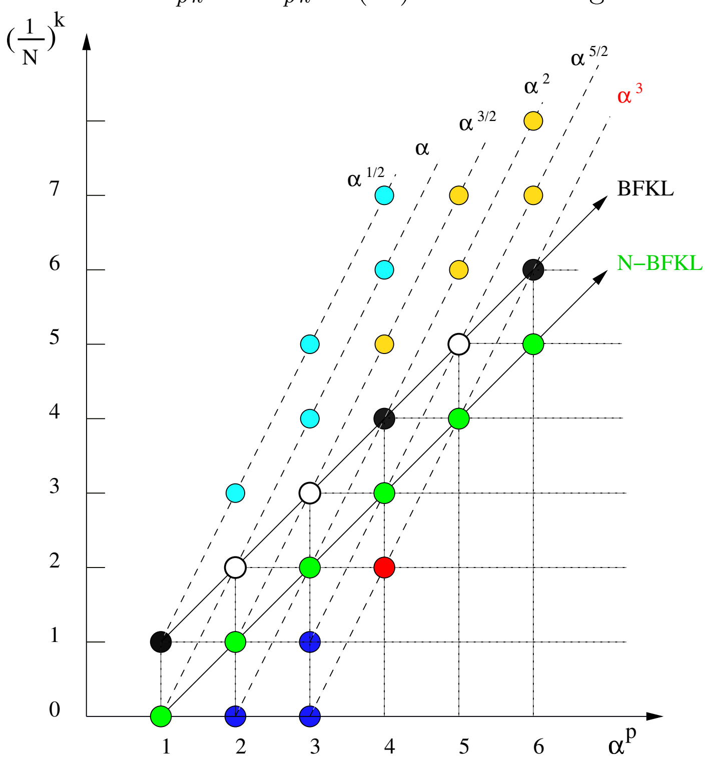
$$\gamma_-[N,\alpha_s] = \frac{\bar{\alpha}_s}{N} - \bar{\alpha}_s a, \quad a = \frac{11}{12} + \frac{n_f}{6N_c^3}, \tag{38a}$$

analogously produces the next-to-leading (MLLA) expression for γ_+,

$$\gamma_+^{\rm MLLA}[N,\alpha_s] \;=\; \gamma_+^{\rm DLA}[N,\alpha_s] - \bar{\alpha}_s\frac{a}{2}\left(1+\frac{N}{\sqrt{N^2+8\bar{\alpha}_s}}\right), \tag{38b}$$

(correct modulo a β-function term) [21], etc.

The space-like anomalous dimension is known to three loops. Adding to it the known infinite BFKL and next-to-BFKL [23] series of $1/N$-enhanced terms and plugging this information into the RRE, one should be able to obtain *five* successive terms in the $\sqrt{\alpha_s}$ expansion for the time-like anomalous dimension, from the leading $\sqrt{\alpha_s}$ downto $\alpha_s^{5/2}$. The coefficients $s_{p\,k}$ and $t_{p\,k}$ in (36) can be arranged into series as depicted by the following chart.



This chart demonstrates interdependence between perturbative series for γ_- (solid lines) and γ_+ (dashed). Black circles show the BFKL terms $\bar{\alpha}_s^p/N^p$. Empty circles on the same line stand for BFKL terms that are "accidentally" zero. The parallel (green) line collects the next-to-BFKL terms $\bar{\alpha}_s^p/N^{p-1}$.
Not knowing the (N–N–BFKL) point $p=4, k=2$ prevents us from finding the N-N-N-N-N-LL time-like anomalous dimension term $\mathcal{O}(\alpha_s^3)$.

V. CONCLUSIONS

Getting hold of the MS-bar space-like anomalous dimension in three loops (with the time-like case on the way) was a great achievement. But equally is it a cause for depression: how much physics would you be able to enjoy while browsing through some 100K of formulae?

We formulated the quest for simplifying perturbative expansions and discussed a number of internal symmetries the parton evolution Hamiltonian possesses. In particular, at the LLA level there are excessively many symmetry relations so that the system of two-parton splitting functions turns out to be over-restricted! This observation is rather suggestive and encourages one to look for definite simplifications when attacking higher orders.

In the present lecture I put emphasis on exploring, and exploiting, inter-relation between DIS and e^+e^-, between space- and time-like parton evolution. We have constructed together the Gribov–Lipatov–Reciprocity respecting equation (21) and demonstrated that it is well suited for elucidating the physics of the large-x limit. The RRE also seems to offer an intriguing possibility of mirroring two "accidental" zeroes in the BFKL series with the absence of corrections to exact angular ordering in small-x time-like cascades [24].

Large x being a realm of the Low–Burnett–Kroll wisdom, the main terms in $q \to q$ and $g \to g$ anomalous dimensions in all orders are basically governed by the *first loop*, provided we cast the series in terms of the *physical coupling* $\alpha_{\text{phys}} \equiv A_{(q)}/C_F = A_{(g)}/C_A$. Given such reshuffling, non-trivial quantum effects may enter only at the level of corrections suppressed w.r.t. to the classical result as $\mathcal{O}((1-x)^2)$, modulo logarithms.

We discussed the terms $\ln(1-x)$ and $(1-x)^0$ present in diagonal quark and gluon anomalous dimensions. In the LBK nomenclature, these terms fall "in between" classical and quantum contributions: they are less singular than the former, $\mathcal{O}((1-x)^{-1})$, and more singular than the latter, $\mathcal{O}(1-x)$. Therefore they must be "trivial", *inherited* from classical physics. Indeed, the RRE framework allowed us to guess these *sub-singular* contributions in all orders.

Pushing the *inheritance* idea to the extreme I would dare to propose a heretic "letter of intent" and suggest, what one should aim at searching for *ideal* perturbation theory. Stepping up the order of the perturbative expansion, $n \to n+1$, we add a parton to the system of n partons that determined the parton Hamiltonian in the n^{th} order, $P^{(n)}$. If a "quantum object" (quark or hard gluon) is added, the phase space volume consideration tells us that the next order anomalous dimension will acquire an extra suppression factor, $P^{(n+1)}(x) \propto (1-x)P^{(n)}(x)$. Adding a *soft gluon* as the $(n+1)^{\text{st}}$ parton would avoid the phase space suppression. However, this specific contribution we should be able to reduce to the previous order dynamics by exploiting the classical nature of soft gluon radiation.

So, if we were smart enough to systematically relate all soft gluon effects to classical *inheritance* and formulate parton dynamics as *quantum* dynamics, then the *genuine* higher order contributions to the anomalous dimension would have followed the ideal pattern[2]

$$\gamma^{(n)}_{\text{quantum}} \sim (1-x)^n.$$

For such an ambitious programme to succeed, it is not enough to merely switch to "physical coupling". It will also be necessary to design a physically motivated separation between the anomalous dimension (Hamiltonian) and the coefficient function (short-distance cross section), in other words, to properly choose factorisation scheme. This battle the conventional MS-bar prescription is bound to lose. As it was remarked in [12], an alternative calculation based on the cut vertex formalism naturally produced reciprocity respecting γ and C in two loops [26]. Systematic physically motivated renormalisation/factorisation method is wanted.

Physics of soft gluons is that of classical radiation. The depth of this statement was not granted attention it rightfully deserved. "Classical" implies *manageable*, *solvable*, *simple*. Let us list some examples of soft gluon governed phenomena where these key words happen to materialise:

- *Manageability* of the so-called "maximum helicity violating" (MHV) multi-gluon amplitudes (Parker–Taylor amplitudes are literally the soft LBK ones).
- Picking up from the QCD parton Hamiltonian the so-called "maximal transcendentality" structures (driven by soft gluon radiation physics) produces the anomalous dimension of the $N = 4$ SUSY (see [27] and references therein), the QFT model known to be exactly *solvable*.
- The second loop soft gluon radiative correction to parton scattering matrix elements turned out to be "surprisingly *simple*" [28]. Actually, non-existent: the first loop takes it all in (recall the "physical coupling" message).

To add to the list, let me mention a recently observed unexpected property of the "soft anomalous dimension" occurring in the gluon–gluon scattering. The programme of resumming logarithmic effects due to large angle soft gluon emission in hadron–hadron collisions was pioneered by Botts and Sterman [29] and developed in a series of papers [30]. In [31] it was demonstrated that this anomalous dimension possesses a mysterious symmetry between

[2] Such programme was carried out for heavy quark fragmentation functions to two loops in [25] where the *genuine* second loop contribution was shown to be negligibly small, uniformly in x.

internal and external variables of the problem, linking the rank of the gauge group with scattering kinematics,

$$\frac{1}{N_c} \quad \Longleftrightarrow \quad \frac{\ln(t/u)}{\ln(t/s)+\ln(u/s)}.$$

The origin of such a weird symmetry is very unlikely to be understood from within the QFT framework and seems to call for an enveloping "theoretical theory" wisdom which may shed additional light onto hidden beauty of parton dynamics.

Good luck.

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