A new look at Parton Evolution and $\mathcal{N}=4$ SYM as a tool for QCD

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plan

Parton Dynamics Revisited

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Parton Dynamics Revisited with Giuseppe (Pino) Marchesini

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- Innovative Bookkeeping
  - QCD in Kbytes
  - Relating *Space*- and *Time*-like Evolutions
  - New "*wrong but smart*" Parton Evolution Equations
  - First checks

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  - *Clagons* and *Quagons*
  - $\mathcal{N}=4$ SYM as QCD playing ground
  - Soft gluons and "*transcedentality*"
  - Higher loops, subleading twist(s)

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  - QCD as $\mathcal{N}=4$ SYM++
  - "*... two loops = one loop too many ...* "

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  - "*... two loops = one loop too many ...*" (for provocation sake)

3rd loop non-singlet a.d.

$$P_{\rm ns}^{(2)+}(x) = 16 C_A C_F n_f \Bigg(\frac{1}{6} p_{\rm qq}(x) \Bigg[\frac{10}{3}\zeta_2 - \frac{209}{36} - 9\zeta_3 - \frac{167}{18}\mathrm{H}_0 + 2\mathrm{H}_0\zeta_2 - 7\mathrm{H}_0$$

$$+3\mathrm{H}_{1,0,0} - \mathrm{H}_3 \Bigg] + \frac{1}{3} p_{\rm qq}(-x) \Bigg[\frac{3}{2}\zeta_3 - \frac{5}{3}\zeta_2 - \mathrm{H}_{-2,0} - 2\mathrm{H}_{-1}\zeta_2 - \frac{10}{3}\mathrm{H}_{-1,0} - \mathrm{H}_-$$

$$+2\mathrm{H}_{-1,2} + \frac{1}{2}\mathrm{H}_0\zeta_2 + \frac{5}{3}\mathrm{H}_{0,0} + \mathrm{H}_{0,0,0} - \mathrm{H}_3 \Bigg] + (1-x)\Bigg[\frac{1}{6}\zeta_2 - \frac{257}{54} - \frac{43}{18}\mathrm{H}_0 -$$

$$-(1+x)\Bigg[\frac{2}{3}\mathrm{H}_{-1,0} + \frac{1}{2}\mathrm{H}_2 \Bigg] + \frac{1}{3}\zeta_2 + \mathrm{H}_0 + \frac{1}{6}\mathrm{H}_{0,0} + \delta(1-x)\Bigg[\frac{5}{4} - \frac{167}{54}\zeta_2 + \frac{1}{20}\zeta_2$$

$$+16 C_A {C_F}^2 \Bigg(p_{\rm qq}(x) \Bigg[\frac{5}{6}\zeta_3 - \frac{69}{20}{\zeta_2}^2 - \mathrm{H}_{-3,0} - 3\mathrm{H}_{-2}\zeta_2 - 14\mathrm{H}_{-2,-1,0} + 3\mathrm{H}_{-2,0}$$

$$-4\mathrm{H}_{-2,2} - \frac{151}{48}\mathrm{H}_0 + \frac{41}{12}\mathrm{H}_0\zeta_2 - \frac{17}{2}\mathrm{H}_0\zeta_3 - \frac{13}{4}\mathrm{H}_{0,0} - 4\mathrm{H}_{0,0}\zeta_2 - \frac{23}{12}\mathrm{H}_{0,0,0} + 5\mathrm{H}$$

$$-24\mathrm{H}_1\zeta_3 - 16\mathrm{H}_{1,-2,0} + \frac{67}{9}\mathrm{H}_{1,0} - 2\mathrm{H}_{1,0}\zeta_2 + \frac{31}{3}\mathrm{H}_{1,0,0} + 11\mathrm{H}_{1,0,0,0} + 8\mathrm{H}_{1,1,0,0}$$

3rd loop, more

$$+\frac{67}{9}\mathrm{H}_2 - 2\mathrm{H}_2\zeta_2 + \frac{11}{3}\mathrm{H}_{2,0} + 5\mathrm{H}_{2,0,0} + \mathrm{H}_{3,0}\Big] + p_{\mathrm{qq}}(-x)\Big[\frac{1}{4}{\zeta_2}^2 - \frac{67}{9}\zeta_2 + \frac{31}{4}\zeta$$

$$-32\mathrm{H}_{-2}\zeta_2 - 4\mathrm{H}_{-2,-1,0} - \frac{31}{6}\mathrm{H}_{-2,0} + 21\mathrm{H}_{-2,0,0} + 30\mathrm{H}_{-2,2} - \frac{31}{3}\mathrm{H}_{-1}\zeta_2 - 42\mathrm{H}$$

$$-4\mathrm{H}_{-1,-2,0} + 56\mathrm{H}_{-1,-1}\zeta_2 - 36\mathrm{H}_{-1,-1,0,0} - 56\mathrm{H}_{-1,-1,2} - \frac{134}{9}\mathrm{H}_{-1,0} - 42\mathrm{H}_{-1}$$

$$+32\mathrm{H}_{-1,3} - \frac{31}{6}\mathrm{H}_{-1,0,0} + 17\mathrm{H}_{-1,0,0,0} + \frac{31}{3}\mathrm{H}_{-1,2} + 2\mathrm{H}_{-1,2,0} + \frac{13}{12}\mathrm{H}_0\zeta_2 + \frac{29}{2}\mathrm{H}$$

$$+13\mathrm{H}_{0,0}\zeta_2 + \frac{89}{12}\mathrm{H}_{0,0,0} - 5\mathrm{H}_{0,0,0,0} - 7\mathrm{H}_2\zeta_2 - \frac{31}{6}\mathrm{H}_3 - 10\mathrm{H}_4\Big] + (1-x)\Big[\frac{133}{36} +$$

$$-\frac{167}{4}\zeta_3 - 2\mathrm{H}_0\zeta_3 - 2\mathrm{H}_{-3,0} + \mathrm{H}_{-2}\zeta_2 + 2\mathrm{H}_{-2,-1,0} - 3\mathrm{H}_{-2,0,0} + \frac{77}{4}\mathrm{H}_{0,0,0} - \frac{20}{6}$$

$$+4\mathrm{H}_{1,0,0} + \frac{14}{3}\mathrm{H}_{1,0}\Big] + (1+x)\Big[\frac{43}{2}\zeta_2 - 3{\zeta_2}^2 + \frac{25}{2}\mathrm{H}_{-2,0} - 31\mathrm{H}_{-1}\zeta_2 - 14\mathrm{H}_{-1,-}$$

$$+24\mathrm{H}_{-1,2} + 23\mathrm{H}_{-1,0,0} + \frac{55}{2}\mathrm{H}_0\zeta_2 + 5\mathrm{H}_{0,0}\zeta_2 + \frac{1457}{48}\mathrm{H}_0 - \frac{1025}{36}\mathrm{H}_{0,0} - \frac{155}{6}\mathrm{H}_2$$

3rd loop, and more

$$+2\mathrm{H}_{2,0,0} - 3\mathrm{H}_4\Big] - 5\zeta_2 - \frac{1}{2}{\zeta_2}^2 + 50\zeta_3 - 2\mathrm{H}_{-3,0} - 7\mathrm{H}_{-2,0} - \mathrm{H}_0\zeta_3 - \frac{37}{2}\mathrm{H}_0\zeta_2$$

$$-2\mathrm{H}_{0,0}\zeta_2 + \frac{185}{6}\mathrm{H}_{0,0} - 22\mathrm{H}_{0,0,0} - 4\mathrm{H}_{0,0,0,0} + \frac{28}{3}\mathrm{H}_2 + 6\mathrm{H}_3 + \delta(1-x)\Big[\frac{151}{64} +$$

$$-\frac{247}{60}{\zeta_2}^2 + \frac{211}{12}\zeta_3 + \frac{15}{2}\zeta_5\Big]\Big) + 16\, {C_A}^2 C_F \Big(p_{\mathrm{qq}}(x)\Big[\frac{245}{48} - \frac{67}{18}\zeta_2 + \frac{12}{5}{\zeta_2}^2 + \frac{1}{2}$$

$$+\mathrm{H}_{-3,0} + 4\mathrm{H}_{-2,-1,0} - \frac{3}{2}\mathrm{H}_{-2,0} - \mathrm{H}_{-2,0,0} + 2\mathrm{H}_{-2,2} - \frac{31}{12}\mathrm{H}_0\zeta_2 + 4\mathrm{H}_0\zeta_3 + \frac{389}{72}$$

$$-\mathrm{H}_{0,0,0,0} + 9\mathrm{H}_1\zeta_3 + 6\mathrm{H}_{1,-2,0} - \mathrm{H}_{1,0}\zeta_2 - \frac{11}{4}\mathrm{H}_{1,0,0} - 3\mathrm{H}_{1,0,0,0} - 4\mathrm{H}_{1,1,0,0} + 4\mathrm{H}$$

$$+\frac{11}{12}\mathrm{H}_3 + \mathrm{H}_4\Big] + p_{\mathrm{qq}}(-x)\Big[\frac{67}{18}\zeta_2 - {\zeta_2}^2 - \frac{11}{4}\zeta_3 - \mathrm{H}_{-3,0} + 8\mathrm{H}_{-2}\zeta_2 + \frac{11}{6}\mathrm{H}_{-2,0}$$

$$-3\mathrm{H}_{-1,0,0,0} + \frac{11}{3}\mathrm{H}_{-1}\zeta_2 + 12\mathrm{H}_{-1}\zeta_3 - 16\mathrm{H}_{-1,-1}\zeta_2 + 8\mathrm{H}_{-1,-1,0,0} + 16\mathrm{H}_{-1,-1,2}$$

$$-8\mathrm{H}_{-2,2} + 11\mathrm{H}_{-1,0}\zeta_2 + \frac{11}{6}\mathrm{H}_{-1,0,0} - \frac{11}{3}\mathrm{H}_{-1,2} - 8\mathrm{H}_{-1,3} - \frac{3}{4}\mathrm{H}_0 - \frac{1}{6}\mathrm{H}_0\zeta_2 - 4$$

3rd loop, and again

$$-3\mathrm{H}_{0,0}\zeta_2 - \frac{31}{12}\mathrm{H}_{0,0,0} + \mathrm{H}_{0,0,0,0} + 2\mathrm{H}_2\zeta_2 + \frac{11}{6}\mathrm{H}_3 + 2\mathrm{H}_4\Big] + (1-x)\Big[\frac{1883}{108} - \frac{1}{2}$$

$$-\mathrm{H}_{-2,-1,0} + \frac{1}{2}\mathrm{H}_{-3,0} - \frac{1}{2}\mathrm{H}_{-2}\zeta_2 + \frac{1}{2}\mathrm{H}_{-2,0,0} + \frac{523}{36}\mathrm{H}_0 + \mathrm{H}_0\zeta_3 - \frac{13}{3}\mathrm{H}_{0,0} - \frac{5}{2}\mathrm{H}$$

$$-2\mathrm{H}_{1,0,0}\Big] + (1+x)\Big[8\mathrm{H}_{-1}\zeta_2 + 4\mathrm{H}_{-1,-1,0} + \frac{8}{3}\mathrm{H}_{-1,0} - 5\mathrm{H}_{-1,0,0} - 6\mathrm{H}_{-1,2} - \frac{13}{3}$$

$$-\frac{43}{4}\zeta_3 - \frac{5}{2}\mathrm{H}_{-2,0} - \frac{11}{2}\mathrm{H}_0\zeta_2 - \frac{1}{2}\mathrm{H}_2\zeta_2 - \frac{5}{4}\mathrm{H}_{0,0}\zeta_2 + 7\mathrm{H}_2 - \frac{1}{4}\mathrm{H}_{2,0,0} + 3\mathrm{H}_3 + \frac{3}{4}$$

$$+\frac{1}{4}\zeta_2{}^2 - \frac{8}{3}\zeta_2 + \frac{17}{2}\zeta_3 + \mathrm{H}_{-2,0} - \frac{19}{2}\mathrm{H}_0 + \frac{5}{2}\mathrm{H}_0\zeta_2 - \mathrm{H}_0\zeta_3 + \frac{13}{3}\mathrm{H}_{0,0} + \frac{5}{2}\mathrm{H}_{0,0,0}$$

$$-\delta(1-x)\Big[\frac{1657}{576} - \frac{281}{27}\zeta_2 + \frac{1}{8}\zeta_2{}^2 + \frac{97}{9}\zeta_3 - \frac{5}{2}\zeta_5\Big]\Big) + 16\, C_F n_f^2 \Big(\frac{1}{18}p_{qq}(x)\Big[\mathrm{H}_{0,0}$$

$$+(1-x)\Big[\frac{13}{54} + \frac{1}{9}\mathrm{H}_0\Big] - \delta(1-x)\Big[\frac{17}{144} - \frac{5}{27}\zeta_2 + \frac{1}{9}\zeta_3\Big]\Big) + 16\, C_F{}^2 n_f \Big(\frac{1}{3}p_{qq}(x)\Big[$$

3rd loop, and still some more

$$-\frac{55}{16}+\frac{5}{8}\mathrm{H}_0+\mathrm{H}_0\zeta_2+\frac{3}{2}\mathrm{H}_{0,0}-\mathrm{H}_{0,0,0}-\frac{10}{3}\mathrm{H}_{1,0}-\frac{10}{3}\mathrm{H}_2-2\mathrm{H}_{2,0}-2\mathrm{H}_3\Big]+\frac{2}{3}$$

$$-\frac{3}{2}\zeta_3+\mathrm{H}_{-2,0}+2\mathrm{H}_{-1}\zeta_2+\frac{10}{3}\mathrm{H}_{-1,0}+\mathrm{H}_{-1,0,0}-2\mathrm{H}_{-1,2}-\frac{1}{2}\mathrm{H}_0\zeta_2-\frac{5}{3}\mathrm{H}_{0,0}-$$

$$-(1-x)\Big[\frac{10}{9}+\frac{19}{18}\mathrm{H}_{0,0}-\frac{4}{3}\mathrm{H}_1+\frac{2}{3}\mathrm{H}_{1,0}+\frac{4}{3}\mathrm{H}_2\Big]+(1+x)\Big[\frac{4}{3}\mathrm{H}_{-1,0}-\frac{25}{24}\mathrm{H}_0+$$

$$+\frac{7}{9}\mathrm{H}_{0,0}+\frac{4}{3}\mathrm{H}_2-\delta(1-x)\Big[\frac{23}{16}-\frac{5}{12}\zeta_2-\frac{29}{30}{\zeta_2}^2+\frac{17}{6}\zeta_3\Big]\Big)+16\,{C_F}^3\Big(p_{\mathrm{qq}}(x)\Big[$$

$$+6\mathrm{H}_{-2}\zeta_2+12\mathrm{H}_{-2,-1,0}-6\mathrm{H}_{-2,0,0}-\frac{3}{16}\mathrm{H}_0-\frac{3}{2}\mathrm{H}_0\zeta_2+\mathrm{H}_0\zeta_3+\frac{13}{8}\mathrm{H}_{0,0}-2\mathrm{H}_0$$

$$+12\mathrm{H}_1\zeta_3+8\mathrm{H}_{1,-2,0}-6\mathrm{H}_{1,0,0}-4\mathrm{H}_{1,0,0,0}+4\mathrm{H}_{1,2,0}-3\mathrm{H}_{2,0}+2\mathrm{H}_{2,0,0}+4\mathrm{H}_{2,1}$$

$$+4\mathrm{H}_{3,0}+4\mathrm{H}_{3,1}+2\mathrm{H}_4\Big]+p_{\mathrm{qq}}(-x)\Big[\frac{7}{2}{\zeta_2}^2-\frac{9}{2}\zeta_3-6\mathrm{H}_{-3,0}+32\mathrm{H}_{-2}\zeta_2+8\mathrm{H}_{-2}$$

$$-26\mathrm{H}_{-2,0,0}-28\mathrm{H}_{-2,2}+6\mathrm{H}_{-1}\zeta_2+36\mathrm{H}_{-1}\zeta_3+8\mathrm{H}_{-1,-2,0}-48\mathrm{H}_{-1,-1}\zeta_2+40\mathrm{H}$$

3rd loop, and UFF

$$+48\mathrm{H}_{-1,-1,2} + 40\mathrm{H}_{-1,0}\zeta_2 + 3\mathrm{H}_{-1,0,0} - 22\mathrm{H}_{-1,0,0,0} - 6\mathrm{H}_{-1,2} - 4\mathrm{H}_{-1,2,0} - 32$$

$$-\frac{3}{2}\mathrm{H}_0\zeta_2 - 13\mathrm{H}_0\zeta_3 - 14\mathrm{H}_{0,0}\zeta_2 - \frac{9}{2}\mathrm{H}_{0,0,0} + 6\mathrm{H}_{0,0,0,0} + 6\mathrm{H}_2\zeta_2 + 3\mathrm{H}_3 + 2\mathrm{H}_{3,0}$$

$$+(1-x)\Big[2\mathrm{H}_{-3,0} - \frac{31}{8} + 4\mathrm{H}_{-2,0,0} + \mathrm{H}_{0,0}\zeta_2 - 3\mathrm{H}_{0,0,0,0} + 35\mathrm{H}_1 + 6\mathrm{H}_1\zeta_2 - \mathrm{H}_{1,}$$

$$+(1+x)\Big[\frac{37}{10}{\zeta_2}^2 - \frac{93}{4}\zeta_2 - \frac{81}{2}\zeta_3 - 15\mathrm{H}_{-2,0} + 30\mathrm{H}_{-1}\zeta_2 + 12\mathrm{H}_{-1,-1,0} - 2\mathrm{H}_{-1,0}$$

$$-24\mathrm{H}_{-1,2} - \frac{539}{16}\mathrm{H}_0 - 28\mathrm{H}_0\zeta_2 + \frac{191}{8}\mathrm{H}_{0,0} + 20\mathrm{H}_{0,0,0} + \frac{85}{4}\mathrm{H}_2 - 3\mathrm{H}_{2,0,0} - 2\mathrm{H}_3$$

$$-\mathrm{H}_4\Big] + 4\zeta_2 + 33\zeta_3 + 4\mathrm{H}_{-3,0} + 10\mathrm{H}_{-2,0} + \frac{67}{2}\mathrm{H}_0 + 6\mathrm{H}_0\zeta_3 + 19\mathrm{H}_0\zeta_2 - 25\mathrm{H}_{0,0}$$

$$-2\mathrm{H}_2 - \mathrm{H}_{2,0} - 4\mathrm{H}_3 + \delta(1-x)\Big[\frac{29}{32} - 2\zeta_2\zeta_3 + \frac{9}{8}\zeta_2 + \frac{18}{5}{\zeta_2}^2 + \frac{17}{4}\zeta_3 - 15\zeta_5\Big]\Big)$$

music of the spheres

2×2 anomalous dimension matrix occupies

1 st loop: $1/10$ page

music of the spheres

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music of the spheres

2×2 anomalous dimension matrix occupies

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3 rd loop: 100 pages (200 K asci)

Moch, Vermaseren and Vogt

[waterfall of results launched
March 2004, and counting]

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$$V \sim \begin{cases} 10^{\frac{N(N-1)}{2}-1} \\ 10^{2^{N-1}-2} \end{cases}$$

facing music of the spheres

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not too encouraging a trend ...

Fighting complexity

How to reduce complexity ?

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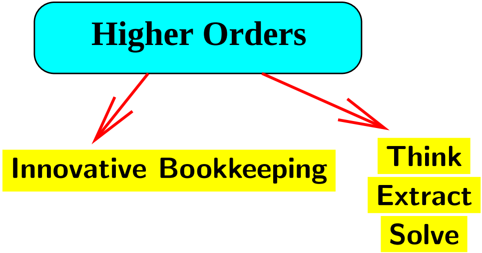
Guidelines



Fighting complexity

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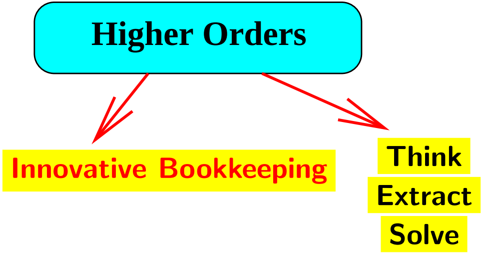
✓ exploit internal properties :

- Drell–Levy–Yan relation
- Gribov–Lipatov reciprocity



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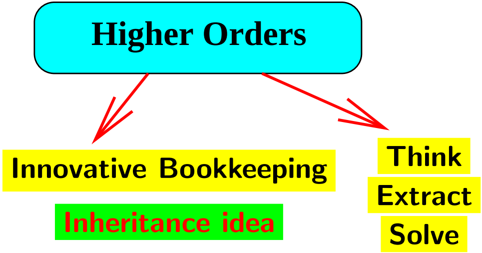

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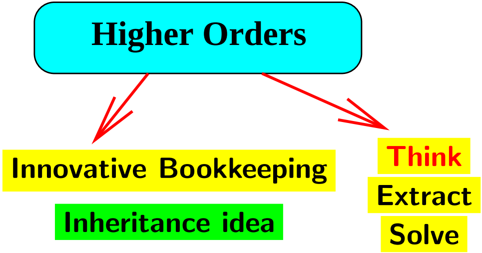

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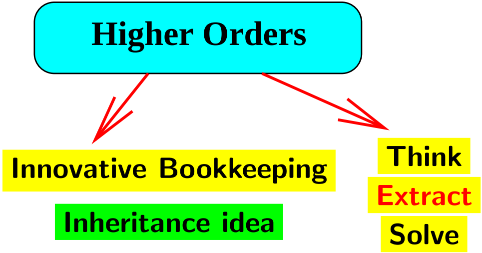

An essential part of gluon dynamics is Classical. (F.Low)

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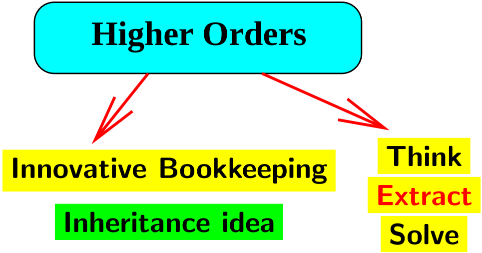

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"Classical" does not mean "Simple".
However, it has a good chance to be Exactly Solvable.

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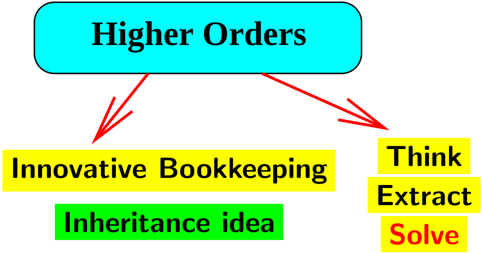

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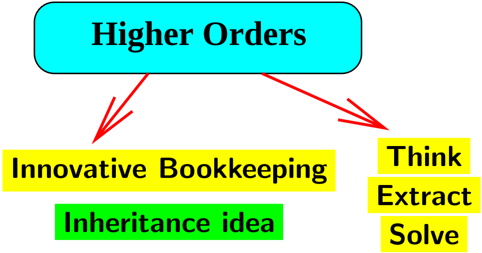

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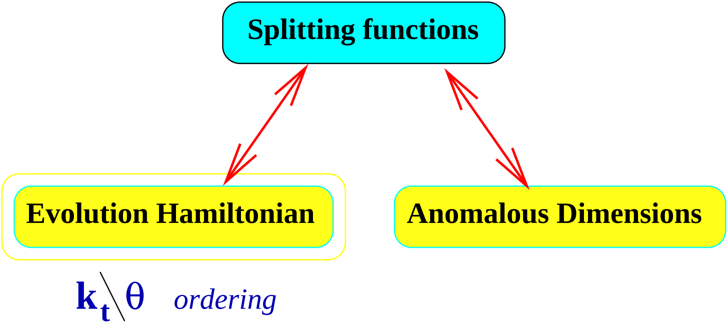
➥ A playing ground for theoretical theory: SUSY, AdS/CFT, . . .

Innovative Bookkeeping

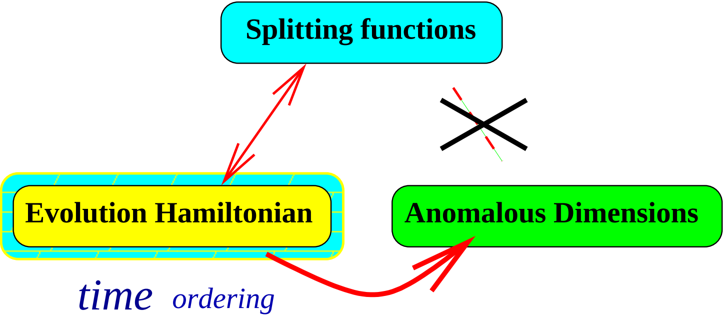
In the standard approach,



- parton splitting functions are equated with anomalous dimensions;
- they are different for DIS and e^+e^- evolution;
- "clever evolution variables" are different too

Innovative Bookkeeping

In the new approach,



- splitting functions are disconnected from the anomalous dimensions;
- the evolution kernel is identical for space- and time-like cascades (Gribov–Lipatov reciprocity relation true in all orders);
- unique evolution variable — parton fluctuation time

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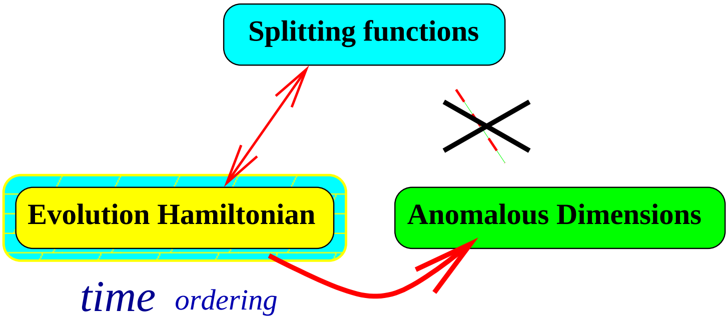

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old new evolution — Innovative Bookkeeping

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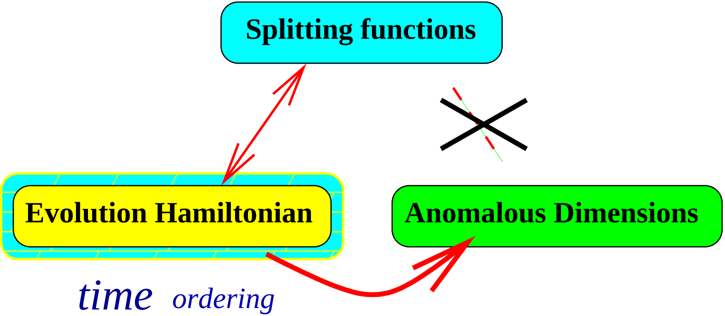

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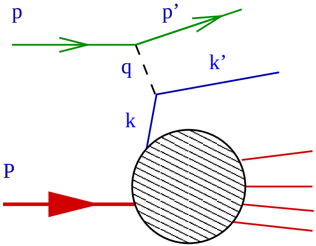
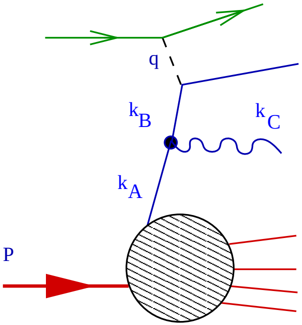
Long-living partons fluctuations



Kinematics of the parton splitting $A \to B + C$

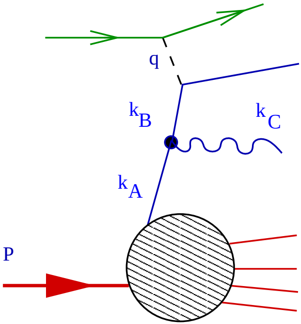
Long-living partons fluctuations



Kinematics of the parton splitting $A \to B + C$

$$k_B \simeq x \cdot P\,, \quad k_A \simeq \frac{x}{z} \cdot P$$

Long-living partons fluctuations



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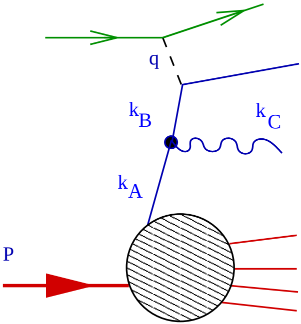

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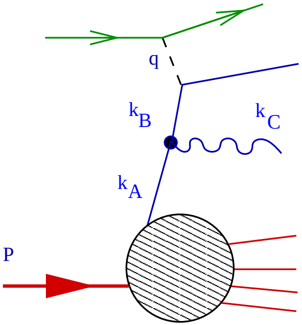

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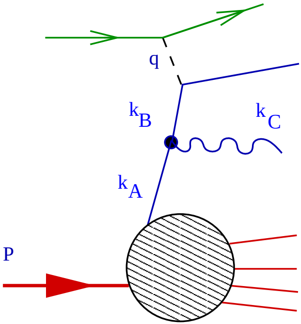

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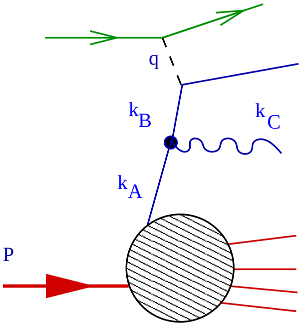

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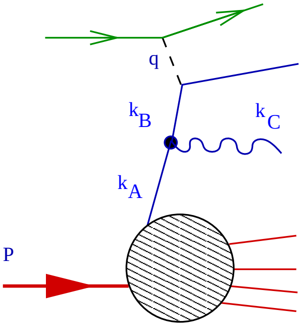

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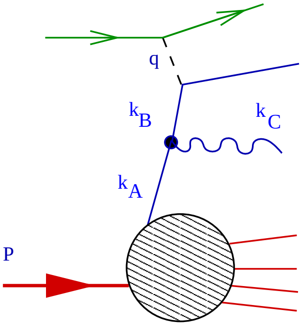

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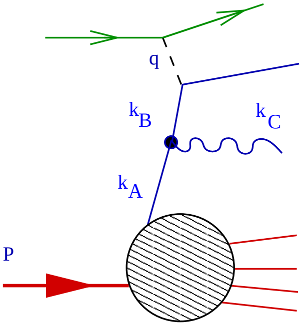

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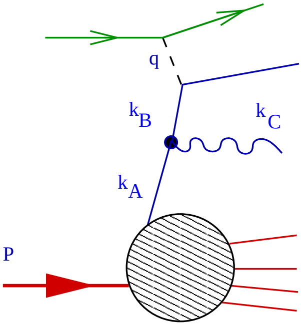

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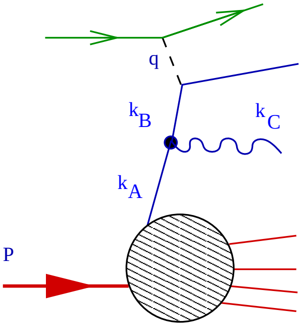

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strongly ordered *lifetimes* of successive parton fluctuations !

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Beyond the 1st loop, it starts to matter how does one order successive parton splittings that is, what one chooses for "parton evolution time".

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Each of these two clever choices — consequence of taking into full consideration soft gluon coherence in order to prevent explosively large terms $(\alpha_s \ln^2 x)^n$ from appearing in higher loop anomalous dimensions.

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Mark the different meaning of x in the two channels!

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$$\frac{dD^A(x,Q^2)}{d\ln Q^2} = \int_0^1 \frac{dz}{z}\,\mathcal{P}_B^A(z;\alpha_s)\,D^B\left(\frac{x}{z}, z^\sigma Q^2\right)$$

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$$P_N \equiv \int_0^1 \frac{dz}{z}\,P(z)\,z^N \qquad \Longrightarrow \qquad \gamma_N \cdot D_N(Q^2) = \mathcal{P}_{N+\sigma d}\cdot D_N(Q^2)$$

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the *evolution kernel* $\mathcal{P}$ emerges with the *differential operator* for argument.

Expanding, get an equation for the an.dim. γ, one for both channels

$$\gamma[\alpha] = \mathcal{P} + \dot{\mathcal{P}}\cdot(\sigma\gamma + \beta/\alpha) + \tfrac{1}{2}\ddot{\mathcal{P}}\cdot\left[\gamma^2 + \sigma(2\beta/\alpha\,\gamma + \beta\partial_\alpha\gamma) + \beta/\alpha\,\partial_\alpha\beta\right] + \mathcal{O}(\alpha^4).$$

GLR beyond the 1st loop

Examine the "reciprocity respecting equation" (RRE) by feeding in the one-loop parton "Hamiltonian", $\mathcal{P}(\alpha) \simeq \alpha P_1$:

$$\begin{aligned}\gamma[\alpha] &= \mathcal{P} + \dot{\mathcal{P}} \cdot \left(\sigma\gamma + \beta/\alpha\right) + \tfrac{1}{2}\ddot{\mathcal{P}} \cdot \left[\gamma^2 + \sigma(2\beta/\alpha\,\gamma + \beta\partial_\alpha\gamma) + \beta/\alpha\,\partial_\alpha\beta\right] + \dots \\ &= \alpha P_1 + \alpha^2 \cdot (\sigma\, P_1 \dot{P}_1 + \beta_0) \qquad\qquad + \mathcal{O}\left(\alpha^3\right).\end{aligned}$$

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The difference between time- and space-like anomalous dimensions,

$$\tfrac{1}{2}\left[P^{(T)} - P^{(S)}\right] = \alpha^2 \cdot P_1 \dot{P}_1 + \mathcal{O}(\alpha^3)\,,$$

in the x-space corresponds to the convolution

$$\frac{1}{2}\left[P_{qq}^{(2),T} - P_{qq}^{(2),S}\right] = \int_0^1 \frac{dz}{z} \left\{P_{qq}^{(1)}\left(\frac{x}{z}\right)\right\}_+ \cdot\; P_{qq}^{(1)}(z) \ln z\,,$$

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$\Longrightarrow$ the genuine $\mathcal{P}_2$ does not contain σ, is GLR respecting

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More generally, a *renormalization scheme transformation* as a cure for/against GLR violation was proposed by Stratmann & Vogelsang (1996)

large x

Another important aspect of the RREE is the "double nature" of the perturbative expansion — in α_{phys} and, at the same time, in $(1-x)$:

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A gap between *classical radiation* (Low–Burnett–Kroll wisdom)

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and *quantum fluctuations*

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$D = -\sigma A\,B + \mathcal{O}(\beta)$ — another all-order relation

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inherited from previous loops !

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This QFT has a good chance to be *solvable* — "integrable".
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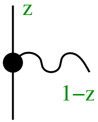
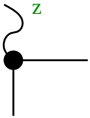
Recall an old hint from QCD ...

Relating parton splittings



$$= C_F \cdot \frac{1+z^2}{1-z}$$



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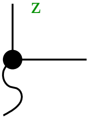
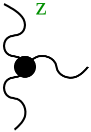


$$= T_R \cdot \left[\, z^2 + (1-z)^2 \,\right]$$

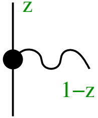


$$= N_c \cdot \frac{1+z^4+(1-z)^4}{z(1-z)}$$

Four "parton splitting functions"

$${}^{q[g]}_{q}(z)\,, \qquad {}^{g[q]}_{q}(z)\,, \qquad {}^{q[\bar{q}]}_{g}(z)\,, \qquad {}^{g[g]}_{g}(z)$$

Relating parton splittings



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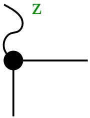

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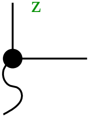

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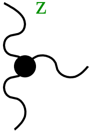

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- Exchange the decay products : $z \to 1-z$

$^{q[g]}_{q}(z)$ $^{g[q]}_{q}(z)$ $^{q[\bar{q}]}_{g}(z)$ $^{g[g]}_{g}(z)$

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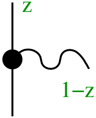

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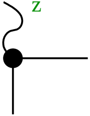

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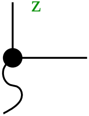

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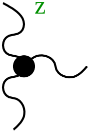

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- Exchange the decay products : $z \to 1-z$
- Exchange the parent and the offspring : $z \to 1/z$ (GLR)

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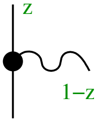

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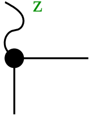

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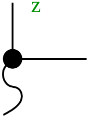

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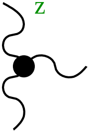

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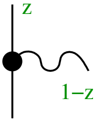
Three (QED) "kernels" are inter-related; gluon self-interaction stays put :

$$\boxed{ {}_{q}^{q[g]}(z)\,, \quad {}_{q}^{g[q]}(z)\,, \quad {}_{g}^{q[\bar{q}]}(z) } \quad ; \quad \boxed{ {}_{g}^{g[g]}(z) }$$

Relating parton splittings



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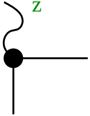

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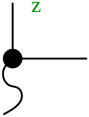

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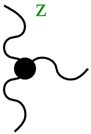

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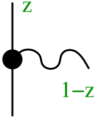

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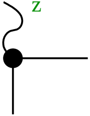

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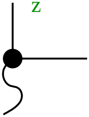

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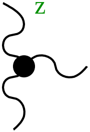

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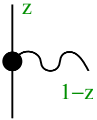

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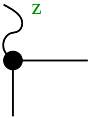

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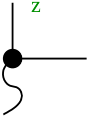

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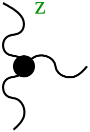

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The integrability feature manifests itself already in *certain sectors* of QCD, in specific problems where one can *identify* QCD with SUSY-QCD :

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WHY and **WHAT FOR** ?

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And here we arrive at the second — Divide and Conquer — issue

LBK wisdom

Recall the diagonal first loop anomalous dimensions:

$$\tilde{\gamma}_{q\to q(x)+g} = \frac{C_F\alpha_s}{\pi}\left[\frac{x}{1-x} + (1-x)\cdot\frac{1}{2}\right],$$

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Let us look at the rôles these animals play on the QCD stage

Gluenatomy

Clagons :

- ✗ Classical Field
- ✓ infrared singular, $d\omega/\omega$
- ✓ define the physical coupling
- ✓ responsible for
  - ➥ DL radiative effects,
  - ➥ reggeization,
  - ➥ QCD/Lund string (gluers)
- ✓ play the major rôle in evolution

Quagons :

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In addition,

✗ Tree multi-clagon (Parke–Taylor) amplitudes are *known exactly*

✗ It is clagons which dominate in all the *integrability cases*

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In spite of having many states ($s = 0, \frac{1}{2}, 1$), the SYM-4 parton dynamics is built of a single "universal" anomalous dimension:

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as well as multiple indices — *nested sums*

$$S_{m,\vec{\rho}}(N) = \sum_{k=1}^N \frac{S_{\vec{\rho}}(k)}{k^m} \qquad (\vec{\rho} = (m_1, m_2, \dots, m_i)),$$

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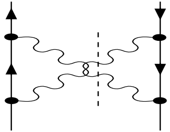
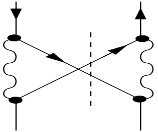


(a)



(b)

$(a) \leftrightarrow (b)$

$P \rightarrow -P$

$x \rightarrow -x$

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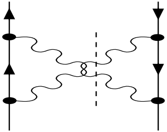
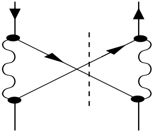


(a)



(b)

$(a) \leftrightarrow (b)$

$P \;\to\; -P$

$x \;\to\; -x$

$$p_{q\bar{q}}(x) \;=\; \alpha_s^2 \left(\tfrac{1}{2} C_A - C_F\right) p_{qq}(-x) \cdot \phi_2(x), \quad p_{qq}(x) = \frac{1+x^2}{2(1-x)}.$$

negative indices

Starting from the 2nd loop, one encounters also *negative indices*,

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$x \;\to\; -x$

$\frac{x}{1-x} \cdot \ln^2 x \to S_3(N) \qquad \frac{x}{1+x} \cdot \phi_2(x) \to Y_{-3}(N)$

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"classicality" and "transcedentality"

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(direct calculation by Kotikov & Lipatov, 2000)

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$$\gamma_\sigma(N) = \mathcal{P}(N + \sigma\gamma_\sigma(N))$$

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generates positives and simplifies negatives.

RR evolution kernel

In terms of the perturbative expansion in the physical coupling,

$$a_{\rm ph} = a\left(1 - \tfrac{1}{2}\zeta_2 a + \tfrac{11}{20}\zeta_2^2 a^2 + \ldots\right),$$

$$\begin{aligned}
\mathcal{P}_1 &= -\ S_1; \\
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Notation:

$$\hat{Y}_{-m}(N) = (-1)^N \, \mathbf{M}\left[\frac{x}{1+x}\,\phi_{m-1}(x)\right],$$

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Twist-3

The $\mathfrak{sl}(2)$ sector of planar $\mathcal{N}=4$ SYM contains single trace states which are linear combinations of the basic operators

$$\mathrm{Tr}\left\{(\mathcal{D}^{s_1} Z) \cdots (\mathcal{D}^{s_L} Z)\right\}, \quad s_1 + \cdots + s_L = N,$$

where Z is one of the three complex scalar fields and $\mathcal{D}$ is a light-cone covariant derivative. The numbers $\{s_i\}$ are non-negative integers and N is the total spin. The number L of Z fields is the twist of the operator, *i.e.* the classical dimension minus spin.

The anomalous dimensions of these states are the eigenvalues $\gamma_L(N; g)$ of the dilatation operator — integrable Hamiltonian.

These values were obtained by solving numerically the Bethe Ansatz equations (BAE), order by order in g^2, and guessing the answer in terms of harmonic sums of transcedentality $\tau = 2n-1$, at n loops.

Since *wrapping problems*, delayed by supersymmetry, appear at $L+2$ loop order for twist-L operators, the BAE for twist-3 are reliable up to *four loops* (including, at the fourth loop, the dressing factor).

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Twist-3 : Answer

$$
\begin{aligned}
\gamma_3^{(1)} &= 4\,S_1\\
\gamma_3^{(2)} &= -2\,(S_3 + 2\,S_1 S_2)\\
\gamma_3^{(3)} &= 5\,S_5 + 6\,S_2\,S_3 - 8\,S_{3,1,1} + 4\,S_{4,1} - 4\,S_{2,3} + S_1(4\,S_2^2 + 2\,S_4 + 8\,S_{3,1})\\
\gamma_3^{(4)} &= \tfrac{1}{2}\,S_7 + 7\,S_{1,6} + 15\,S_{2,5} - 5\,S_{3,4} - 29\,S_{4,3} - 21\,S_{5,2} - 5\,S_{6,1}\\
&\quad -40\,S_{1,1,5} - 32\,S_{1,2,4} + 24\,S_{1,3,3} + 32\,S_{1,4,2} - 32\,S_{2,1,4} + 20\,S_{2,2,3}\\
&\quad +40\,S_{2,3,2} + 4\,S_{2,4,1} + 24\,S_{3,1,3} + 44\,S_{3,2,2} + 24\,S_{3,3,1} + 36\,S_{4,1,2}\\
&\quad +36\,S_{4,2,1} + 24\,S_{5,1,1} + 80\,S_{1,1,1,4} - 16\,S_{1,1,3,2} + 32\,S_{1,1,4,1}\\
&\quad -24\,S_{1,2,2,2} + 16\,S_{1,2,3,1} - 24\,S_{1,3,1,2} - 24\,S_{1,3,2,1} - 24\,S_{1,4,1,1}\\
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&\quad -8\,\beta\,S_1\,S_3\,.
\end{aligned}
$$

The last term, with $\beta = \zeta_3$, is the contribution from the dressing factor that appears in the BAE at the fourth loop.

Twist-3 : features

The twist-3 anomalous dimension has two characteristic features:

1. All harmonic functions $S_{\vec{a}}$ are evaluated at half the spin, $S_{\mathbf{a}} \equiv S_{\mathbf{a}}(N/2)$. On the integrability side, this does not look unwarranted, since only *even* N belong to the non-degenerate ground state of the magnet.
2. No negative indices appear at twist-3, while in the case of twist-2 negative index sums were present starting from the second loop.

At the $N \to \infty$ limit, the *minimal* anomalous dimension γ (corresponding to the ground state) must exhibit the universal (LBK-classical) $\ln N$ behaviour which depends neither on the twist, nor on the nature of fields under consideration. Computing analytically the large N asymptotics yields

$$\frac{\gamma_3(N)}{\ln N} = 4\,g^2 - \frac{2\pi^2}{3}\,g^4 + \frac{11\pi^4}{45}\,g^6 - \left(4\zeta_3^2 + \frac{73\pi^6}{630}\right)\,g^8 + \mathcal{O}(g^{10})\,,$$

which matches the four-loop cusp anomalous dimension — the *physical coupling*. This is a non-trivial check, since the derivation was based on experimenting with finite values of the spin N.

Twist-3 : features

The twist-3 anomalous dimension has two characteristic features:

1. All harmonic functions $S_{\vec{a}}$ are evaluated at half the spin, $S_{\mathbf{a}} \equiv S_{\mathbf{a}}(N/2)$. On the integrability side, this does not look unwarranted, since only *even* N belong to the non-degenerate ground state of the magnet.
2. No negative indices appear at twist-3, while in the case of twist-2 negative index sums were present starting from the second loop.

At the $N \to \infty$ limit, the *minimal* anomalous dimension γ (corresponding to the ground state) must exhibit the universal (LBK-classical) ln N behaviour which depends neither on the twist, nor on the nature of fields under consideration. Computing analytically the large N asymptotics yields

$$\frac{\gamma_3(N)}{\ln N} = 4\,g^2 - \frac{2\pi^2}{3}\,g^4 + \frac{11\pi^4}{45}\,g^6 - \left(4\zeta_3^2 + \frac{73\pi^6}{630}\right)g^8 + \mathcal{O}(g^{10})\,,$$

which matches the four-loop cusp anomalous dimension — the *physical coupling*. This is a non-trivial check, since the derivation was based on experimenting with finite values of the spin N.

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Twist-3 : Evolution Kernel (rough)

After processing thru $\gamma = \mathcal{P}(N + \frac{1}{2}\gamma)$, in series in $g^2 = \frac{N_c \alpha}{2\pi}$,

$$
\begin{aligned}
P^{(1)} &= 4\, S_1, \\
P^{(2)} &= -2\, S_3 - 4\, \zeta_2\, S_1, \\
P^{(3)} &= S_5 + 2\, \zeta_2\, S_3 + 4\, \left(S_{3,2} + S_{4,1} - 2\, S_{3,1,1}\right) \\
&\quad + 4\, S_1 \left(2\, S_{3,1} - S_4 + 4\, \zeta_4\right) - 4\, S_1^2 \left(S_3 - \zeta_3\right).
\end{aligned}
$$

The fourth loop kernel we split into two terms: $P^{(4)} = P_S^{(4)} + P_\zeta^{(4)}$.

$$
\begin{aligned}
P_S^{(4)} &= -8\left[S_{3,3} + S_{1,5} + 2S_{2,4} - 4(S_{2,1,3} + S_{1,2,3} + S_{1,1,4}) + 8S_{1,1,1,3}\right] S_1 \\
&+ \frac{3}{2}\, S_7 - 16\left(S_{1,6} + S_{4,3}\right) - 24\left(S_{2,5} + S_{3,4}\right) \\
&\quad + 48\left(S_{1,1,5} + S_{1,3,3} + S_{3,1,3}\right) + 64\left(S_{2,2,3} + S_{2,1,4} + S_{1,2,4}\right) \\
&\quad - 128\left(S_{1,1,1,4} + S_{2,1,1,3} + S_{1,2,1,3} + S_{1,1,2,3}\right) + 256\, S_{1,1,1,1,3}\,, \\
P_\zeta^{(4)} &= 8\zeta_4\, \mathcal{S}_1^3 - 4\left[\zeta_2\zeta_3 + 8\zeta_5\right] \mathcal{S}_1^2 - \left[4(\zeta_3 + 2\beta)\mathcal{S}_3 + 49\zeta_6\right] \mathcal{S}_1 \\
&\quad + \left(8\mathcal{S}_{1,1,3} - 4\mathcal{S}_{1,4} - 4\mathcal{S}_{2,3} - \mathcal{S}_5\right)\zeta_2 - 8\mathcal{S}_3\, \zeta_4.
\end{aligned}
$$

RR harmonic functions

Let $\vec{m} = \{m_1, m_2, \ldots, m_\ell\}$, and examine the recurrence relation

$$\tilde{\Phi}_{b,\vec{m}}(x) = -[\Gamma(b)]^{-1} \frac{x}{x-1} \int_x^1 \frac{dz\,(z+1)}{z^2} \ln^{b-1} \frac{z}{x} \cdot \tilde{\Phi}_{\vec{m}}(z),$$

where the single index function coincides with the image of the standard harmonic sum,

$$\tilde{\Phi}_a(x) = [\Gamma(a)]^{-1} \frac{x}{x-1} \ln^{a-1} \frac{1}{x} = \tilde{\mathcal{S}}_a(x).$$

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At the base of the recursion, we have (the *weight* $w \equiv \tau - \ell$)

$$\tilde{\Phi}_a(x) = \left(-x\,\tilde{\Phi}_a(x^{-1})\right) \cdot (-1)^{a-1} \equiv \left(-x\,\tilde{\Phi}_a(x^{-1})\right) \cdot (-1)^{w[a]}.$$

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An iteration increases transcedentality $\tau = \sum_{i=1}^{\ell} |m_i|$ of the function by b, and the length ℓ of the index vector by one, so that

$$w[\vec{m}] + b - 1 = w[b, \vec{m}].$$

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Twist-3 : Evolution Kernel (beautified)

Then, in terms of the physical coupling,

$$\mathbf{g}_{\mathrm{ph}}^2 \equiv \frac{N_c\,\alpha_{\mathrm{ph}}}{2\pi} = g^2 - \zeta_2\, g^4 + \tfrac{11}{5}\zeta_2^2\, g^6 - \left(\tfrac{73}{10}\zeta_2^3 + \zeta_3^2\right) g^8 + \ldots,$$

the perturbative series for the kernel, $\mathcal{P} = \sum_{n=1} \mathbf{g}_{\mathrm{ph}}^{2n}\, \mathcal{P}_{\mathrm{ph}}^{(n)}$, becomes

$$\begin{aligned}
\mathcal{P}_{\mathrm{ph}}^{(1)} &= 4\,\mathcal{S}_1, \\
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\widehat{\mathcal{A}}_4 &= 2\,\widehat{\Phi}_{1,1,1,3} - \big(\widehat{\Phi}_{1,5} + \widehat{\Phi}_{3,3}\big) - \zeta_3\,\widehat{\mathcal{S}}_3, \\
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Twist-3 vis a vis Twist-2

This result can be compared with the evolution kernel that generates the twist-2 universal anomalous dimension :

$$\begin{aligned}
\mathcal{P}_{\rm ph}^{(1)} &= 4\,\mathcal{S}_1; \\
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\mathcal{P}_{\rm ph}^{(3)} &= 8\,\mathcal{S}_5 - 24\,\Phi_{1,1,1,-2} - 8\,\zeta_2\,\mathcal{S}_3 \\
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Remark : in general, the GL parity is

$$\tilde{\Phi}_{\vec{m}}(x) = \Big(-x\,\tilde{\Phi}_{\vec{m}}(x^{-1})\Big) \cdot (-1)^{w[\vec{m}]} \cdot (-1)^{\#\text{ of negative indices}}$$

since

$$\frac{x}{x-1} \Longrightarrow \frac{x}{x+1}$$

Logs in γ and $\mathcal{P}$

General structure of the RR Evolution Kernel

$$\mathcal{P}(N) = \mathcal{S}_1 \cdot \left(\alpha_{\text{ph}} + \widehat{\mathcal{A}}\right) + \mathcal{B}, \qquad \widehat{\mathcal{A}} = \mathcal{O}\left(1/N^2\right).$$

This feature is in a marked contrast with the anomalous dimension *per se*, whose large N expansion includes growing powers of $\log N$:

$$\gamma(N) = a \ln N + \sum_{k=0}^{\infty} \frac{1}{N^k} \sum_{m=0}^{k} a_{k,m} \ln^m N.$$

Easy to see from

$$\gamma_\sigma = \mathcal{P}(N + \sigma\gamma) \quad \Longrightarrow \quad \gamma_\sigma(N) = \sum_{k=1}^{\infty} \frac{1}{k!} \left(\sigma \frac{d}{dN}\right)^{k-1} \left[\mathcal{P}(N)\right]^k,$$

Physically, the reduction of singularity of the large N expansion shows that the tower of subleading logarithmic singularities in the anomalous dimension is actually *inherited* from the first loop — the LBK-classical $\gamma^{(1)} = \mathcal{P}^{(1)} \propto S_1$, and the RREE generates them automatically !

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Fresh results

- RRE as a natural consequence of the conformal invariance
  "Anomalous dimensions of high-spin operators beyond the leading order"
  Benjamin Basso & Gregory Korchemsky
  Nucl.Phys. **B775** *(07) 1 [hep-th/0612247]*
- *"$\mathcal{N}=4$ SUSY Yang–Mills: three loops made simple(r)"*
  D-r & Pino Marchesini *Phys.Lett.* **B 646** *(07) 189 [hep-th/0612248]*
- *"Anomalous dimensions at twist-3 in the sl(2) sector of $\mathcal{N}=4$ SYM"*
  Matteo Beccaria *JHEP* **0706** *(07) 044 [0704.3570]*
- Bethe Ansatz fails ("maximally") at 4 loops for twist-2
  "Dressing and Wrapping"
  Kotikov, Lipatov, Rej, Staudacher & Velizhanin
  J.Stat.Mech. **0710** *(07) P10003 [0704.3586]*
- twist-3 gaugino = twist-2 "universal"
  "Universality of three gaugino anomalous dimensions in $\mathcal{N}=4$ SYM"
  Beccaria *JHEP* **0706** *(07) 054 [0705.0663]*
- *"Twist 3 of the sl(2) sector of $\mathcal{N}=4$ SYM and reciprocity respecting evolution"*
  Beccaria, D-r & Marchesini *Phys.Lett.* **B652** *(07) 194 [0705.2639]*

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$$\frac{\text{clever 2nd loop}}{\text{clever 1st loop}} < 2\% \qquad \left(\begin{array}{c} \text{Heavy quark fragmentation} \\ \text{D-r, Khoze \& Troyan , PRD 1996} \end{array} \right)$$

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Employ $\mathcal{N} = 4$ SYM to simplify the essential part of the QCD dynamics

Conclusions

- A steady progress in high order perturbative QCD calculations is worth accompanying by reflections upon the origin and the structure of higher loop correction effects
- Reformulation of parton cascades in terms of Gribov–Lipatov reciprocity respecting evolution equations (RREE)
  - reduces complexity by (at leat) an order of magnitude
  - improves perturbative series (less singular, better "converging")
  - links interesting phenomena in the DIS and e^+e^- annihilation channels
- The Low theorem should be part of theor.phys. curriculum, worldwide
- Complete solution of the $\mathcal{N}=4$ SYM QFT should provide us with a *one-line-all-orders* description of the major part of QCD parton dynamics
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  - reduces complexity by (at leat) an order of magnitude
  - improves perturbative series (less singular, better "converging")
  - links interesting phenomena in the DIS and e^+e^- annihilation channels
- The Low theorem should be part of theor.phys. curriculum, worldwide
- Complete solution of the $\mathcal{N}=4$ SYM QFT should provide us with a *one-line-all-orders* description of the major part of QCD parton dynamics
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EXTRAS

Extras

small x : 2 puzzles

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DIS (space-like evolution). Look at small x that is, $N \ll 1$

$$\text{BFKL} \quad : \quad \gamma_N = \frac{\alpha_s}{N} + \left(\frac{\alpha_s}{N}\right)^2 + \left(\frac{\alpha_s}{N}\right)^3 + \left(\frac{\alpha_s}{N}\right)^4 + \ldots$$

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so-called "Malaza puzzle"

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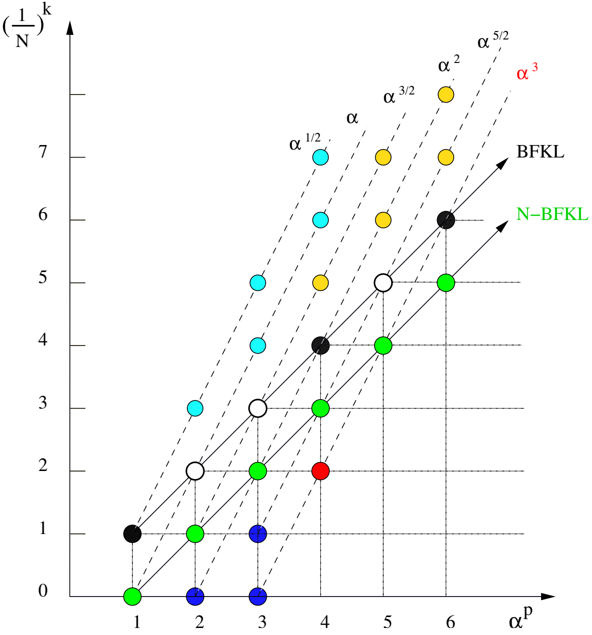
small x chart



Solid – BFKL (black) and N-BFKL (green) known in all orders.

Dashed blue – γ_+ terms generated by α/N and α.

Yellow – unknown.

physical coupling

$$A = \sum_{1}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^n A_n, \qquad \frac{A^{(g)}}{C_A} = \frac{A^{(q)}}{C_F} \qquad P_{a\to a[x]+g}(x) = \frac{A(\alpha_s)}{1-x}$$

$$\begin{aligned}
\frac{A_1}{C} &= 4 \\
\frac{A_2}{C} &= 8\left[\left(\frac{67}{18} - \zeta_2\right) C_A - \frac{5}{9}\, n_f\right] \\
\frac{A_3}{C} &= 16 C_A^2 \left(\frac{245}{24} - \frac{67}{9}\,\zeta_2 + \frac{11}{6}\,\zeta_3 + \frac{11}{5}\,\zeta_2^2\right) \\
&\quad +16 C_F n_f \left(-\frac{55}{24} + 2\,\zeta_3\right) \\
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= *universal* magnitude of double-log enhanced contributions.

Enters in :

large-N asymptotics of anomalous dimensions *and* coefficient functions,

Sudakov quark and gluon form factors,

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threshold resummation,

singular ($x \to 1$) part of the Drell–Yan K-factor,

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non-diagonal transitions

Second loop $G \to G$ [quark box] $(n_f T_R C_F)$

$$P_G^{(S)} = 8x - 16 + \frac{20}{3}x^2 + \frac{4}{3}x^{-1} - (6 + 10x)\ln x - 2(1+x)\ln^2 x,$$

$$P_G^{(T)} = 12x - 4 - \frac{164}{9}x^2 + \frac{92}{9}x^{-1} + (10 + 14x + \frac{16}{3}[x^2 + x^{-1}])\ln x + 2(1+x)\ln^2 x;$$

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2- and 3-prong colour antennae are sort of "trivial": coherence being taken care of, the answers turned out to be essentially additive

The case of $2 \to 2$ hard parton scattering is more involved (4 emitters), especially so for gluon–gluon scattering.
Here one encounters 6 (5 for $SU(3)$) colour channels that mix with each other under soft gluon radiation

The difficult quest of sorting out large angle gluon radiation in all orders in $(\alpha_s \log Q)^n$ was set up and solved by George Sterman and collaborators.

Recent (fall 2005) addition to the problem (G.Marchesini & YLD)

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$$\frac{\partial}{\partial \ln Q} M \propto \left\{ -N_c \ln\left(\frac{t\,u}{s^2}\right) \cdot \hat{\Gamma} \right\} \cdot M, \qquad \hat{\Gamma} V_i = E_i V_i .$$

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Three "ain't-so-simple" ones were found to satisfy the cubic equation:

$$\left[E_i - \frac{4}{3}\right]^3 - \frac{(1+3b^2)(1+3x^2)}{3}\left[E_i - \frac{4}{3}\right] - \frac{2(1-9b^2)(1-9x^2)}{27} \; = \; 0,$$

where

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Mark the *mysterious symmetry* w.r.t. to $x \to b$: interchanging internal (group rank) and external (scattering angle) variables of the problem . . .

Integrability

1. anomalous dimensions $\Rightarrow$ eigenvalues of the dilatation operator
2. subset of composite operators su(2) = trace(XXXYYXYXXXYYY) can be mapped onto a spin 1/2 system (X = spin up, Y = spin down)
3. At one loop, it is the Hamiltonian of the integrable XXX spin 1/2 chain
4. At higher loops, a more complicated spin chain, but with spins interacting at neighbouring sites (up to a certain distance)
5. At all loops, there are conjectures for the all loop spin Hamiltonian, exploiting the string results, assuming AdS/CFT duality.
6. Integrability = an infinite number of invariants (conserved quantities).