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On the integrability of correspondences associated to integral curves

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Abstract

We analyze the integrability of birational maps of the plane having rational invariants of various degrees. We show by explicit examples that the maps turn out to be additions on elliptic curves. We also examine the correspondences defined by the integral curves.

Keywords: Integrable discrete systems

1. Introduction

A number of tools have been derived in the recent years to analyze the integrability of discrete systems. Most of the effort has been put on characterization of integrability. We may quote the singularity confinement [1], the Nevanlinna analysis [2], the complexity approach [3–6], and recently a new arithmetic test [7]. All these tests "measure" the integrability of maps. What we want to do here concerns maps of the 2-dimensional plane which are known to be integrable. Our aim is consequently not to detect their integrability, but to further describe some of their properties, and propose new ways of constructing correspondences from their integral curves.

One of the very first results on integrable mappings (which was obtained before any integrability detector had been proposed and thus served as testing ground for these subsequent developments) was the one of Quispel et al. [8,9]. These authors introduced a five-parameter family of second order mappings of the form:

$$x_{n+1} = \frac{f_1(x_n) - x_{n-1} f_2(x_n)}{f_2(x_n) - x_{n-1} f_3(x_n)}, \tag{1}$$

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where the f_i are specific polynomials of order not higher than four. This, so-called symmetric, QRT mapping possesses an invariant of the form:

$$(\alpha_0 + K\alpha_1)x_{n+1}^2 x_n^2 + (\beta_0 + K\beta_1)x_{n+1}x_n(x_{n+1} + x_n) + (\gamma_0 + K\gamma_1)\big(x_{n+1}^2 + x_n^2\big) \\ + (\epsilon_0 + K\epsilon_1)x_{n+1}x_n + (\zeta_0 + K\zeta_1)(x_{n+1} + x_n) + (\mu_0 + K\mu_1) = 0, \tag{2}$$

where K plays the role of the integration constant. Moreover, it was shown that the solution of the mapping can be expressed in terms of elliptic functions, of which it is just a sampling over a discrete, equidistant set of points.

A generalisation of the mapping (1) to an 8-parameter one has been proposed by QRT under the name of "asymmetric". It is a system of two first-order mappings and possesses an invariant which is a ratio of two biquadratic polynomials. Its integration was given recently [10–12] and again it turned out that the solution can be expressed in terms of elliptic functions. Further extensions of the QRT mapping were proposed by Roberts and Iatrou in [10,11].

The use, a posteriori, of the integrability criteria on the QRT mapping has given the expected results. Both symmetric and asymmetric QRT mappings possess the singularity confinement property and have zero entropy: the degree growth of the iterates is quadratic [13]. The arithmetic test of [7] also agrees.

It was shown that the symmetric QRT mapping is the only one that satisfies the singularity confinement criterion under some assumptions concerning the structure of singularity patterns [14]. No analogous result exists for the asymmetric case. Indeed there exist many more integrable equations written as systems of two first order mappings than the ones captured by the asymmetric QRT parametrization. A recent paper on spin edge models offers a wealth of such examples [15].

2. A possibly non-QRT integrable map of the plane

While investigating third-order mappings, Kimura et al. [16] obtained systems which could be integrated to second-order mappings which possessed a biquartic invariant. Here is an example:

$$(x_n x_{n+1} - 1)(x_n x_{n-1} - 1) = \frac{(x_n - a)(x_n - 1/a)(x_n^2 - 1)}{p^2 x_n^2 - 1} \tag{3}$$

with invariant

$$K = \frac{((x_n - x_{n-1})^2 - p^2(x_n x_{n-1} - 1)^2)((x_n + x_{n-1} - a - 1/a)^2 - p^2(x_n x_{n-1} - 1)^2)}{(x_n x_{n-1} - 1)^2}. \tag{4}$$

Relation (3) may be put in the form (1) but the polynomials f_i are not of the specific form required for QRT mapping.

Many more mappings with biquartic invariants were obtained in [14] through autonomisation of discrete Painlevé equations. As explained above, the integration of the QRT mapping was given explicitly in terms of elliptic functions. With the existence of mappings with biquartic invariants, it was natural to wonder as to the nature of their solutions and to the precise method of their integration. First note that the mapping is an automorphism of infinite order. We thus expect the invariant curve

$$\big((x - y)^2 - p^2(xy - 1)^2\big)\big((x + y - b)^2 - p^2(xy - 1)^2\big) - K(xy - 1)^2 = 0, \tag{5}$$

where $b = a + 1/a$, to be of genus 0 or 1. Computing the genus of this curve can be performed following the algorithm proposed by van Hoeij [17]. It turns out that the genus of (5) is 1. This curve is birationally equivalent to a curve of the form $v^2 - 4u^3 + \alpha u + \beta = 0$. The precise method for the construction of the canonical form follows again the method proposed by van Hoeij. One chooses a point x, y which in turns fixes the value of K. Following the algorithm of [17] one constructs $u(x, y)$ and $v(x, y)$ which satisfy the canonical relation

$$v^2 - 4u^3 + \alpha u + \beta = 0. \tag{6}$$

This last relation is parametrized in terms of the Weierstraß elliptic function $u = \wp$ and its derivative $v = \wp'$. The action of the initial mapping interpreted at the level of the canonical form (6) is just a shift from $\wp(z)$ to $\wp(z + \delta)$ (where the step δ is not curve-independent). One can also give a nice geometrical construction of the point of coordinates $(\wp(z + \delta), \wp'(z + \delta))$ once the points $(\wp(z), \wp'(z))$ and $(\wp(\delta), \wp'(\delta))$ are known. This construction is just the geometrical interpretation of the well-known identity for Weierstraß functions

$$\begin{vmatrix} \wp(a) & \wp'(a) & 1 \\ \wp(b) & \wp'(b) & 1 \\ \wp(c) & \wp'(c) & 1 \end{vmatrix} = 0 \tag{7}$$

for $a + b + c = 0$.

Once the solution of (6) in terms of elliptic functions is given, one can construct the parametrization of the initial curve (5) using the inverse transformation $x = x(u, v)$, $y = y(u, v)$. This construction is also obtained through the van Hoeij algorithm. We have performed the derivation of these transformations for various choices of curves, i.e., for various values of the invariant K. None will be exhibited here: they fill a number of pages and can be performed only with the help of efficient symbolic computation programs. Still, the important result is that they do exist: they show that the solutions of the mapping (3) can be given in terms of elliptic functions.

As a matter of fact all algebraically integrable mappings can be treated this way (see [15] for another example).

3. An example amenable to QRT form

We start from a map having an invariant of degree[1] higher than that of QRT, taken from [5,18]. Consider the birational transformations on $\mathbb{C}\mathrm{P}^2$, written in homogeneous coordinates $[x, y, z]$:

$$\begin{aligned} x &\to 2yz + (q^2 - 1)xz + (q^2 - 1)xy, \\ y &\to 2yz + (q - 1)xz - (q + 1)xy, \\ z &\to 2yz - (q + 1)xz + (q - 1)xy. \end{aligned} \tag{8}$$

The invariant can in this case be written

$$K = \frac{(y + z)(x(y + z) - 2yz)(2x - y - z)}{(y - z)^2(x(y + z)(q^2 - 2) + 2x^2 + 2yz)}. \tag{9}$$

A first change of coordinates

$$[x, y, z] \to [X, Y, Z] = [y(x - y), (x - y)(y - z), y(y - z)]$$

brings the invariant to the form

$$K = \frac{(2X - Y)(-2X + Y - Z)(2X + Z)}{-2XYq^2Z + q^2Y^2Z - 2Y^2Z - 2Z^2Xq^2 + q^2YZ^2 - 2XY^2}. \tag{10}$$

We introduce inhomogeneous variables through $[X, Y, Z] \to [\xi + 1/2, 1, \eta]$ and write the integral curve:

$$(-2\xi q^2\eta - 2\eta - 2q^2\xi\eta^2 - 2\xi - 1)K + 2\xi(2\xi + \eta)(2\xi + 1 + \eta) = 0. \tag{11}$$

[1] Of course, the degree of an invariant is not a canonical notion and may be affected by changes of coordinates.

It is now easy to transform (11) to the Weierstraß canonical form $v^2 - 4u^3 + \alpha u + \beta = 0$, via the transformation

$$u = -\frac{(4K^2q^2 - 20K - 1 + 2Kq^2 - K^2q^4)\xi + 6K(-1 + Kq^2)(1 + 2\eta)}{3\xi}, \tag{12a}$$

$$v = \frac{4K(-1 + Kq^2)(Kq^2 - 8K - 1)\xi - 4K^2(-1 + Kq^2)(1 + 2\eta)}{\xi^2}. \tag{12b}$$

The inverse transform can also be easily computed. The action of the mapping, seen on the Weierstraß canonical form, is just the addition of the point:

$$u = 1/3 + q^2(q + 10)(q + 2)K^2/3 - 2\big(q^2 + 6q + 2\big)K/3, \tag{13a}$$

$$v = 8(q + 2)^2q^3K^3 - 16\big(q^2 + 2q + 2\big)qK^2 + 8Kq. \tag{13b}$$

As a matter of fact, the transformation of the cubic integral curve to a canonical Weierstraß form can be performed in a way simpler in principle, albeit more complicated in practice, involving just homographic transformations.

The remarkable result is that it is possible to express the solutions of the mapping (8) rationally in terms of elliptic functions. This suggests that the mapping is just a QRT in disguise.

This turns out to be the case indeed. Taking $y = 1$ from the onset we obtain two first-order mappings for x and z and eliminating x we find the second-order recursion:

$$z_{n+1}z_{n-1}\big(z_n(1 - q)^2 - (1 + q)^2\big) - (z_{n+1} + z_{n-1})(z_n - 1)\big(1 - q^2\big) + z_n(1 + q)^2 - (1 - q)^2 = 0. \tag{14}$$

We can further simplify (14) by setting

$$z = \frac{1 - w\sqrt{q}}{1 + w\sqrt{q}}$$

and finally find:

$$(w_{n-1}w_n - 1)(w_nw_{n+1} - 1) = 1 - qw_n^2 \tag{15}$$

a mapping which is QRT, in one of the canonical forms given in [9].

At this point one may wonder whether all integrable second order mappings with a rational invariant can be brought to a QRT form by a birational change of coordinates of the 2-plane. The question is open at this moment.

4. From invariants curves to correspondences

Up to this point we have considered a relation like (5) as an invariant curve associated to some mappings, like Eq. (3). This means that the evolution, i.e., the computation of the iterates is given by the mapping itself. However, there exists another interpretation of (5) and, as a matter of fact, of (2) as well. These relations can be considered as defining correspondences. In this case one starts with a given x and solves the invariant equation (here Eq. (5)) for y. Since the relation is not linear in y one obtains more than one solution for y. Next, one injects these values of y into the equation and solves for x. Again more than one solutions result (one of which is the value of x at the previous step) and so on. Geometrically, this construction means that one intersects the curve defined by the invariant first with a vertical line then by horizontal ones at each intersection point, then vertical and so on. Clearly this is a different kind of evolution than the one defined by the mapping, although the latter is one of the solutions of the correspondence.

Correspondences appear naturally in various settings. For instance, while performing the duality transformations proposed in [19] which converts constants of motions to coupling constants and vice versa, one may well end up with the dual of some mapping being a correspondence. Correspondences may be obtained from the application of Miura transformations to discrete Painlevé equations [20]. More naively, if one eliminates one variable in a two-component, asymmetric QRT mapping, one usually ends up with a correspondence. In what follows we shall address the question of the integrability of such correspondences. To be more precise, is the correspondence obtained from the invariant curves of an integrable mapping (as we described above), integrable *per se*?

The simplest example of a correspondence is provided by the invariant of the symmetric QRT mapping. Starting with initial values $x = x_n$, $y = x_{n+1}$, we compute the value of the invariant $K(x, y)$. To iterate we look for the w's such that $K(x, w) = K(x, y)$. Since K is the ratio of quadratic polynomials in w there exist two solutions. One is clearly $w = y = x_{n+1}$ and the other solution is x_{n-1}. (Invariance means $K(x_n, x_{n+1}) = K(x_{n-1}, x_n)$ and by symmetry the latter is $K(x_n, x_{n-1})$.) The correspondence leads to the set $\{x_{n-1}, x_{n+1}\}$ at this stage. From $y = x_{n+1}$ we start from $K(y, v) = K(x, y)$ and get two solutions, one being $y = x_{n+2}$ and the other x_n, by symmetry. From x_{n-1} we find similarly x_{n-2} and once more x_n leading to the set $\{x_{n-2}, x_n, x_{n+2}\}$. Proceeding in the same way it is clear that the number of images grows linearly with the number of iterations. Applying the criterion of slow (i.e., polynomial) growth of the number of images predicts the integrability of the correspondence. This is in agreement with the results of Veselov who examined the integrability of the biquadratic 2–2 correspondence also from the point of view of the growth of the number of images. The case of the correspondence associated to the asymmetric QRT mapping was treated in [21]. It was again shown that, with the appropriate interpretation of the evolution, this correspondence is integrable like its symmetric counterpart.

We examine now the biquartic case (5). Starting from a given point (x, y) and iterating, following the procedure described above we find that the successive number of images is $4, 13, 40, 121, 364, \ldots$. These numbers, manifestly, follow the recursion relation $N_{n+1} = 3N_n + 1$ which is the maximal growth one can obtain in the biquartic case. This exponential growth of the number of images is an indication of the non-integrability of (5) considered as a correspondence. The fact that there exists a parametrization of (5) through elliptic functions is not in contradiction with the non-integrability: the elliptic function solution describes just one branch among the exponentially many branches of the evolution (5).

Next we turn to the case of the invariant (10) and try to introduce a correspondence in an appropriate way. We start by remarking that the pencil of cubics (10) has eight base points (i.e., points which are common to all curves of the pencil). To each of the base point we may associate a rational involution in the following way: choose a base point B of the pencil of cubics. Consider a running point $M = [X, Y, Z]$. Through M passes one curve Σ of the pencil (10) and one straight line BM. Both intersect at a unique third point, given rationally in terms of $[X, Y, Z]$. The map constructed in this way is a (bi)-rational involution on the plane.

The base points are

$$[1/2 - 1/2q, -q, -1 + q], \quad [-1/2 - 1/2q, -q, 1 + q], \quad [-1, -2, 1],$$
$$[-1/2 - 1/2q, -q, 1], \quad [1/2q - 1/2, q, 1], \quad [-1, -1, 1], \quad [0, 1, 0], \quad [0, 0, 1].$$

From these points, we get eight involutions $(i_1, \ldots, i_8)$, which are not independent, and can all be expressed in terms of i_1, i_2, i_3.

$$\begin{aligned} i_1(M) = \big[&(2qX - qY + Y)(2X - qY - qZ - Y + Z), -2(2qX - qY + Y)q(Y + Z), \\ &- 4XY + 4q^2XZ + 4Xq^2Y - 2q^2YZ - 2Y^2q^2 + 2Y^2 - 2YZ\big]. \end{aligned}$$

i_2 is obtained from i_1 by the substitution $q \leftrightarrow -q$, and

$$i_3(M) = [X - Y, -Y, Y + Z].$$

These involutions verify the following relations:

$$i_1 \cdot i_2 = i_2 \cdot i_1, \qquad i_4 = i_3 \cdot i_1 \cdot i_3, \qquad i_5 = i_1 \cdot i_2 \cdot i_3 \cdot i_1 \cdot i_3,$$
$$i_6 = i_1 \cdot i_3 \cdot i_2 = i_2 \cdot i_3 \cdot i_1, \qquad i_7 = i_3 \cdot i_2 \cdot i_3 \cdot i_1 \cdot i_3, \qquad i_8 = i_1 \cdot i_2 \cdot i_3.$$

The two infinite order maps one may construct form the i's, namely $\varphi = i_1 \cdot i_3$ and $\psi = i_2 \cdot i_3$ are not independent. They verify $\varphi^2 = \psi^2$. The group generated by $i_1, \ldots, i_8$ is made of the iterates of φ and its inverse, dressed with a finite group.

It is interesting to see what the straight lines we just used are in the original coordinate system $[x, y, z]$. Since the coordinate transformation is quadratic, the lines become conics: to each of the eight pencils of lines in $[X, Y, Z]$ is associated a pencil of conics in $[x, y, z]$. These conics pass through some base points of the original pencil of invariant curves (9). Through any generic point $m = [x, y, z]$ of the plane, passes one conic of each of the pencil of conics and one of the invariant curves (9). They intersect at m, some fixed base points and at a single other point, which is given rationally in term of m. The latter my be taken as the image of m, and the correspondences reduce to maps.

Another example can be provided by the integral curve obtained in [15] where various spin edge models were analyzed:

$$b(x-y)^2(x+y)^2 - \left(x^3 + ax - ax^2y - y\right)\left(ax + xy^2 - y - ay^3\right) = 0. \tag{16}$$

The growth of the number of images is identical to the one obtained in the case of (5). However there exists a way to appropriately define a correspondence for this integral curve as well. Consider the linear pencil of invariant curves (16), which has eight base points (with $\mathrm{i}^2 = -1$):

$$[1, 0, 0], \quad [0, 0, 1], \quad [a, 0, 1], \quad [0, 1, 0], \quad [1, -\mathrm{i}, -1], \quad [1, \mathrm{i}, -1], \quad [1, -1, 1], \quad [1, 1, 1].$$

It is possible to construct two basic correspondences as follows.

Take the straight line D passing through the base point $[a, 0, 1]$ and a running point $m = [x, y, z]$ of P_2. Through the point m passes one curve Σ of the pencil (16). The line D cuts the curve Σ in the six following points.

$$[a, 0, 1], \quad m, \quad i_1(m), \quad i_2(m), \quad i_3(m),$$

where the point $[a, 0, 1]$ counts twice since it is a double point of Σ, and $i_1(m)$, $i_2(m)$, and $i_3(m)$ define a three-valued rational correspondence.

$$\begin{aligned}
i_1(m) &= \left[z(-az + x), -y(xa - z), (-az + x)x\right],\\
i_2(m) &= \left[-y^2a^2x - xy^2 + x^2z - 2z^2xa + 2zy^2a + z^3a^2, \left(-z^2a - a^2zx + xz + y^2a^2 - y^2 + ax^2\right)y,\right.\\
&\qquad \left. y^2z + zy^2a^2 - 2xay^2 - z^2xa^2 + 2x^2az - x^3\right],\\
i_3(m) &= \left[y^2az^2 + y^2a^3z^2 + x^4 - 3x^3az + 3x^2a^2z^2 + 2x^2ay^2 - xz^3a^3 - xy^2z - 3xzy^2a^2,\right.\\
&\qquad \left(x^3 - x^2az - xa^3y^2 - z^2xa^2 + xay^2 + z^3a^3 + zy^2a^2 - y^2z\right)y,\\
&\qquad \left. -3xz^3a^2 + x^2y^2a^2 + z^4a^3 + 2z^2y^2a^2 - xza^3y^2 + x^2y^2 + 3x^2az^2 - 3xzy^2a - zx^3\right].
\end{aligned}$$

The three maps i_1, i_2, i_3 are rational involutions. They commute since

$$i_1 \cdot i_2 \cdot i_3 = 1 = i_1^2 = i_2^2 = i_3^2.$$

Consider now the straight lines passing through the base point $[0, 1, 0]$. A similar construction yields another three valued rational correspondence, with images $k_1(m)$, $k_2(m)$, and $k_3(m)$:

$$\begin{aligned}
k_1(m) &= [x, -y, z],\\
k_2(m) &= \left[y(xa - z)x, z(-az + x)x, y(xa - z)z\right],\\
k_3(m) &= \left[-y(xa - z)x, z(-az + x)x, -y(xa - z)z\right].
\end{aligned}$$

Again

$$k_1 \cdot k_2 \cdot k_3 = 1 = k_1^2 = k_2^2 = k_3^2.$$

There are simple commutation relations between the two correspondences: k_1 commutes with k_2, k_3, i_1, i_2, i_3, and i_1 commutes with i_2, i_3, k_1, k_2, k_3. The group generated by the i's and the k's is the product of the iterates of the infinite order birational mapping $i_2 \cdot k_2$ and its inverse $k_2 \cdot i_2$, with the finite (commuting) group generated by i_1 and k_1. From the invariant curves (16), we have defined a rational integrable correspondence which reproduces the original map of Ref. [14].

5. The commutation criterion for correspondences

In [22], Veselov has proposed another criterion for the integrability of correspondences: if given a correspondence $\Phi(x, y) = 0$ one can find another correspondence $\Psi(x, y) = 0$ which commutes with it, then both are integrable. The example used in [22] in order to illustrate this approach is constructed from modular functions. Let $\Phi_n(x, y) = 0$ be the modular equation satisfied by $x = j(z)$, $y = j(nz)$ where $j(z)$ is the modular function (see, for example, the lecture notes of I. Dolgachev [23], with a misprint in Φ_3 corrected below). The equations for $n = 2$ and $n = 3$ are, respectively

$$\begin{aligned}\Phi_2(x, y) = {} & x^3 + y^3 - x^2y^2 + 2^4 \cdot 3 \cdot 31xy(x + y) - 2^4 \cdot 3^4 \cdot 5^3\left(x^2 + y^2\right) + 3^4 \cdot 5^3 \cdot 4027xy \\ & + 2^8 \cdot 3^7 \cdot 5^6(x + y) - 2^{12} \cdot 3^9 \cdot 5^9 \end{aligned} \tag{17}$$

and

$$\begin{aligned}\Phi_3(x, y) = {} & x^4 + y^4 - x^3y^3 + 2^3 \cdot 3^2 \cdot 31\left(x^3y^2 + y^3x^2\right) - 2^2 \cdot 3^3 \cdot 9907\left(x^3y + y^3x\right) \\ & + 2 \cdot 3^4 \cdot 13 \cdot 193 \cdot 6367x^2y^2 + 2^{16} \cdot 3^5 \cdot 5^3 \cdot 17 \cdot 263\left(x^2y + xy^2\right) \\ & + 2^{15} \cdot 3^2 \cdot 5^3\left(x^3 + y^3\right) + 2^{30} \cdot 3^3 \cdot 5^6\left(x^2 + y^2\right) - 2^{31} \cdot 5^6 \cdot 22973xy + 2^{45} \cdot 3^3 \cdot 5^9(x + y). \end{aligned} \tag{18}$$

One can verify in a straightforward way that Φ_2 commutes with Φ_3: this is equivalent to checking that the resultant of $\Phi_2(x, t)$ and $\Phi_3(y, t)$ with respect to t is symmetric in (x, y). In fact, any Φ_n commutes with any Φ_m since they form a non-linear realization of the multiplication of integers. According to the commutation criterion of Veselov these correspondences are thus integrable. This is in agreement with the fact that they can be parametrized in terms of modular functions. On the other hand if we apply the criterion of the number of images to Φ_2, say, we find the sequence $N = 3, 7, 15, 31, 63, 127, \ldots$ which clearly exhibits exponential growth ($N_n = 2N_{n-1} + 1$). Thus we are here in the presence of two different criteria which yield different answers. One would be tempted to adopt the commutation criterion of Veselov but we must remark that: (a) no constructive method for the derivation of the commuting correspondence is proposed, and (b) the case at hand is very special in the sense that the curves $\Phi_n(x, y)$ have a rational parametrization. The cases we examined above are not rational curves and thus, the parallel to Veselov's case is questionable.

6. Perspectives

We have shown how to explicitly integrate a second order mapping with rational invariant in terms of elliptic functions, *whatever the degree of the invariant is*. The question of whether all these mappings can be transformed into QRT ones (with bi-quadratic invariants) remains open at this stage. We have also shown that, although the curve is associated to the invariant of some (integrable) mapping, the natural correspondences are generally not integrable. This result was obtained through the study of the growth of the number of images of a given point. In the case of modular functions, we have analyzed the integrability of correspondences using also the criterion proposed by Veselov. The result was that the two criteria give contradicting answers. This is an indication that they may be related to different kinds of integrability, and such a possibility will be studied elsewhere.

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