HAMILTONIAN STRUCTURES AND LAX EQUATIONS ☆

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We show that any hamiltonian system, which is integrable in the sense of Liouville, admits a Lax representation, at least locally at generic points in phase space. We introduce the most general Poisson bracket ensuring the involution property of the integrals of motion and existence of a Lax pair. We give examples of the structure we describe.

1. Introduction

Lax pairs are the main tool available in the present to produce equations of evolution possessing conserved quantities [1]. Their existence does not refer a priori to any symplectic structure or Poisson bracket. However, the use of Lax pairs in the realm of hamiltonian systems [2], proves extremely useful to produce systems which are integrable in the sense of Liouville [3,4]. We will place ourselves in this setting, and consider hamiltonian systems. A Lax pair L, M then consists of two functions on the phase space of the system, with values in some Lie algebra $\mathscr{G}$, such that the (hamiltonian) evolution equations may be written

$$\frac{dL}{dt} = [L, M] \tag{1}$$

($[\ ,\ ]$ denotes the bracket in the Lie algebra $\mathscr{G}$). We will denote by G the connected Lie group having $\mathscr{G}$ as a Lie algebra.

The interest in the existence of such a pair originates in the fact that it allows for an easy construction of conserved quantities. Indeed, the solution of eq. (1) is of the form

$$L(t) = g^{-1}(t)L(0)g(t) ,$$

where $g(t) \in G$ is determined by the equation

$$M = g^{-1}\frac{dg}{dt} .$$

It follows that if I is an Ad-invariant function on $\mathscr{G}$, then $I(L(t))$ is a constant of the motion. Integrability of the system in the sense of Liouville demands that these conserved quantities be in involution, i.e. Poisson commute.

It is an open question to know if all integrable systems have an associated Lax pair, and what is the degree of generality of the algebraic structures commonly used in the domain, as for example the r-matrix and the classical Yang–Baxter equation [2,5,6].

We first show that any hamiltonian system with a finite number of degrees of freedom, which is integrable in the sense of Liouville, admits a Lax pair, at least locally at generic points in phase space. Conversely, we introduce the most general Poisson bracket ensuring the involution property and existence of a Lax pair. We show that standard Poisson brackets like the Kirillov and the r-matrix brackets, are special cases of the structure we describe. We also present two especially interesting examples, related to the theory of multihamiltonian systems.

We will not dwell here on the infinite dimensional case, although the generality and the algebraic nature of the structures we describe guarantee that they will bear on the case. These structures actually already appeared precisely in the same form in the work of

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Maillet [7] on non-ultralocal two-dimensional field theories. More examples will appear shortly [8].

2. Existence of a Lax pair

Consider a finite dimensional hamiltonian system, with n degrees of freedom, Poisson bracket $\{\ ,\ \}$ and hamiltonian h. Suppose it is integrable in the sense of Liouville, i.e. it possesses n integrals of the motion F_i, $i=1, ..., n$, which are in involution. The Liouville theorem states that there exists, at least locally and outside of critical points, a system of conjugate coordinates I_i, θ_i, $i=1, ..., n$, where the I_j are functions of the F's only. In these coordinates, the equations of motion take the very simple form

$$\dot{I}_j=0\,, \tag{2}$$

$$\dot{\theta}_j=\frac{\partial h}{\partial I_j}\,. \tag{3}$$

To prove the existence of a Lax pair, it is sufficient to exhibit one such pair. This is straightforward in the action-angle coordinate system as we now show. Introduce the Lie algebra $\mathscr{G}$ generated by $\{H_i, E_i, i=1, ..., n\}$ with relations

$$[H_i, H_j]=0\,,$$
$$[H_i, E_j]=2\delta_{ij}E_j\,,$$
$$[E_i, E_j]=0\,. \tag{4}$$

Set

$$L=\sum_{j=1}^{n} I_jH_j+2I_j\theta_jE_j\,,$$

$$M=\sum_{j=1}^{n}\frac{\partial h}{\partial I_j}E_j\,.$$

The equation

$$\dot{L}=[L,M]$$

is equivalent to eq. (2,3).

Remarks.

(1) A Lax pair is by no means unique: even the Lie algebra $\mathscr{G}$ may be changed.

(2) The Lie algebra $\mathscr{G}$ given by eq. (4) has a natural representation by $2n\times 2n$ matrices.

(3) There is a natural gauge transformation group acting on the Lax pair:

$$L\to g^{-1}Lg\,,$$

$$M\to g^{-1}Mg+g^{-1}\frac{\mathrm{d}g}{\mathrm{d}t}\,,$$

where g is a G-valued function on phase space.

3. Poisson structure

A Lax pair provides us with conserved quantities without referring to a Poisson structure. The notion of Liouville integrability requires the knowledge of a Poisson structure together with the involution property of the conserved quantities. We shall now describe the general form of Poisson structures which ensure the involution property for the conserved quantities furnished by the Lax pair. Suppose we are given a Lax pair L, M in some matrix representation of some Lie algebra $\mathscr{G}$. Assume that the matrix L may be brought to a diagonal form Λ by some gauge transformation. In other words we have the matrix relation

$$L=S^{-1}\Lambda S\,. \tag{5}$$

The matrix elements λ_k of the diagonal matrix Λ are the conserved quantities. We will not care here about the independence of these quantities.

Let us introduce some notations. Let X_μ be a basis of the Lie algebra $\mathscr{G}$. We can write

$$L=\sum_\mu L^\mu X_\mu\,.$$

The L^μ are functions on phase space. We may evaluate the Poisson brackets $\{L^\mu, L^\nu\}$ and gather the results as follows: set

$$L_1=L\otimes 1=\sum_\mu L^\mu(X_\mu\otimes 1)\,,$$

$$L_2=1\otimes L=\sum_\mu L^\mu(1\otimes X_\mu)\,,$$

$$\{L_1, L_2\}=\sum_{\mu\nu}\{L^\mu, L^\nu\}X_\mu\otimes X_\nu\,,$$

and if $\alpha\in\mathscr{G}\otimes\mathscr{G}$, denote

$$\alpha=\alpha_{12}=\sum_{\mu\nu}\alpha^{\mu\nu}X_\mu\otimes X_\nu\,,$$

$$\alpha_{21}=\sum_{\mu\nu}\alpha^{\mu\nu}X_\nu\otimes X_\mu\,.$$

Proposition. The involution property of the eigenvalues of L is equivalent to the existence of functions over phase space a and b with values in $\mathcal{G}\otimes\mathcal{G}$ such that:

$$\{L_1, L_2\} = [a_{12}, L_1] + [b_{12}, L_2] . \tag{6}$$

Proof. We use eq. (5) in a matrix representation of $\mathcal{G}$ (and G). Assume first that $\{\lambda_i, \lambda_j\}=0$. Since S is a function on the phase space, we may compute brackets like $S_1^{-1}S_2^{-1}\{S_1, S_2\}$ or $S_1^{-1}\{S_1, \Lambda_2\}$. We get

$$\{L_1, L_2\} = \{S_1^{-1}\Lambda_1 S_1, S_2^{-1}\Lambda_2 S_2\}$$
$$= [a_{12}, L_1] + [b_{12}, L_2] ,$$

with

$$a_{12} = -q_{12} + \tfrac{1}{2}[k_{12}, L_2] ,$$
$$b_{12} = q_{21} + \tfrac{1}{2}[k_{12}, L_1] ,$$

where we have defined

$$q_{12} = S_1^{-1}S_2^{-1}\{S_1, \Lambda_2\}S_2 ,$$
$$k_{12} = S_1^{-1}S_2^{-1}\{S_1, S_2\} .$$

We have used the freedom to change a and b by

$$a_{12} \to a_{12} + [c_{12}, L_2] , \tag{7}$$
$$b_{12} \to b_{12} - [c_{12}, L_1] , \tag{8}$$

for any $c \in \mathcal{G}\otimes\mathcal{G}$, to distribute evenly the term containing k_{12} between a and b.

Conversely, suppose we have

$$\{L_1, L_2\} = [a_{12}, L_1] + [b_{12}, L_2] ,$$

then in any matrix representation

$$\{L_1^n, L_2^m\} = [a_{12}^{n,m}, L_1] + [b_{12}^{n,m}, L_2] \tag{9}$$

with

$$a_{12}^{n,m} = \sum_{p=0}^{n-1}\sum_{q=0}^{m-1} L_1^{n-p-1}L_2^{m-q-1}a_{12}L_1^pL_2^q ,$$

$$b_{12}^{n,m} = \sum_{p=0}^{n-1}\sum_{q=0}^{m-1} L_1^{n-p-1}L_2^{m-q-1}b_{12}L_1^pL_2^q .$$

Taking the trace of eq. (9), we get the desired involution property.

Proposition. Suppose we have

$$\{L_1, L_2\} = [a_{12}, L_1] + [b_{12}, L_2] .$$

If we take as a hamiltonian $\mathrm{tr}(L^n)$, then the equations of motion have a Lax representation.

Proof. Set $m=1$ in eq. (9), take the trace of it over the first space, and get $\dot{L} = [L, \ M]$ with

$$M = -n\,\mathrm{tr}_1(L_1^{n-1}b_{12}) .$$

4. Properties of the Poisson structure

4.1. Action of the gauge group

The form of the bracket is preserved by gauge transformations, i.e. if

$$\{L_1, L_2\} = [a_{12}, L_1] + [b_{12}, L_2] ,$$

and

$$L' = g^{-1}Lg \tag{10}$$

then

$$\{L'_1, L'_2\} = [a'_{12}, L'_1] + [b'_{12}, L'_2] .$$

A direct calculation shows that

$$a'_{12} = g_1^{-1}g_2^{-1}\,(a_{12} - \{g_1, L_2\}g_1^{-1} + \tfrac{1}{2}[u_{12}, L_2])g_1g_2 ,$$
$$b'_{12} = g_1^{-1}g_2^{-1}\,(b_{12} - \{L_1, g_2\}g_2^{-1} + \tfrac{1}{2}[u_{12}, L_1])g_1g_2 ,$$

where

$$u_{12} = \{g_1, g_2\}g_1^{-1}g_2^{-1} .$$

4.2. Antisymmetry

The antisymmetry of the Poisson bracket

$$\{L_1, L_2\} = -\{L_2, L_1\}$$

implies

$$[a_{12} + b_{21}, L_1] + [a_{21} + b_{12}, L_2] = 0 .$$

Consequently if we set

$$d_{12} = \tfrac{1}{2}(a_{12} - b_{21})$$

then the bracket reads

$$\{L_1, L_2\} = [d_{12}, L_1] - [d_{21}, L_2] . \tag{11}$$

In this form the antisymmetry property of the bracket is explicit, although d has no special symmetry property. This form contains the same information as eq.

(6). It is the one we will use in the sequel.

Notice that the ambiguity (7), (8) gives the possibility of redefining d by

$$d_{12} \to d_{12} + [\sigma_{12}, L_2] , \qquad (12)$$

where σ is symmetric, without changing the Poisson bracket.

4.3. Dualization

Suppose $\mathscr{G}$ is a Lie algebra equipped with a non degenerate invariant scalar product (,). We will use a basis $\{X_\mu\}$ of $\mathscr{G}$, and denote $g_{\mu\nu} = (X_\mu, X_\nu)$ and $g^{\mu\nu}$ its inverse. L may be viewed as a linear form on $\mathscr{G}$.

$$X \in \mathscr{G} \to L(X) = (L, X) .$$

We may as well view d as a linear map $D: \mathscr{G} \to \mathscr{G}$. If

$$d_{12} = \sum_{\mu\nu} d^{\mu\nu} X_\mu \otimes X_\nu ,$$

then

$$D(X) = \sum_{\mu\nu} d^{\mu\nu} X_\mu (X_\nu, X) , \qquad (13)$$

and eq. (11) also reads

$$\{L(X), L(Y)\} = L([X, Y]_D) \qquad (14)$$

with

$$[X, Y]_D = [D(X), Y] + [X, D(Y)] .$$

Notice that no particular symmetry of d is assumed here.

4.4. Jacobi identity

It is straightforward to write the Jacobi identity on the Poisson bracket. One gets the following constraint on d (see ref. [7]):

$$\begin{aligned} &[L_1, [d_{12}, d_{13}] + [d_{12}, d_{23}] + [d_{32}, d_{13}]] \\ &\quad + [L_1, \{L_2, d_{13}\} - \{L_3, d_{12}\}] \\ &\quad + \text{cyclic permutations} = 0 . \qquad (15) \end{aligned}$$

Using the previous dualization, we may write an equivalent equation on D:

$$\begin{aligned} &L([X, [D(Y), D(Z)] - D([Y, Z]_D)]) \\ &\quad + L([X, \{L(Y), D(Z)\} - \{L(Z), D(Y)\}]) \\ &\quad + \text{cyclic permutations} = 0 . \qquad (16) \end{aligned}$$

Solving this equation amounts to classifying integrable hamiltonian systems. We will comment on this equation in specific examples.

Remark. Eq. (15) was already obtained in ref. [7]. Notice that the first terms in eq. (15), which are the only remaining ones if d happens to be constant are very similar, but not quite identical to the usual classical Yang–Baxter equation. They are if d is not only constant, but also antisymmetric.

5. Examples

We give here a few examples of Poisson brackets taking the form we have described.

5.1. Classical r-matrix

In the approach of classical hamiltonian integrable systems developed by the Leningrad School, the key equation is [2,5]

$$\{L_1, L_2\}_r = [r_{12}, L_1 + L_2] . \qquad (17)$$

We may assume r antisymmetric i.e. $r_{12} = -r_{21}$, since the symmetric part of r does not contribute to the Poisson bracket. Eq. (17) is a particular case of eq. (11), for a constant antisymmetric d.

An example of this situation is provided by our existence theorem of section 2. Recall that for each degree of freedom

$$L = IH + 2IOE ,$$

where $[H, E] = 2E$. It is straightforward to check that $d = H \otimes E - E \otimes H$. Notice that d is constant and is an antisymmetric solution of the classical Yang–Baxter equation.

This shows that all integrable systems have an r-matrix. However the statement is not globally true over phase space unless there is a *global* action-angle coordinate system.

5.2. Kirillov bracket

Suppose $\mathscr{G}$ is a Lie algebra equipped with a non degenerate invariant scalar product as in paragraph (4.3).

Proposition.

$$d_K = \frac{1}{2} \sum_{\mu\nu} g^{\mu\nu} X_\mu \otimes X_\nu$$

corresponds to the Kirillov bracket.

Proof. By dualization, we get $D = \frac{1}{2}$, and eq. (14) yields the result

$$\{L(X), L(Y)\}_K = L([X, Y]) .$$

5.3. Two quadratic Poisson brackets

In the multihamiltonian approach to integrable systems, one uses the notion of compatible Poisson brackets, i.e. pairs of brackets such that any linear combination of them still verifies the Jacobi identity. We will use an antisymmetric solution of the modified Yang–Baxter equation to produce such pairs of brackets. Recall that $r \in \mathscr{G} \otimes \mathscr{G}$ is an antisymmetric solution of the modified Yang–Baxter equation if

$$r = r^{\mu\nu} X_\mu \otimes X_\nu, \quad r^{\mu\nu} = -r^{\nu\mu} ,$$

and

$$[R(X), R(Y)] - R([R(X), Y] + [X, R(Y)])$$
$$= -[X, Y] \quad \forall X, Y \in \mathscr{G} ,$$

where $R : \mathscr{G} \to \mathscr{G}$ is defined by

$$R(X) = \sum_{\mu\nu} r^{\mu\nu} X_\mu (X_\nu, X) ,$$

as in eq (13). We will denote $R_+ = \frac{1}{2}(R+1)$ and $R_- = \frac{1}{2}(R-1)$.

We will assume that $\mathscr{G}$ is the Lie algebra of some associative algebra $\mathscr{A}$, i.e.

$$[X, Y] = XY - YX ,$$

and the existence of a trace tr on $\mathscr{A}$ such that

$$\mathrm{tr}(XY) = \mathrm{tr}(YX) .$$

In what follows we understand $L \in \mathscr{A}$ and $\mathrm{d} \in \mathscr{A} \otimes \mathscr{A}$.

– The first quadratic bracket is the one appearing in the r-matrix formalism [2]:

$$\{L_1, L_2\}_+ = [r_{12}, L_1 L_2] .$$

To write it in the form of eq. (11), it suffices to define

$$d_{12} = \frac{1}{2}(-L_2 r_{21} + r_{12} L_2) .$$

If r is antisymmetric, this reduces to

$$d_{12} = \frac{1}{2}(L_2 r_{12} + r_{12} L_2) .$$

In the dual form

$$D^+(X) = \frac{1}{2} R(LX + XL)$$

or

$$D^+(X) = R_+(LX) + R_-(XL) , \tag{18}$$

if we use the freedom (12) with $\sigma = d_K$.

– The second bracket has been introduced in ref. [9]. It reads

$$\{L(X), L(Y)\}_- = \mathrm{tr}_{12}([X_1, L_1] r_{12} [Y_2, L_2])$$
$$+ \mathrm{tr}(L^2 [X, Y]) .$$

In the standard matrix basis $\{E_{ij}\}$ we have

$$d_{12} = \frac{1}{2}[r_{12}, L_2] + \sum_{ijp} L_{ij} E_{ip} \otimes E_{pj} .$$

It may be written in the form (14) with

$$D^-(X) = \frac{1}{2} R([L, X]) + LX ,$$

or

$$D^-(X) = R_+(LX) - R_-(XL) . \tag{19}$$

We know that the brackets defined by eq. (18), and eq. (19) verify the Jacobi identity. Consequently D^+ and D^- verify eq. (16), as can be checked directly.

Proposition. (Refs. [6,9].) The bracket $\{\ ,\ \}_+$ is compatible with $\{\ ,\ \}_r$ and the bracket $\{\ ,\ \}_-$ is compatible with $\{\ ,\ \}_K$

Proof. Let $f_X(L) = \mathrm{tr}(LX)$. Then

$$(\{f_X, f_Y\}_+ + \lambda \{f_X, f_Y\}_r)(L) = \{f_X, f_Y\}_+(L + \lambda \mathbf{1}) ,$$

and similarly for the brackets $\{\ ,\ \}_-$ and $\{\ ,\ \}_K$.

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