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# Singularity confinement and degree growth

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ABSTRACT. We discuss the singularity confinement property using polynomial maps in projective variables. It is shown through examples that singularity confinement alone does not guarantee integrability, but in addition the degree growth of the map must be slow enough (polynomial rather than exponential). We also present a method by which one can easily find out when a non-autonomous map has its first cancellation.

## 1. Introduction

The Painlevé test has become an important tool for identifying differential equations whose solutions behave in a regular manner. It is quite natural to try to search for a similar test for difference equations or maps. In [**1**], a singularity confinement test was proposed as the analogue of the Painlevé test for discrete equations. This test turned out to be very useful in identifying well behaving maps, including discrete analogues of Painlevé equations [**2**].

However, the very concept of integrability is more problematic for discrete equations than for continuous ones, and as a consequence the singularity confinement test was met with some reserve. It was generally accepted that while singularity confinement is necessary it was not at all clear whether it is sufficient.

In [**3**] the present authors reported an example which passed the singularity confinement test but which nevertheless showed numerical chaos in a typical way. In this same paper attention was directed on the degree growth of the map and it was proposed that as an additional requirement the degree of an integrable map should not grow faster than polynomially (see also [**4**]). This means that when the map is viewed as a polynomial map in projective space there should be projective cancellations. These cancellations are intimately related to singularity confinement and for polynomial growth one sometimes just needs more of them.

## 2. The singularity confinement test

We will choose our examples from the class of maps

$$x_{n+1} + x_{n-1} = a_n x_n + b_n + \frac{c_n}{x_n} + \frac{d_n}{x_n^2}, \tag{1}$$

where $a_n, \ldots, d_n$ are some functions of $n$. (We assume that $c_n$ and/or $d_n$ are nonzero, otherwise the map is linear.) It is clear that this equation can be iterated equally well in both directions and that potential problems arise only if for some $k$ we obtain $x_k = 0$ (with $x_{k-1}$ finite and nonzero). This would mean $x_{k+1} = \infty$ and then $x_{k+2} = \infty$ but in computing $x_{k+3}$ we would encounter the undefined expression $\infty - \infty$.

The idea of the singularity confinement test is to check whether this potential singularity can be circumvented. Let us therefore assume that $x_1 = u$ which is finite and nonzero and that $x_0 = \epsilon$, which is a small number and we will let it approach zero later. For this example let us further assume that in (1) $a_n = d_n = 1$, $b_n = c_n = 0$, $\forall n$, i.e.,

$$x_{n+1} + x_{n-1} = x_n + \frac{1}{x_n^2}.$$

We find

$$\begin{aligned}
x_{-1} &= u, \\
x_0 &= \epsilon, \\
x_1 &= \epsilon^{-2} - u + \epsilon, \\
x_2 &= \epsilon^{-2} - u + \epsilon^4 + \ldots, \\
x_3 &= -\epsilon + 2\epsilon^4 + \ldots, \\
x_4 &= u + 3\epsilon + \ldots
\end{aligned}$$

Thus a closer analysis indicates that in this case "$\infty - \infty = 0$". Furthermore one can emerge from the singularity with the initial information $u$.

Let us now write the system as a polynomial map in $\mathbb{C}P^2$. First we write is as a first order system,

$$\begin{cases}
x_{n+1} = x_n - y_n + \dfrac{1}{x_n^2}, \\
y_{n+1} = x_n,
\end{cases}$$

and then we homogenize it by substituting

$$x_n = u_n/f_n,\ y_n = v_n/f_n,$$

which yields

$$\begin{cases} \dfrac{u_{n+1}}{f_{n+1}} &= \dfrac{u_n - v_n}{f_n} + \dfrac{f_n^2}{u_n^2}, \\[2ex] \dfrac{v_{n+1}}{f_{n+1}} &= \dfrac{u_n}{f_n}, \end{cases}$$

Here we can eliminate denominators if we choose $f_{n+1} = f_n u_n^2$, this leads to

$$\begin{cases} u_{n+1} &= -u_n^2 v_n + u_n^3 + f_n^3, \\ v_{n+1} &= u_n^3, \\ f_{n+1} &= f_n u_n^2. \end{cases}$$

This map is in $\mathbb{C}P^2$ and therefore we have the identification $(u, v, f) \sim (\lambda u, \lambda v, \lambda f)$. The apparent degree growth of (2) is $3^n$, where $n$ is the iteration step. However, the identification rule means that common factors can be canceled and therefore the actual degree growth may be slower.

Let us now see how the singularity confinement looks from this point of view. In $\mathbb{C}P^2$ we have the following sequence

$$\begin{pmatrix} 0 \\ u \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

In this case the first four points are regular points in the projective space but the last one is not in $\mathbb{C}P^2$ at all and therefore singular. To figure out how to proceed we must next study the behavior near it.

Starting from the point corresponding to $x_{-1} = u$, $x_0 = \epsilon$ we obtain

$$\begin{pmatrix} \epsilon \\ u \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 - \epsilon^2 u + \epsilon^3 \\ \epsilon^3 \\ \epsilon^2 \end{pmatrix} \to \begin{pmatrix} 1 - 3\epsilon^2 u + \dots \\ 1 - 3\epsilon^2 u + \dots \\ \epsilon^2 - 2\epsilon^4 u + \dots \end{pmatrix} \to$$

$$\to \begin{pmatrix} -\epsilon^3 + \dots \\ 1 - 9\epsilon^2 u + \dots \\ \epsilon^2 + \dots \end{pmatrix} \to \begin{pmatrix} \epsilon^8 u + \dots \\ -\epsilon^9 + \dots \\ \epsilon^8 + \dots \end{pmatrix} = \begin{pmatrix} u + \dots \\ -\epsilon + \dots \\ 1 + \dots \end{pmatrix}$$

It is important here to note how we emerged from the singularity. The initial value was temporarily buried under $\epsilon^2$, however, the cancellation of $\epsilon^8$ from the vector (that was initially considered a singular point) made it finite and at the same time brought forward the initial value $u$. This is important: emergence from singularity is possible only if we can cancel enough factors of $\epsilon$, but at the same time the cancellations also reduce the degree growth. The amount of cancellations required for singularity confinement is just not always enough to guarantee integrability, without any cancellations the degree of the $n$-th iterate would have been $3^n$ but for integrability one needs so many cancellations that the degree growth slows down from exponential to polynomial.

For the map we are discussing (2) we get the following sequence of degrees: 1, 3, 9, 27, 73, 195, 513, 1347, 3529, … The first four degrees follow the $3^n$ rule but then we have the first cancellation: the fifth term would have been 91 but is only 73, corresponding to a cancellation of 8. This corresponds exactly to the factorization of $\epsilon^8$ in the above calculation. (The later cancellations appearing in this sequence are not so easy to identify using $\epsilon$ expansion.)

From these first nine numbers of the degree sequence we can construct a generating function for the degrees, it has the form

$$\begin{aligned} g(x) &= \frac{1+3\,x^3}{(1-x)\,(1+x)\,(x^2-3\,x+1)} \\ &= 1+3\,x+9\,x^2+27\,x^3+73\,x^4+195\,x^5+513\,x^6+ \\ &\quad +1347\,x^7+3529\,x^8+9243\,x^9+24201\,x^{10}+\ldots \end{aligned} \tag{2}$$

(There is an argument in [**5**] by which one can actually prove that this is the generating function.)

Let $g_n$ be the degree of the $n$-th iterate, and $\lambda$ the root of the denominator of $g$ with smallest modulus. Thus $\lambda$ determines the radius of convergence of $g$ and therefore asymptotically $g_{n+1}/g_n \approx 1/\lambda$, and hence $g_n \approx \lambda^{-n}$. For (2) $1/\lambda = (3+\sqrt{5})/2 \approx 2.618$, and therefore we have exponential growth. Numerical studies confirm the prediction of non-integrability and show typically chaotic behavior [**3**].

Let us compare the above with a truly integrable system, e.g., [**2**]

$$x_{n+1}+x_{n-1} = \frac{c}{x_n}+\frac{d}{x_n^2},$$

which is related to $d-P_I$. This is also a third degree map in projective coordinates, but calculations show that cancellations are much stronger and produce the degrees 1, 3, 9, 19, 33, 51, 73, 99, 129, 163, 201, … The generating function for this sequence is

$$g = \frac{1+3x^2}{(1-x)^3},$$

and in fact the degree growth follows the polynomial rule

$$g_n = 2n^2+1.$$

As another integrable example consider d-$P_I$

$$x_{n+1}+x_{n-1}=b+\frac{c}{x_n}, \tag{3}$$

with degree sequence 1, 2, 4, 7, 12, 25, 34, 44, 55, 68 the generating function

$$g=\frac{1+x^2+2x^4}{(1-x^3)(1-x)^2},$$

Again the roots of the denominator have unit modulus.

## 3. Non-autonomous systems

The above discussion was concerned with autonomous maps but we can use the same approach when trying to find those deautonomizations for which the system still passes the singularity confinement test.

Let us consider (1) but now with $a_n=-1$, $b_n=b, d_n=0$, (d-P$_I$)

$$x_{n+1}+x_n+x_{n-1}=b+\frac{c_n}{x_n}. \tag{4}$$

As a polynomial map in $\mathbb{C}P^2$ it reads

$$\begin{cases} u_{n+1} &= -u_n(u_n+v_n)+f_n(c_nf_n+bu_n),\\ v_{n+1} &= u_n^2,\\ f_{n+1} &= f_nu_n. \end{cases}$$

The singularity pattern of the original equation is

$$x_{-1}=x,\ x_0=0,\ x_1=\infty, x_2=\ ?$$

and correspondingly in $\mathbb{C}P^2$ we have the sequence

$$\begin{pmatrix}0\\x\\1\end{pmatrix}\to\begin{pmatrix}1\\0\\0\end{pmatrix}\to\begin{pmatrix}1\\-1\\0\end{pmatrix}\to\begin{pmatrix}0\\1\\0\end{pmatrix}\to\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Expanding around the singularity gives the sequence

$$\begin{pmatrix}\epsilon\\x\\1\end{pmatrix}\to\begin{pmatrix}c_0+(b-x)\epsilon+\dots\\ \epsilon^2\\ \epsilon\end{pmatrix}\to\begin{pmatrix}-c_0^2+\epsilon c_0(2x-b)+\dots\\ c_0^2+2\epsilon c_0(-x+b)+\dots\\ \epsilon c_0+\epsilon^2(-x+b)+\dots\end{pmatrix}\to$$

$$\to\begin{pmatrix}\epsilon^2c_0^2(-c_0+c_1+c_2)+\dots\\ c_0^4+2\epsilon c_0^3(-2x+b)\dots\\ -\epsilon c_0^3+\epsilon^2c_0^2(3x-2b)+\dots\end{pmatrix}\to\begin{pmatrix}\epsilon^2c_0^6C_3+\epsilon^3c_0^5K+\dots\\ \epsilon^4c_0^4C_2^2+\dots\\ -\epsilon^3c_0^5C_2+\dots\end{pmatrix}.$$

where we have used the shorthand notation $C_2:=c_2+c_1-c_0$ and $C_3:=c_3-c_2-c_1+c_0$, $K=b(4C_3+c_0-c_2)-x(6C_3+c_0)$.

We now observe that we can in any case cancel $\epsilon^2$. But from the singularity confinement point of view the crucial observation is that *if*

$C_3 = 0$, $C_2 \neq 0$ then we can cancel $\epsilon^3$ and then take the $\epsilon \to 0$ limit and get (after also factoring out $-c_0^5$)

$$\begin{pmatrix} c_0(x-b)+c_2 b \\ 0 \\ c_3 \end{pmatrix}.$$

Thus we have emerged from the singularity and in particular recovered the initial data $x$. The condition $C_3 = 0$ reads

$$c_{n+3} - c_{n+2} - c_{n+1} + c_n = 0.$$

This is a third order linear map with the general solution

$$c_n = \alpha + \beta\, n + \gamma\, (-1)^n.$$

which yields an integrable map.

Later emergence from the singularity is also possible. If $C_3 \neq 0$ then at the seventh step we find

$$\begin{pmatrix} \epsilon^2 C_0^{14} C_2^2 C_3^6 C_6 + \epsilon^3 C_0^{13} C_2 C_3^5 (bM + xN + C_6(\dots)) + \dots \\ \epsilon^4 C_0^{12} C_2^4 C_3^4 C_5^2 + \dots \\ \epsilon^3 C_0^{13} C_2^3 C_3^5 C_5 + \dots \end{pmatrix}.$$

where $M$ and $N$ are some polynomials of $C_n$. If now $C_6 := c_6 - c_5 - c_4 + c_3 - c_2 - c_1 + c_0 = 0$ we can divide out $\epsilon^3$ and exit from the singularity with the initial datum $x$, as long as $N = C_3(4C_2C_3 - 2C_0C_3 - C_0C_2) \neq 0$.

The condition $C_6 = 0$ has the solution $c_n = \sum_{\nu=1}^{6} \alpha_\nu X_\nu^n$ where $X_\nu$ is a root of

$$X^6 - X^5 - X^4 + X^3 - X^2 - X + 1 = 0.$$

Note that 1 is not a root of this equation, furthermore, there is a root of modulus $< 1$ and since this expression appears in the denominator of the generating function [**6**] we have exponential growth.

The conclusion here is that there are many ways to deautonomize an integrable autonomous system while preserving the singularity confinement property. However, a deautonomized map with a later confinement is probably chaotic. It seems that integrability is only associated with confinement at the first possibility, in any case no counterexamples are known for this conjecture.

## 4. Condition for the first cancellation

We have shown above that the number of steps needed for singularity confinement can vary. In any case one needs some cancellations, and we will next derive the condition for the first cancellation to take place after $N$ iteration steps.

We shall consider the map

$$x_{n+1} + x_{n-1} = a_n x_n + b_n + \frac{c_n}{x_n},$$

where $x_n$ is the dynamical variable and $a, b, c$ are some fixed functions of $n$, of which we assume that $c_n \neq 0$, $\forall n$. In projective coordinates the map reads

$$\left\{ \begin{array}{lcl} u_{n+1} & = & -v_n u_n + a_n u_n^2 + b_n u_n f_n + c_n f_n^2, \\ v_{n+1} & = & u_n^2, \\ f_{n+1} & = & u_n f_n. \end{array} \right. \tag{5}$$

When this map it iterated the degrees grow like $2^n$, unless some cancellations take place. The iterations are supposed to start with $u_0 = x, v_0 = y, f_0 = z$ which are the indeterminates. The assumption that the first cancellation takes place at the $N$'th iteration means

$$\gcd(u_N, v_N, f_N) \neq 1, \text{ but } \gcd(u_k, v_k, f_k) = 1, \forall k < N.$$

Let us first derive some lemmas that will be useful in the analysis.

Lemma 1. $\gcd(u_k, f_k) = 1 \iff \gcd(u_{k+1}, v_{k+1}, f_{k+1}) = 1.$

Proof. From (5) we find $\gcd(u_k, f_k)$ divides $\gcd(u_{k+1}, v_{k+1}, f_{k+1})$, from which follows "$\Leftarrow$". On the other hand, if $\gcd(u_k, f_k) = 1$ then $\gcd(v_{k+1}, f_{k+1}) = u_k$, but if $c_k \neq 0$ then $u_k \nmid u_{k+1}$ and therefore $\gcd(u_{k+1}, v_{k+1}, f_{k+1}) = 1.$ $\square$

Using this lemma we find that the first cancellation takes place at $N$ if

$$\gcd(u_k, f_k) = 1, \forall k < N-1, \quad \gcd(u_{N-1}, f_{N-1}) \neq 1.$$

Lemma 2.

$$\gcd(u_k, f_k) = 1 \Rightarrow \gcd(u_{k+1}, f_{k+1}) = \gcd(-v_k + a_k u_k, f_k).$$

Proof. Recalling that if $\gcd(u, f) = 1$ and $\gcd(u, b) = 1$ then $\gcd(ux + bf, fu) = \gcd(x, f)$, it then follows that

$$\begin{array}{rcl} \gcd(u_{k+1}, f_{k+1}) & = & \gcd(u_k(-v_k + a_k u_k + b_k f_k) + a_k f_k^2, u_k f_k), \\ & = & \gcd(-v_k + a_k u_k + b_k f_k, f_k) \\ & = & \gcd(-v_k + a_k u_k, f_k). \end{array}$$

$\square$

This lemma again takes us one step forward so that we now find the condition for first cancellation at step $N$ to be

$$\begin{array}{l} \gcd(-v_k + a_k u_k, f_k) = 1, \forall k < N-2, \\ \gcd(-v_{N-2} + a_{N-2} u_{N-2}, f_{N-2}) \neq 1. \end{array}$$

LEMMA 3. *If* $\gcd(u_k, f_k) = 1$ *then*

$$\gcd(\alpha_{k+1}v_{k+1} + \beta_{k+1}u_{k+1}, f_{k+1}) = \begin{cases} u_k, & \text{if } \beta_{k+1} = 0, \\ \gcd(\alpha_k v_k + \beta_k u_k, f_k) & \text{otherwise} \end{cases}$$

*where*

$$\begin{cases} \alpha_{n-1} &= -\beta_n, \\ \beta_{n-1} &= \alpha_n + a_{n-1}\beta_n. \end{cases} \tag{6}$$

PROOF. If $\beta_{k+1} = 0$ then $\gcd(\alpha_{k+1}v_{k+1}, f_{k+1}) = \gcd(\alpha_{k+1}u_k^2, u_k f_k)$ $= u_k$. If $\beta_{k+1} \neq 0$ then

$$\begin{aligned} &\gcd(\alpha_{k+1}v_{k+1} + \beta_{k+1}u_{k+1}, f_{k+1}) = \\ &\gcd(u_k(\alpha_{k+1}u_k + \beta_{k+1}(-v_k + a_k u_k + b_k f_k)) + \beta_{k+1}a_k f_k^2, f_k u_k) = \\ &\gcd(-\beta_{k+1}v_k + (\alpha_{k+1} + \beta_{k+1}a_k)u_k, f_k). \end{aligned}$$

□

Collecting the results of the 3 lemmas we get

THEOREM 1. *First cancellation takes place after $N$ steps iff in the map (6) the initial values $\beta_{N-2} = a_{N-2}$, $\alpha_{N-2} = -1$ imply $\beta_1 = 0$ and $\beta_k \neq 0, \forall 1 < k \leq N-2$.*

As the first result we find that we can have the first cancellation at $N = 3$ provided that $\beta_1 = a_1 = 0$. If this is to hold with any starting point $n$ we must take $a_n \equiv 0$, this is what happens with the map (3).

Later cancellation is possible for other values of $a_n$. We find from (6) that $\beta_{N-3} = -1 + a_{N-3}a_{N-2}$ and thus first cancellation is at $N = 4$ if $a_{n+1}a_n = 1$. This happens for (4), which has $a_n = -1$. The condition for the first cancellation at the $N = 5$ step is $a_{n+1}a_n a_{n-1} = a_{n+1} + a_{n-1}$, one solution of which is $c = \sqrt{2}$.

If we only consider constant $a_n$ then $a_n = 2, \forall n$ is special in that then $\alpha_n = -1$, $\beta_n = 1, \forall n$ and therefore there will be *no cancellations ever*. This may therefore be considered as a maximally non-integrable map.

It should be noted that a cancellation of factors is necessary but not sufficient for singularity confinement. Thus this result can be used just to narrow down the set of maps that are candidates for singularity confinement.

## 5. Conclusions

The above discussions show that the singularity confinement question can be best studied by viewing the rational map as a polynomial map in projective space. It is then clear that:

- Some cancellations are possible even without singularity confinement.
- For singularity confinement one needs so many cancellations that the initial data can be recovered.
- For integrability one needs so many cancellations that the degree of the map grows only polynomially.

It would be interesting to find more examples of maps satisfying only some of the listed properties.

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