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# Complexity and integrability

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Abstract. We describe the relations between singularity, complexity, and integrability of discrete systems with a rational evolution.

## 1. Introduction

Characterizing the integrability of evolution equations is still an open problem. For continuous systems, apart from the analysis of the hamiltonian structure and the direct proof of integrability in the sense of Liouville [**1, 2**], there exist other approaches like the singularity analysis [**3, 4**]. For discrete systems, which are the ones we are interested with, an analog of the Painlevé analysis was proposed in [**5, 6**].

We describe here a method one can use for discrete systems having a rational evolution, i.e: the state of the system belongs to some finite dimensional space ($R^n$ or preferably projective space $\mathbb{P}_n$, real or complex), and the state at time $t+1$ is expressible in terms of the state at time $t$ through rational functions, or polynomial functions if one uses the homogeneous coordinates of projective space.

We associate to any such system an entropy (algebraic entropy) which measures the complexity of the evolution map, and propose to use a non-vanishing entropy as an index of non-integrability. In such a general setting the proper notion of integrability will be that the motion reduces to a shift on some abelian variety. Basic considerations linking complexity with singularity and integrability of rational maps go back to [**7**]. Further elaborations followed, and we describe some of them here.

## 2. Definitions

Let $\varphi$ be a rational evolution map. The action of $\varphi$ represents one time-step. We use homogeneous coordinates $[x_0, x_1, \ldots, x_n]$ over $\mathbb{P}_n$, so that $\varphi$ is given in terms of $n+1$ homogeneous polynomials of degree $d$. If any common factor appears among these polynomials, we ought to factor it out. The degree $d$ is thus well defined, but of course not invariant by changes of coordinates (see later).

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Definition 1: Consider the sequence $\{d_k\}$ of degrees of the iterates $\varphi^{(k)}$ of $\varphi$. We shall use as an index of complexity, and call algebraic entropy of $\varphi$ the limit:

$$\epsilon(\varphi) = \lim_{k\to\infty} \frac{1}{k} Log(d_k) \tag{1}$$

Claim: This limit always exists. It is a straightforward consequence of the inequality on the degree of composed rational maps:

$$d(f\cdot g) \leq d(f)\cdot d(g) \tag{2}$$

Definition 2: We shall make use of the generating function of the sequence of degrees

$$\eta_\varphi(z) = \sum_{k=0}^{\infty} d_k z^k \tag{3}$$

Remark 1: The degree $d_k$ of the $k-th$ iterates of $\varphi$ is bounded above by $d^k$, and the entropy is thus bounded by $Log(d)$. The possible simplifications of the fractions (or equivalently the presence of common factors in the homogeneous description) are responsible for the drop of the degree. This drop may be so important that the growth falls from its generic exponential rate to a polynomial rate.

Remark 2: The entropy of $\varphi$ is the logarithm of the smallest modulus of the poles of $\eta_\varphi$. If the entropy is non vanishing, the growth of the degree is exponential. If the entropy vanishes, the growth is polynomial, and the degree of this polynomial may be taken as a secondary index.

Remark 3: There is a similarity between $\eta$ and the dynamical $\zeta$ function introduced by Artin-Mazur and Smale [**8, 9**]. Some precaution however has to be taken with the invariance of all these objects under changes of coordinates. If one studies homeomorphisms (or diffeomorphisms) like in [**8, 9**] or [**10**], the (natural) changes of coordinates should themselves be diffeomorphisms. The multiplicity of fixed points for example is unchanged by such coordinate transformations. If, and this is what we do here, one is interested in rational transformations, the natural group of coordinate transformations is the group of birational transformations, and neither the order of fixed points, nor the degree are invariant under such changes. Neither the function $\zeta$ nor the function $\eta$ are invariant. However, and this is one of its important features, our entropy is left unchanged by these coordinate transformations. There are also relations between the asymptotic behaviours of degrees, and number of fixed points, if the latter is finite [**11**]. The drawback of the $\zeta$ function is that it is not defined if the number of fixed points of some order is infinite. This happens quite often when the map we consider is integrable, in particular if there exists any rational invariant [**12**].

Remark 4: The definition may straightforwardly be extended to sequences of maps, allowing to consider non-autonomous maps, provided for example the degree of the successive maps is bounded.

## 3. Examples

We will study a number of elementary quadratic maps in $P_2$. Here is a list:

a) The Hénon map: $[x,y,z] \longrightarrow [x^2, x^2+xz-\alpha y^2, \beta xy]$
b) $\phi : [x,y,z] \longrightarrow [-xy+\gamma z^2+x^2, x^2, xz]$

c) Alternating sequence with odd terms $\varphi_o : [x, y, z] \longrightarrow [-xy + \gamma z^2 + \beta xz + x^2, x^2, xz]$ and even terms $\varphi_e : [x, y, z] \longrightarrow [-xy + \gamma z^2 + \beta xz, x^2, xz]$, with $\beta \neq 0$
d) The same as c) with an exchange of odd and even
e) $\varphi : [x, y, z] \longrightarrow [x^3 + \alpha z^3 - yx^2, x^3, x^2 z]$

## 4. A phenomenological approach

Calculating the full expressions of the iterates of $\varphi$ is beyond reach, even with a good machine. We rely on the simple fact that their degree is also the degree of the curve obtained by successive transformations of a generic straight line in $\mathbb{P}_n$ [**7**]. The rest of the task will be to analyze the sequences of integers (see [**13**] for an extensive study).

We get the following sequence of numbers:

a) 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, ...
b) 1, 2, 4, 8, 13, 20, 28, 38, 49, 62, 76, 92, 109, ...
c) 1, 2, 4, 7, 14, 22, 44, 70, 140, 224, 448, 716, ...
d) 1, 2, 4, 8, 14, 27, 42, 84, 133, 266, 426, 852, 1362, 2724, 4352, 8704, 13908, ...
e) 1, 3, 9, 27, 73, 195, 513, 1347, ...

The dots mean of course: guess the next terms, like in IQ tests. Sequences of integers are known to have arbitrary continuations if no hint is given about their construction rule. What is remarkable is that we can actually infer what the rule is, because all these sequences have common stringent features.

To evaluate the growth of the degrees, it is sometimes sufficient to calculate the discrete derivatives of the sequence $(d_k)$.

To do better it is useful to compute a generating functional, e.g using Pade-type methods, i.e. approximate the (truncated) series $\eta$ with a rational fraction.

What is remarkable is that in all the cases which have been studied, what comes out is *a rational generating function with integer coefficients*, i.e. the Pade approximation stabilizes, and the untruncated series is given by a rational function. This has two consequences (yet to be proven in general, but no counterexample is available at this moment):

-1- The algebraic entropy is the logarithm of an algebraic number.

-2- The sequences of degrees verify a *finite recurrence relation with integer coefficients.*

For the above list of examples, we gave a sufficient information in the listings of the various sequences, by cutting them after they stabilize.

a) Elementary: the sequence was $d_k = 2^k$.
b) Call b the sequence of degrees. If we compute the the discrete derivatives b1, and b2 of b, we get:
   b1: $1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, \ldots$
   b2: $1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, \ldots$
   The degree $d_k$ grows like a polynomial of degree 2 in $k$, and the entropy vanishes. The answer to the test was:

$$\ldots, 109, 128, 148, 170, 193, 218, 244, 272, 301, \ldots$$

   The generating function has poles on the unit circle.

$$\eta_b = -(1 + 2s^3)/(s + 1)/(s - 1)^3$$

c) Calculating the discrete derivatives yields:
c1: $1, 2, 3, 7, 8, 22, 26, 70, 84, 224, 268, \ldots$
c2: $1, 1, 4, 1, 14, 4, 44, 14, 140, 44, \ldots$
c3: $0, 3, -3, 13, -10, 40, -30, 126, -96, \ldots$
etc, and we do not see much of the structure, but, if we use the Pade method, we immediately get the generating function:

$$\eta_c = -\frac{\left(s^2 - s + 1\right)\left(s + 1\right)^3}{2\,s^6 + 3\,s^2 - 1}$$

This can be checked against the next calculable terms:
$\ldots, 716, 1432, 2288, 4576, 7312, 14624, 23368, 46736, 74680$
Entropy is non vanishing, and is given by the root of the denominator of $\eta_c$ with smallest modulus.

d) The discrete derivatives are :
d1: $1, 2, 4, 6, 13, 15, 42, 49, 133, 160, 426, 510, 1362, 1628, 4352, 5204, \ldots$
d2: $1, 2, 2, 7, 2, 27, 7, 84, 27, 266, 84, 852, 266, 2724, 852, \ldots$
etc, and is again not illuminating. Pade gives $\eta_d = P/Q$, with

$$\begin{aligned} P &= -1 - 2s - s^2 - 2s^3 - 2s^4 - 3s^5 + 2s^6 + s^7 + s^8 + 2s^9 + s^{10}, \\ Q &= 2\,s^6 + 3\,s^2 - 1. \end{aligned}$$

This agrees with the next computed terms:
$\ldots, 13908, 27816, 44448, 88896.$

Remark: The two cases c) and d) have the same entropy, because they are related to the same non-autonomous iterations.

The only integrable map in that list is case b). Our experience from numerous examples (it is indeed an experimental fact, but we turned it into a conjecture) is that if the entropy is non vanishing, the map is not integrable. We have found no counterexample yet.

In other words, the Pade approximation should always stabilize to a rational fraction with integer coefficients. Phrased differently: there always exists a finite recurrence relation with integer coefficients governing the sequence of degrees of the iterates (see section (6)). If the characteristic polynomial of this recurrence relation has a root of modulus smaller than 1, then the map is not integrable.

## 5. Degree growth versus "singularity confinement"

It is simple to understand the relation between the lowering of the degree and the singularity structure. It suffices to look at the geometry behind the factorization process, and more precisely the blow-downs and blow-ups familiar to algebraic geometers. A picture explains the process (this is a *virtual* but plausible example).

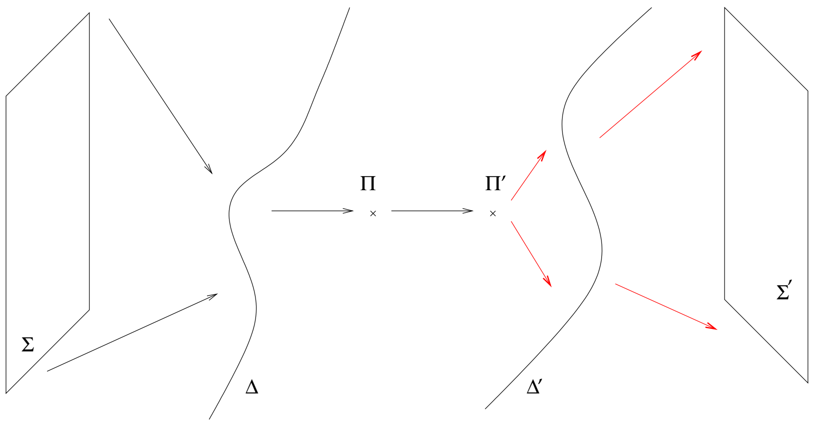


Figure 1: A possible blow-down blow-up scheme in $\mathbb{P}^3$.

Suppose a variety $\Sigma$ of codimension 1 is blown down by $\varphi$. Further transformations by $\varphi$ may blow the image down again (if $n \geq 3$), reaching a singular point of $\varphi$ (i.e. a point which blows up). This means that all points of $\Sigma$ are sent to $[0, 0, \ldots, 0]$ by $\varphi^4$. In other words the equation of $\Sigma$ factorizes in the expression of $\varphi^4$, implying a drop of the degree. In the above picture, there is more: there is a "resolution of the singularity" (alias confinement) by $\varphi^5$ since points of $\Sigma$ have an image on some other hypersurface $\Sigma'$, after all common factors, coming from the successively encountered singularities, have been removed. The removal of factors plays the role of the regularization and local exploration performed in the singularity confinement approach [**5, 6, 14**].

One should point out that there is no equivalence between resolution of singularities and absence of entropy. There is room between the two [**15, 14, 16**]. The main result is that there exist non-integrable (chaotic) maps which verify singularity confinement. There also exist integrable maps which do not verify confinement. It seems that the singularity confinement is not strong enough if it happens "too late" or if the number of dimensions of the phase space of the system is too large.

## 6. The meaning of $\eta$

The fact that $\eta$ is rational says that the degrees verify, at least after a certain rank, a recurrence relation. Indeed suppose this function is

$$\eta = \frac{P}{Q} = \frac{\sum_{i=0}^{p} \alpha_i z^i}{\sum_{j=0}^{q} \beta_j z^j}.$$

Then since by definition

$$\eta_\varphi(z) = \sum_{k=0}^{\infty} d_k z^k,$$

the degrees verify at least for large enough $r$,

$$\sum_{j=0}^{q} \beta_j d_{r-j} = 0,$$

and this recurrence controls their asymptotic behaviour.

## 7. Comments

In some instances, we have been able to prove the recurrence relation between the degrees [**16**].

The algebraic entropy is exactly calculable by constructing the generating function of the sequence.

The measure of entropy is more sensitive to non-integrability than singularity confinement [**14, 15**].

The algebraic entropy exists even in cases where the Kolmogorov-Sinai entropy cannot be defined (as it does not need the knowledge a priori of an ergodic measure).

From section (5) we see that the resolution of singularities is a reversibility requirement for the system: one requests that the indeterminations created by the possible non-invertibilities of the evolution map be lifted after a finite number of steps. What appears in [**16, 14, 15**] is that the process helps not having entropy, but is not the end of the story.

A final comment: it was pointed out by M. Kruskal that the Painlevé criterion is by essence an infinitesimal one (not a local one nor a global one). The infinitesimal scale seems indeed out of reach of discrete systems. However, if one wants to recover this scale, one has to think of smaller and smaller time steps, of which the number increases indefinitely. One thus has to examine large (discrete) time behaviour, and this is precisely what we do.

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