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A classification of four-state spin edge Potts models

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Abstract

We classify four-state spin models with interactions along the edges according to their behaviour under a specific group of symmetry transformations. This analysis uses the measure of complexity of the action of the symmetries, in the spirit of the study of discrete dynamical systems on the space of parameters of the models, and aims at uncovering solvable ones. We find that the action of these symmetries has low complexity (polynomial growth, zero entropy). We obtain natural parametrizations of various models, among which an unexpected elliptic parametrization of the four-state chiral Potts model, which we use to localize possible integrability conditions associated with high genus curves.

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1. Introduction

Our purpose is to describe the structure of the set of spin models with interactions along the edges, and propose a classification of the matrix of Boltzmann weights of such models.

The main tool we use is the action of a group of transformations which was exhibited in previous works concerning integrable models [1–3], and which has its origin in the so-called inversion relations [4]. They are generated by basic involutions, realized as birational transformations of the parameters of the models (we will call this group of transformations **I**-symmetries).

We analyse the behaviour of the realizations of **I**-symmetries, especially the iterates of specific infinite order elements of the group. The guiding principle is that a chaotic behaviour of the realization of **I**-symmetries, signalled by a measure of their complexity (entropy), is preventing integrability of the corresponding statistical mechanical model.

We concentrate on the four-state models because of their richness, but the method applies to all values of the number q of states.

The paper is organized as follows: we describe the models we have in mind and the action of **I**-symmetries. We then describe the classification of the models according both to the dimension of their parameter space and to their behaviour under **I**-symmetries. We recover, in the course of this classification, a number of known two-dimensional models which have been studied in the literature, among which are the standard scalar Potts model [5], the Ashkin–Teller model [6], the Kashiwara–Miwa model [7], and the four-state chiral Potts model [8]. One section is devoted to the analysis of definite models. Among other results, we give an unexpected[3] elliptic parametrization of the chiral Potts model. We give a constructive way to write explicitly the algebraic locus where the group of **I**-symmetries degenerates into a finite group, leaving room for solvability. We finally comment on the specificity of four-state models as compared to three and five or more states.

2. Spin edge models, I-symmetries and admissible patterns

The models are defined on a D-dimensional regular lattice (square, triangle, honeycomb, cubic, ...), each vertex bearing a spin σ which can take q values. A model is defined by giving the matrix of Boltzmann weights of local edge configurations: the state of an edge is determined by the values of the spin at its ends. One arranges the weights in a matrix W whose entry $W_{i,j}$ is the weight of the edge configuration with one end in state i and the other end in state j. Note that edges are oriented, and W *is not necessarily symmetric*. One may furthermore distinguish different types of bonds, such as, for example, vertical and horizontal ones on a square lattice, and the most general non-isotropic model is then associated with a number of $q \times q$ matrices W_v, one for each type of bond.

Since the entries of the matrices W are Boltzmann weights, W is defined up to an overall multiplicative factor. The most general matrix is then a collection of q^2 homogeneous entries, defined up to a factor.

Simple transformations may be defined, which act on the matrices W, forming the group of **I**-symmetries. This group is generated by simple involutive generators, namely the matrix inverse I, and the element by element inverse J, both taken up to a overall multiplicative factor. They may be written

$$I : \quad W_{kl} \longrightarrow A_{kl} \tag{1}$$

$$J : \quad W_{kl} \longrightarrow 1/W_{kl} \tag{2}$$

where A_{kl} is the cofactor of W_{kl} in W. To have a well-defined action of J, we assume that entries are non-vanishing.

The two generators are non-commuting involutions. They generate an *infinite* group isomorphic to the dihedral group $Z_2 \ltimes Z$. Its infinite part is made of the iterations of $\varphi = I \cdot J$ and its inverse $\varphi^{-1} = J \cdot I$.

We will build families of models by successive reductions of the number of parameters, obtained by imposing equality relations on the entries of the Boltzmann matrix W.

To illustrate this, consider the chiral Potts model: its Boltzmann weight matrix is a cyclic matrix W, where instead of keeping all q^2 entries independent, one imposes the following equality conditions:

$$W_{u,v} = W_{u+1,v+1} \tag{3}$$

getting for $q = 4$ a pattern of the form

$$W = \begin{pmatrix} w_0 & w_1 & w_2 & w_3 \\ w_3 & w_0 & w_1 & w_2 \\ w_2 & w_3 & w_0 & w_1 \\ w_1 & w_2 & w_3 & w_0 \end{pmatrix} \tag{4}$$

[3] Especially in view of the occurrence of higher genus curves in the solutions of the star–triangle relations [9, 10].

instead of the general 4×4 matrix. The number of parameters has gone down from 16 to 4 because we have imposed 12 equality relations.

The specificity of this type of relations is two-fold. Any relation of the form $W_{a,b} = W_{a',b'}$ for some pairs of indices $\{a, b\}$, $\{a', b'\}$ defines a pattern of matrix which is automatically left stable by the generator J of the group of **I**-symmetries.

Some pattern defined in this way will also be left stable by the matrix inversion I, *but not all of them*. We will say that a pattern is *admissible* if its form is left stable by matrix inversion [1].

Giving a pattern is equivalent to giving a partition of the q^2 entries of W. The number of possible patterns, that is to say the number $P(q^2)$ of partitions of q^2 objects, is given by [1]

$$P(q^2) = \sum_{s=1}^{q^2} \sum_{k=0}^{s-1} (-1)^k \frac{(s-k)^{(q^2-1)}}{k!(s-1-k)!}. \tag{5}$$

It grows extremely rapidly with the number q of spin states. It is 21 147 for $q = 3$, $10\,480\,142\,147 \simeq 10^{10}$ for $q = 4$, and $4638\,590\,332\,229\,998\,592 \simeq 4.6 \times 10^{18}$ for $q = 5$.

Our first step is to determine which of the $P(q^2)$ patterns are admissible. For $q = 4$ this was done by direct inspection of the $P(16) \simeq 10^{10}$ patterns. The outcome is a list of 166 patterns. We have not considered vanishing conditions on the entries, and we restrict ourselves to invertible matrices[4], so that both I and J are defined and invertible.

There are trivial redundancies in this list because any of the $q!$ permutations of the q values of the spin, i.e. the same permutation acting on simultaneously on the rows and columns of W will take an admissible pattern into an admissible pattern: it acts by similarity on W, and commutes with **I**-symmetries. This reduces the list to only 42 different admissible patterns, given in appendix A.

We see that there are no admissible patterns with 9, 11, 12, 13, 14 or 15 free (homogeneous) parameters.

A matrix of the form (4) belongs to the similarity class of pattern 17, which contains three elements (see later):

$$\begin{pmatrix} w_0 & w_1 & w_2 & w_3 \\ w_3 & w_0 & w_1 & w_2 \\ w_2 & w_3 & w_0 & w_1 \\ w_1 & w_2 & w_3 & w_0 \end{pmatrix} \qquad \begin{pmatrix} w_0 & w_1 & w_2 & w_3 \\ w_2 & w_0 & w_3 & w_1 \\ w_1 & w_3 & w_0 & w_2 \\ w_3 & w_2 & w_1 & w_0 \end{pmatrix} \qquad \begin{pmatrix} w_0 & w_1 & w_2 & w_3 \\ w_1 & w_0 & w_3 & w_2 \\ w_3 & w_2 & w_0 & w_1 \\ w_2 & w_3 & w_1 & w_0 \end{pmatrix}.$$

These patterns have four homogeneous independent parameters $[w_0, w_1, w_2, w_3]$. Since they are admissible, there is an action of the homogeneous matrix inversion I on $[w_0, w_1, w_2, w_3]$, which reads, when written for the first of the three representatives (cyclic matrix):

$$\begin{aligned} w_0 &\longrightarrow w_0^3 - 2w_0w_1w_3 - w_0w_2^2 + w_2w_3^2 + w_1^2w_2 \\ w_1 &\longrightarrow -w_1^3 + 2w_0w_1w_2 + w_1w_3^2 - w_3w_0^2 - w_2^2w_3 \\ w_2 &\longrightarrow w_2^3 - 2w_1w_2w_3 - w_2w_0^2 + w_0w_1^2 + w_3^2w_0 \\ w_3 &\longrightarrow -w_3^3 + 2w_0w_2w_3 + w_3w_1^2 - w_1w_2^2 - w_0^2w_1. \end{aligned} \tag{6}$$

The action of J reads, if we use the homogeneity of the entries:

$$[w_0, w_1, w_2, w_3] \longrightarrow [w_1w_3w_2, \quad w_0w_3w_2, \quad w_0w_1w_3, \quad w_1w_2w_0].$$

[4] Neither have we considered equality up to sign between entries [11].

3. Complexity analysis

Since **I**-symmetries are essentially the iterates of $\varphi = I \cdot J$, their action looks like a discrete time-dynamical system defined on the parameter space of the model. The measure of complexity we use is given by the rate of growth of the degrees of the iterates of φ. Indeed once we have reduced the action of I and J to the pattern, the action of φ is a polynomial transformation on the independent parameters. All expressions have a definite degree d. The nth iterate naively has degree d^n, but since we work with homogeneous coordinates, we should factor out any common factor, so that the degree may drop to a lower value d_n.

A measure of the complexity of the map is given by the entropy [12, 13]

$$\epsilon = \lim_{n\to\infty} \frac{1}{n} \log(d_n). \tag{7}$$

If $\epsilon = 0$ the growth of d_n is polynomial in n and d_n behaves like

$$d_n = \alpha n^{\kappa}(1 + O(1/n)) \tag{8}$$

and one may then define a secondary complexity index associated with the map, the integer power κ.

The most compact way to encode the sequence of degrees d_n is to write its generating function:

$$g(s) = \sum_{n=0}^{n=\infty} d_n s^n. \tag{9}$$

This function was conjectured to always be *rational* with integer coefficients, i.e. be of the form:

$$g(s) = \frac{P(s)}{Q(s)} \tag{10}$$

with P and Q some polynomials of degree p and q with integer coefficients. It is sufficient in practice to calculate the $p + q$ first terms of the sequence to infer this generating function[5]. Any further degree calculation serves as a verification.

If the roots of the denominator of the generating function are all of modulus one, then the entropy vanishes, and we may consider the secondary index κ.

To illustrate the method, we may look at pattern 17 as in the previous section. The degree of φ is 9. The first iterations of φ yield the sequence of degrees:

$$[1, 9, 33, 73, 129, 201, 289, 393, 513, 723, \ldots]. \tag{11}$$

The beginning of the sequence (up to d_6) may be fitted with the generating function

$$g(s) = \frac{(1 + 3s)^2}{(1 - s)^3} \tag{12}$$

which is compatible with the above-mentioned conjecture. The subsequent terms are then predicted correctly, and g_{17} is found. We see that the growth of the degree is polynomial ($\epsilon = 0$) and the secondary index is $\kappa = 2$.

For each pattern the different representatives yield the same transformations and consequently the same generating function.

If the entropy ϵ vanishes, the secondary index may be read from the generating function g: if $g(s)$ has a pole of order γ at $s = 1$, then $\kappa + 1 = \gamma$.

We have written the realizations of $\varphi = I \cdot J$ for the admissible patterns, and calculated the generating function for all of them. *The result is that for all 42 admissible patterns the*

[5] The catch is that the values of p and q are not known in advance.

growth of the degree is polynomial, i.e. the entropy vanishes, and the index κ takes values 0, 1, 2 *and* 3 (see the following sections, appendix A and appendix B).

The complexity can also be estimated using the following arithmetical procedure: start with a matrix with random *integer* entries, and iterate φ, dividing all the coefficients by their greatest common divisor. The complexity can be estimated from the growth of a typical entry. In practice the number of digits of the entries grows like λ^n, with $\epsilon = \log \lambda$. Since the entries of the matrix grow fast, one has to use a special representation of the integers allowing the manipulation of arbitrary large values [14]. In practice one performs the iteration with several initial random integer matrices. The average $l(k)$ of the logarithm of the entries is recorded as a function of the number of iterations k. The entropy ϵ or the secondary index κ are easily deduced from $l(k)$.

4. Structure of the set of admissible patterns

There exists a partial order relation on the set of patterns, induced by the partial order on the partitions of the entries. Indeed a partition $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_\mu\}$ may be finer than a partition $\beta = \{\beta_1, \beta_2, \ldots, \beta_\nu\}$, if all parts constituting α are subsets of the constituents of β:

$$\alpha \prec \beta \qquad \text{if} \quad \forall k = 1 \ldots \mu \quad \exists\, l \quad \text{s.t.} \quad \alpha_k \subset \beta_l. \tag{13}$$

If $\alpha \prec \beta$ we will say that β is a descendant of α. We obtain a descendant by adding further equality relations to the entries, i.e. merging parts together.

As an illustration, pattern 3, i.e. the standard scalar Potts model, is a descendant of pattern 17, i.e. the chiral Potts model. All patterns are descendants of pattern 42 which has the finest partition.

The list of immediate descendants of the various patterns can be given with the notation [Pattern number, {list of direct descendants}], where {} indicates an empty list

[4, {2}], [5, {1}], [6, {}], [7, {1, 3}], [8, {3}], [9, {2}], [10, {4, 5}],
[11, {4, 6}], [12, {4}], [13, {6}], [14, {5}], [15, {7}], [16, {5, 7}],
[17, {6, 7}], [18, {9}], [19, {8}], [20, {5, 6}], [21, {12}], [22, {8}],
[23, {4, 7}], [24, {9}], [25, {15}], [26, {16, 17, 20, 25}],
[27, {13, 14, 20}], [28, {19, 22}], [29, {12, 15, 23}], [30, {10, 14, 16, 23}],
[31, {11, 13, 17, 23}], [32, {18, 24}], [33, {10, 11, 20, 21}],
[34, {8, 9, 23}], [35, {21, 25, 29}], [36, {23}],
[37, {26, 27, 30, 31, 33, 35}] [38, {18, 22, 29, 34, 36}], [39, {24, 30, 34}],
[40, {26, 30, 36}], [41, {19, 31, 34, 36}], [42, {28, 32, 37, 38, 39, 40, 41}].

This yields the intricate graph of descent relations (figure 1).

5. Extended permutation classes

If two patterns Π_1 and Π_2 verify $\Pi_2 = L \cdot \Pi_1 \cdot R$, where both L and R are permutation matrices, the realizations of $\varphi_1 = I_1 \cdot J_1$ (resp. $\varphi_2 = I_2 \cdot J_2$) we get from Π_1 (resp. Π_2) are again of the same degree, even if L and R are not inverses of each other. We even have $\varphi_1^2 = \varphi_2^2$. This explains why in the table of generating functions, non-similar patterns may share the same generating function. In the figures, we have used framing boxes to indicate these equivalences, listed in the following table:

No of param.	Classes
2	$\Gamma_1 = \{1, 2, 3\}$
3	$\Gamma_2 = \{4, 5, 6, 7\}$
4	$\Gamma_3 = \{8, 9\}$, $\Gamma_4 = \{10, 17\}$
	$\Gamma_5 = \{11, 16\}$, $\Gamma_6 = \{12, 13, 14, 15\}$
5	$\Gamma_7 = \{18, 22\}$, $\Gamma_8 = \{19, 24\}$
	$\Gamma_9 = \{20, 23\}$, $\Gamma_{10} = \{21, 25\}$
6	$\Gamma_{11} = \{26, 30, 31, 33\}$, $\Gamma_{12} = \{27, 29\}$, $\Gamma_{13} = \{28, 32\}$
7	$\Gamma_{14} = \{34, 36\}$, $\Gamma_{15} = \{35\}$
8	$\Gamma_{16} = \{37\}$
10	$\Gamma_{17} = \{38\}$, $\Gamma_{18} = \{39, 40, 41\}$
16	$\Gamma_{19} = \{42\}$

Remark. φ^2 commutes with matrix transposition t as well. As a consequence, t acts on the classes Γ. It actually acts trivially on these classes.

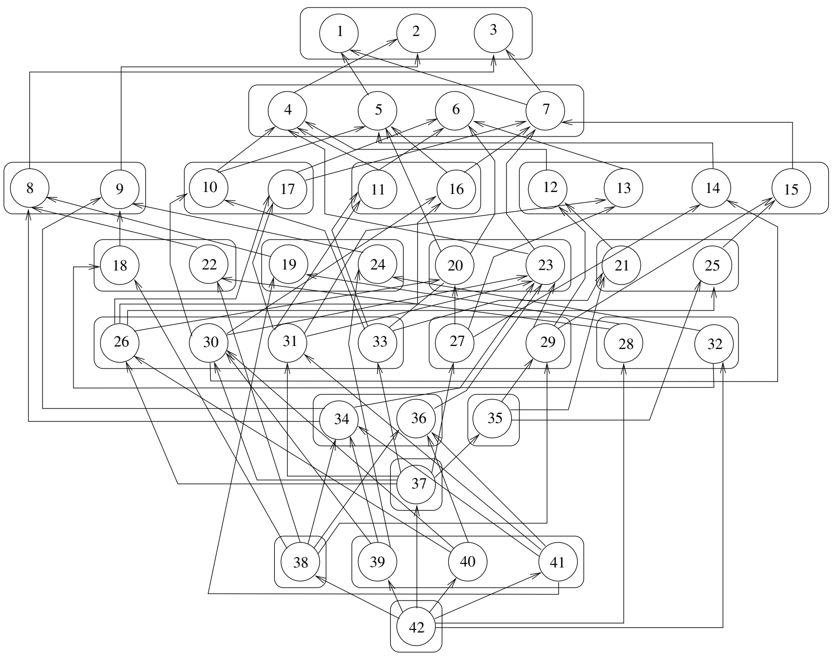

Figure 1. Diagram of descent relations.

6. Gauge equivalence

It is important to take into account the usual gauge symmetries of the Boltzmann weights. Their action may be represented by a similarity transformation:

$$W \longrightarrow W' = g^{-1} \cdot W \cdot g \qquad \text{with} \quad g = \begin{pmatrix} s_1 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \\ 0 & 0 & s_3 & 0 \\ 0 & 0 & 0 & s_4 \end{pmatrix}. \tag{14}$$

They commute with the generator I of $\mathbf{I}$-symmetries, but not with J. However, if two matrices verify (14), then their images by J are automatically gauge related by

$$g' = \begin{pmatrix} 1/s_1 & 0 & 0 & 0 \\ 0 & 1/s_2 & 0 & 0 \\ 0 & 0 & 1/s_3 & 0 \\ 0 & 0 & 0 & 1/s_4 \end{pmatrix}.$$

This means that J has a well-defined induced action on the gauge equivalence classes, although it does not commute with the gauge transformations. Transformations such as (14) yield the following equivalences:

$$\begin{aligned} \Gamma_6 &= \{12, 13, 14, 15\} \longrightarrow \Gamma_2 = \{4, 5, 6, 7\} \\ \Gamma_7 &= \{18, 22\} \longrightarrow \Gamma_3 = \{8, 9\} \\ \Gamma_{12} &= \{27, 29\} \longrightarrow \Gamma_9 = \{20, 23\} \\ \Gamma_{13} &= \{28, 32\} \longrightarrow \Gamma_8 = \{19, 24\}. \end{aligned}$$

Through these equivalences we have reductions by one unit of the number of parameters. Moreover for all these reductions, there exists an I-invariant which can be changed by the action of gauge transformations, and the reduction amounts to fixing the value of this invariant. Take for example pattern 12. One of its representatives has Boltzmann matrix:

$$W_{12} = \begin{pmatrix} w_0 & w_1 & w_2 & w_2 \\ w_1 & w_0 & w_2 & w_2 \\ w_3 & w_3 & w_1 & w_0 \\ w_3 & w_3 & w_0 & w_1 \end{pmatrix}.$$

The quantity

$$\Delta = \frac{1}{2}\left(\frac{w_2}{w_3} + \frac{w_3}{w_2}\right)$$

is I-invariant. It can be brought to the value $\Delta = 1$ by a gauge transformation of the form (14) with $s_1 = s_2$, $s_3 = s_4$, and $w_2 s_3^2 = w_3 s_2^2$. This makes $W'_{12} = g^{-1} \cdot W_{12} \cdot g$ symmetric and yields pattern 4, with Boltzmann matrix:

$$W'_{12} = W_4 = \begin{pmatrix} w_0 & w_1 & w_2 & w_2 \\ w_1 & w_0 & w_2 & w_2 \\ w_2 & w_2 & w_1 & w_0 \\ w_2 & w_2 & w_0 & w_1 \end{pmatrix}.$$

We may picture the gauge equivalences by the diagram given in figure 2.

The usual models which have been studied in the literature are present in the top of this list (they have a low number of parameters). The standard scalar Potts model is pattern 3. The symmetric Ashkin–Teller model is pattern 7. The Ashkin–Teller (alias Kashiwara–Miwa [7] with $N = 4$) model is pattern 16. The chiral Potts model is pattern 17.

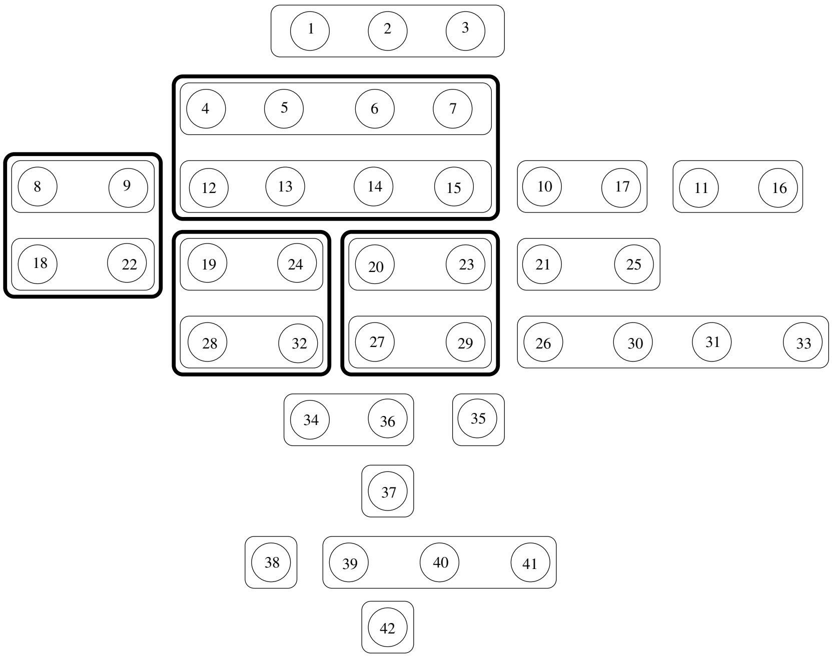

Figure 2. Gauge reductions.

They all share the property that **I**-symmetries have rational invariants, i.e. the orbits of the realization of **I**-symmetries are confined to algebraic varieties. Since we know how crucial is the role played by these varieties in the solvability of the models, it is natural to look for the structure of invariants of the admissible patterns.

7. Invariants of the action of I-symmetries

Since all elements in an extended class yield the same birational realizations, *up to a permutation of coordinates*, invariants may associated with the classes Γ.

The descendance of a given pattern being obtained by adding relations among the entries of W, descendants inherit induced invariants (rational or not). Some invariants will disappear or coalesce in the process. What might also happen is that a transcendent invariant degenerates to a rational one.

The value of κ is related to the existence of conserved algebraic varieties under the action of **I**-symmetries. If there exist invariant algebraic curves, κ is necessarily 0, 1 or 2 [15, 16]. If the curves are rational, $\kappa = 0$ or 1. The values of κ are shown in the diagram in figure 3.

All the values of κ are corroborated by the procedure presented in paragraph 3. A number of iterations of the order of 30 and an averaging over a number of the order of ten initial matrices is enough to get a good precision on the value of κ.

There exists an unbounded algorithm to find rational invariants [12]. There is however no simple way to discover the non-rational ones, and it is an open problem to find all I-invariants. When the number of parameters is small, a graphical approach may be extremely useful (see later with pattern 25). In the next section we will examine in detail the structure of typical patterns.

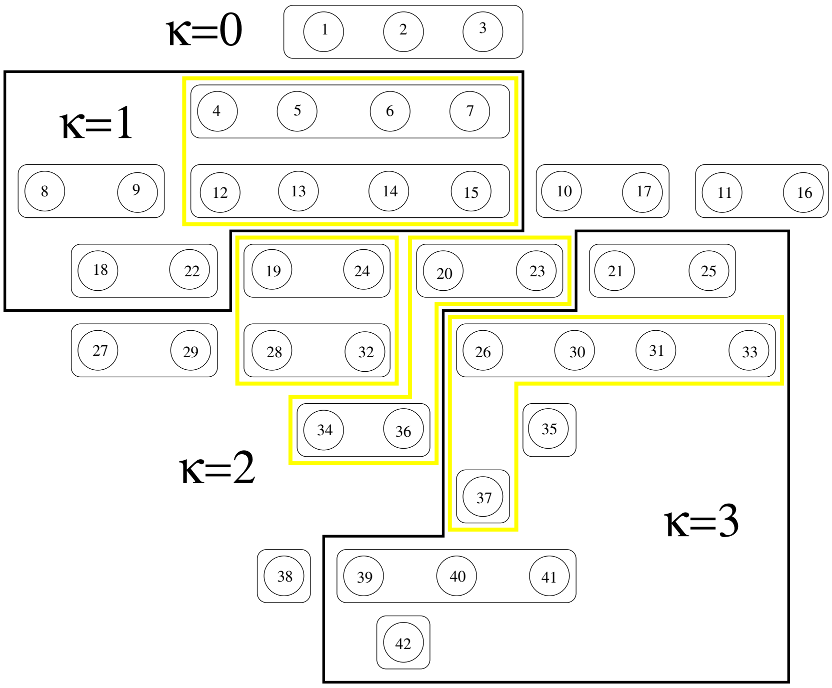

Figure 3. Values of the index κ.

8. Analysis of specific patterns

8.1. Pattern 3: Standard scalar Potts model

The Boltzmann matrix is

$$W_3 = \begin{pmatrix} w_0 & w_1 & w_1 & w_1 \\ w_1 & w_0 & w_1 & w_1 \\ w_1 & w_1 & w_0 & w_1 \\ w_1 & w_1 & w_1 & w_0 \end{pmatrix}.$$

In terms of the parameter $w = w_1/w_0$, the action of $\varphi = I \cdot J$ reduces to a homographic transformation: $I : w \to -2 - w, J : w \to 1/w, \varphi : w \to -2 - 1/w$. In terms of the variable $t = 1/(1 + w)$, φ is the translation $t \longrightarrow t - 1$.

8.2. Pattern 7: Symmetric Ashkin–Teller model

The Boltzmann matrix is

$$W_7 = \begin{pmatrix} w_0 & w_1 & w_2 & w_2 \\ w_1 & w_0 & w_2 & w_2 \\ w_2 & w_2 & w_0 & w_1 \\ w_2 & w_2 & w_1 & w_0 \end{pmatrix}$$

and there is one algebraic invariant

$$\Delta_7 = \frac{w_0 w_1 - {w_2}^2}{w_2(w_1 - w_0)}.$$

The curves $\Delta_7 = a$ have a rational parametrization

$$w_0 = t \qquad w_1 = \frac{1 - at}{t - a} \qquad w_2 = 1.$$

8.3. Pattern 8

The Boltzmann matrix is

$$W_8 = \begin{pmatrix} w_0 & w_1 & w_1 & w_1 \\ w_1 & w_2 & w_3 & w_3 \\ w_1 & w_3 & w_2 & w_3 \\ w_1 & w_3 & w_3 & w_2 \end{pmatrix}. \tag{15}$$

The quantity

$$\Delta_8^{(1)} = \frac{w_2 {w_1}^2 + 2{w_1}^2 w_3 - 2w_0 w_3 w_2 - w_0 {w_3}^2}{w_2 {w_1}^2 - w_0 {w_3}^2} \tag{16}$$

is invariant by J and changes sign under the action of I. The action of φ exchanges the surfaces $\Sigma_+ \left(\Delta_8^{(1)} = a\right)$ and $\Sigma_- \left(\Delta_8^{(1)} = -a\right)$. Since the condition $\Delta_8^{(1)} = a$ may be solved rationally in w_0, we may write the action of φ^2 on Σ_+, with coordinates w_1, w_2, w_3. In terms of the inhomogeneous variables $x = w_1/w_3$ and $y = w_3/(w_2 + w_3)$, it reads

$$x \longrightarrow x' = x \cdot \frac{(b + by - y)}{(2 + y)(by - 2y + b - 1)} \tag{17}$$

$$y \longrightarrow y' = y + b \tag{18}$$

with $b = 2a/(1 + a)$, and the quantity

$$\Delta_8^{(2)} = x\Phi(y) \tag{19}$$

with

$$\Phi(y) = \Gamma\left(\frac{y - 2 + b}{b}\right)\Gamma\left(\frac{y + 1}{b}\right)\Big/\Gamma\left(\frac{y - 1 + b}{b}\right)\Big/\Gamma\left(\frac{y}{b}\right) \tag{20}$$

is invariant by the action of φ^2 on Σ_+. This example, where $\kappa = 1$, possesses a mixture of algebraic and non-algebraic invariants, which can be evaluated exactly. The orbits of φ are confined to non-algebraic curves, and the nth iterate may be written explicitly:

$$y_n = y_0 + nb \qquad x_n = x_0 \frac{\Phi(y_0)}{\Phi(y_n)}. \tag{21}$$

The orbits accumulate to the point $(x_\infty, y_\infty) = \left(\Delta_8^{(2)}(x_0, y_0), \infty\right)$, which is a point on the line $w_2 + w_3 = 0$.

Pattern 8 leads to pattern 3 if $w_3 = w_1$ and $w_2 = w_0$, in which case the above two invariants take definite values: $\Delta_8^{(1)} = \infty$ and $\Delta_8^{(2)} = 1$.

8.4. Pattern 16: Ashkin–Teller model

This model coincides with the $N = 4$ Kashiwara–Miwa model [7]. The Boltzmann matrix is

$$W_{16} = \begin{pmatrix} w_0 & w_1 & w_2 & w_3 \\ w_1 & w_0 & w_3 & w_2 \\ w_2 & w_3 & w_0 & w_1 \\ w_3 & w_2 & w_1 & w_0 \end{pmatrix}. \tag{22}$$

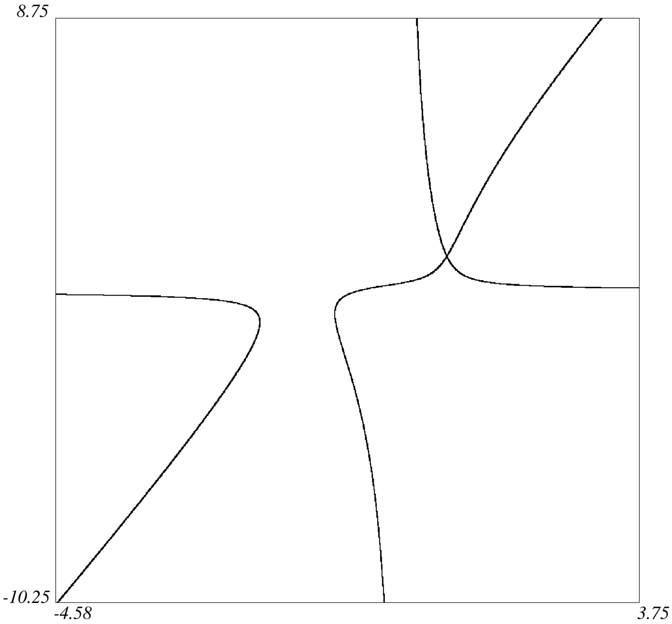

Figure 4. A typical orbit of $\varphi = I \cdot J$ for the chiral Potts model.

There are two independent algebraic I-invariants

$$\Delta_{16}^{(1)} = \frac{w_1 w_3 - w_0 w_2}{w_0 w_1 - w_3 w_2} \qquad \Delta_{16}^{(2)} = \frac{w_1 w_3 - w_0 w_2}{w_0 w_3 - w_1 w_2}. \tag{23}$$

Fixing the values of these two invariants to a_1 and a_2 determines an elliptic curve in the space of parameters. The equation of this curve is

$$a_1 x^2 - a_1 y^2 - a_2 x - a_2 a_1 y + a_2 y^2 x + a_2 a_1 y x^2 = 0$$
$$\text{with} \qquad \frac{w_1}{w_3} = x \qquad \frac{w_2}{w_3} = y \qquad \frac{w_0}{w_3} = \frac{x + a_1 y}{y + a_1 x}. \tag{24}$$

8.5. Pattern 17: chiral Potts model

The chiral Potts model is equivalent to pattern 17. Its Boltzmann matrix was already given above (equation (4)). There exist two algebraically independent invariants

$$\Delta_{\text{Potts}}^{(1)} = \frac{w_3 w_2 - w_0 w_1}{w_1 w_2 - w_0 w_3} \tag{25}$$

$$\Delta_{\text{Potts}}^{(2)} = \frac{\left(w_0 w_2 - w_1{}^2\right)\left(w_0 w_2 - w_3{}^2\right)}{(w_0 w_3 - w_1 w_2)^2}. \tag{26}$$

The condition $\Delta_{\text{Potts}}^{(1)} = a$ may be solved rationally in w_0. Setting $\Delta_{\text{Potts}}^{(2)} = b$, together with $x = w_1/w_2$ and $y = w_3/w_2$, yields the equation of an elliptic curve to which the orbit is confined:

$$(x - y)^2 (x + y)^2 b - (-y x^2 a + a x + x^3 - y)(-y + a x + y^2 x - y^3 a) = 0. \tag{27}$$

We may illustrate the efficiency of the graphical method in this case, since the set of the iterates of φ is dense in a curve[6]. Figure 4 shows a typical orbit in the three-dimensional space of

[6] Contrary to what happens in the example of pattern 8 above.

parameters. The orbit is projected on a coordinate plane. The aspect of the picture is very stable under changes of the starting point of the iteration.

The transformations I and J may be restricted to the quadrics $a = cst$. They may be written on x and y, for fixed a:

$$\begin{aligned} I_a : x \longrightarrow x' &= -\frac{y^3 - 2y^2xa + 2ya + x^2ya^2 - a^2x - x}{1 - ay^2 + a^2yx - xy - a^2 + ax^2} \\ y \longrightarrow y' &= \frac{y^2x - 2x^2ay - y - ya^2 + 2ax + x^3a^2}{1 - ay^2 + a^2yx - xy - a^2 + ax^2} \end{aligned} \tag{28}$$

$$J_a : x \longrightarrow x' = 1/x \qquad y \longrightarrow y' = 1/y. \tag{29}$$

The transformations I_a and J_a leave the curves (27) globally invariant *whatever a and b are*.

The modular invariant of the elliptic curve (27) is given by

$$\begin{aligned} j &= 1728\frac{g_2^3}{g_2^3 - 27g_3^2} \qquad \text{with} \\ g_2 &= 1 + 256b^4 - 512ab^3 - 16b^2a^4 + 288b^2a^2 - 16b^2 + 16a^5b - 32a^3b + 16ab \\ &\quad - 4a^6 + a^8 - 4a^2 + 6a^4 \\ g_3 &= (1 - 32b^2 + 32ab - 2a^2 + a^4)(1 + 16b^2 - 16ab - 2a^2 + a^4) \\ &\quad \times (1 - 8b^2 + 8ab - 2a^2 + a^4)/(3\sqrt{3}). \end{aligned} \tag{30}$$

The modular invariant (30) can be written

$$j = 256\frac{(1 - \mathcal{M} + \mathcal{M}^2)^3}{\mathcal{M}^2(1 - \mathcal{M})^2}$$

where

$$\mathcal{M} = \frac{(1 - 2a + a^2 + 4b)(1 + 2a + a^2 - 4b)}{(1 - a)^2(1 + a)^2} \tag{31}$$

is the square of the modulus of the elliptic functions parametrizing the curve. Note that $\mathcal{M}$ is ambiguously defined by condition (31), and may be replaced by any of the six values $\{\mathcal{M}, 1 - \mathcal{M}, 1/\mathcal{M}, 1 - 1/\mathcal{M}, 1/(1 - \mathcal{M}), \mathcal{M}/(\mathcal{M} - 1)\}$.

There are equivalent symmetric biquadratic forms of the curve (27):

$$\begin{aligned} p^2q^2 - 2\left({J_x}^2{J_y}^2 + {J_y}^2{J_z}^2 + {J_x}^2{J_z}^2\right)pq + 4{J_x}^2{J_y}^2{J_z}^2(p + q) + \left({J_x}^2{J_y}^2 + {J_y}^2{J_z}^2 + {J_x}^2{J_z}^2\right)^2 \\ - 4{J_x}^2{J_y}^2{J_z}^2\left({J_x}^2 + {J_y}^2 + {J_z}^2\right) = 0 \end{aligned}$$

and

$$(J_x - J_y)(p^2q^2 + 1) - (J_x + J_y)(p^2 + q^2) + 4J_z pq = 0$$

with

$$J_x = 2a \qquad J_y = 4b - 2a \qquad J_z = a^2 + 1 \qquad \text{and} \qquad \mathcal{M} = \frac{{J_z}^2 - {J_y}^2}{{J_z}^2 - {J_x}^2}.$$

Changing $\mathcal{M}$ to $1 - \mathcal{M}$ or $1/\mathcal{M}$ amounts to permuting J_x, J_y, and J_z.

Generic orbits of $\varphi = I \cdot J$ are infinite, but we know how important are the degenerate cases where these orbits are finite: for two-dimensional lattice models, this is where non-trivial star–triangle integrability takes place [9, 17]. For given values of a and b, the action of φ is a shift on the curve (27).

It is possible to write down the Weierstrass form of the curve

$$X^3 - \alpha X - \beta + Y^2 = 0 \tag{32}$$

with[7]

$$\alpha = \frac{1}{48}\frac{g_2}{(b-a)^8} \qquad \beta = \frac{\sqrt{3}}{288}\frac{g_3}{(b-a)^{12}} \tag{33}$$

and give an explicit coordinate transformation from the original variables (x, y) to $[X, Y]$. We obtained the transformation through Maple's implementation of the van Hoeij algorithm [18], after setting $b = a + 1/u^2$.

The action of φ, when written on (32) is the addition[8] of the point Π_+ with coordinates $[X_{\pi_+}, Y_{\pi_+}]$:

$$X_{\pi_+} = -1/12\frac{1+16b^2+8ab-2a^2+12b+12a^2b+a^4}{(b-a)^4}$$
$$Y_{\pi_+} = 1/2\frac{b(1+a)^2(1+4b-2a+a^2)}{(b-a)^6}.$$

Although the change of coordinates depends on u, the $[X, Y]$ coordinates of the point Π_+ depend only on u^2 and may be written rationally in terms of a and b.

The inverse map φ^{-1} is the addition of the opposite point $\Pi_- = [X_{\pi_+}, -Y_{\pi_+}]$. Note that both Π_+ and Π_- are obtained from the double point $x = -1/au$, $y = \infty$ of (27) in the desingularization process which leads to the Weierstrass form (32).

The sum of $\Pi_\pm$ with itself, denoted $2 \cdot \Pi_\pm$, is given by

$$\left[-1/12\frac{1+a^4+10a^2-16ab+16b^2}{(b-a)^4}, \pm 1/2\frac{(1+a^2)(2b-a)a}{(b-a)^6}\right] \tag{34}$$

and the multiples $n \cdot \Pi_\pm$ are then easily obtained by recurrence. Their $[X, Y]$ coordinates depend only on u^2 and may be re-expressed rationally in terms of a and b.

The point $2 \cdot \Pi_+$ corresponds to the point

$$(x, y) = \left(\frac{(1+a^2)au}{1-u^2a^3-a^2}, \frac{(2+au^2)a^2u}{1-u^2a^3-a^2}\right)$$

of the original curve (27).

The condition $\varphi^{(m+n)} = \mathrm{id}$ is just

$$X(n \cdot \Pi_+) = X(m \cdot \Pi_-) \qquad \text{and} \qquad Y(n \cdot \Pi_+) = -Y(m \cdot \Pi_-).$$

As an example, the condition that φ is of order 3 is

$$v_3 = (1+a)^2b - a^2 = 0 \tag{35}$$

which is an algebraic condition on the entries of the Boltzmann matrix, and can be checked directly.

The condition that φ is of order 4 is the vanishing of $Y(2 \cdot \Pi_+)$ given in (34), i.e.

$$v_4 = (1+a^2)(2b-a)a.$$

The condition $2b - a = 0$ is precisely the condition appearing in [9, 17], where solutions of the star–triangle relations related to higher genus curves were exhibited. The condition $v_3 = 0$ is not known to yield similar integrability conditions and was not found to yield any transition in the statistics for the random matrix theory approach (see [19]).

[7] The normalizations could be changed, and the powers of $1/(b-a)$ could be absorbed in the definition of X and Y.
[8] To add two points on the cubic (32), draw the straight line through the two points. Compute the third point of intersection of the line with the curve, and take its symmetric under $Y \to -Y$.

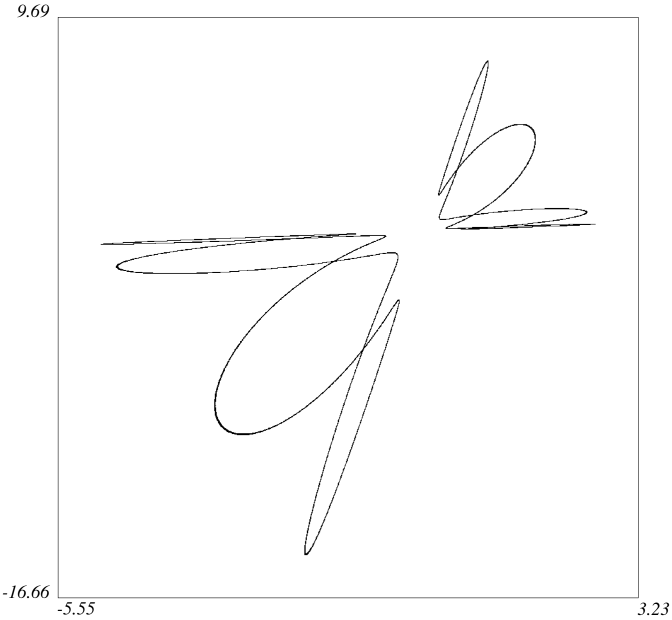

Figure 5. A typical orbit of I-symmetries for pattern 25.

8.6. Pattern 25

This example is a good illustration of the mixed use of algorithmic research of algebraic invariants, graphical method, and the knowledge of the entropy and index κ. The Boltzmann matrix is

$$W_{25} = \begin{pmatrix} w_0 & w_1 & w_2 & w_3 \\ w_4 & w_0 & w_4 & w_2 \\ w_2 & w_3 & w_0 & w_1 \\ w_4 & w_2 & w_4 & w_0 \end{pmatrix}. \qquad (36)$$

The number of inhomogeneous parameters is four. We found two independent algebraic invariants

$$\Delta_{25}^{(1)} = \frac{w_0 w_1 - w_3 w_2}{w_0 w_3 - w_1 w_2} \qquad (37)$$

$$\Delta_{25}^{(2)} = \frac{\left(w_4 w_3 - w_0{}^2\right)\left(w_1 w_4 - w_2{}^2\right)}{\left(w_4 w_3 - w_2{}^2\right)\left(w_1 w_4 - w_0{}^2\right)}. \qquad (38)$$

Drawing orbits of the iterates of φ leads, with a proper choice of starting point, to pictures like figure 5. In contrast to what happens with pattern 17 (figure 4), the shape of the orbit is rather unstable under changes of the starting point. Such a picture indicates that there exists another independent invariant. Were that invariant to be algebraic, the stable curves would be elliptic, and we know that the index κ would then be less than 3. Since $\kappa = 3$ we know this additional invariant has to be non-algebraic.

Equation of the surface $\Sigma_1 : \Delta_{25}^{(1)} = a$ may be solved for w_1. We get an expression of I and J on Σ_1. Changing variables to

$$[x, y, z, t] = \left[w_0^2, w_0 w_2, w_0 w_3, w_3 w_4\right]$$

we obtain

$$\begin{aligned} I_a([x, y, z, t]) = [&x(x + ay - at - t)^2(x + y)(x + ay), \\ &(x + y)(-yx + xta + xt - y^2a)(x + ay)(x + ay - at - t), \\ &-z(x + y)(x^2 + xta^2 - xt - y^2a^2)(x + ay - at - t), \\ &(x + ay)(x - y)t(x^2 + xta^2 - xt - y^2a^2)] \end{aligned} \tag{39}$$

with the remaining algebraic invariant

$$\Delta_{25}^a = \frac{x(t - y)(yt - yx - y^2a + axt)}{(x - y)^2t(x + y)}. \tag{40}$$

Note that in equation (39), the third coordinate z is just multiplied by a factor depending on the other coordinates, and does not appear anywhere else in the induced transformation I_a nor in the invariant Δ_{25}^a. The curves $\Delta_{25}^a = b$ are elliptic curves in the variables (x, y, t), extending to cylinders in $[x, y, z, t]$. The curve appearing in figure 5 is drawn on such a cylinder.

We may examine a particular value of b where the curve $\Delta_{25}^a = b$ degenerates to a rational curve, but there exists a transcendent invariant for I_a. The rational parametrization of the cylinder is

$$[x, y, z, t] = \left[\frac{s^2t(1 + sa)}{s + a}, \frac{st(1 + sa)}{s + a}, z, t\right]. \tag{41}$$

The induced map $(I \cdot J)^2$ may be written

$$S \longrightarrow q^2S \qquad z \longrightarrow z\frac{T(S)}{T(qS)} \tag{42}$$

$$\text{with} \quad T(S) = \frac{(1 - S)(1 - q^3S)}{(1 - q^2S)(1 - q^5S)} \qquad \text{and}$$

$$S = \frac{s - q^{-1}}{s - q} \qquad q = -1/2\frac{a + 1 + \sqrt{(1 - a)(1 + 3a)}}{a}.$$

If we introduce the infinite product

$$\Pi(S) = \prod_{k=0}^{k=\infty} \frac{T(q^{4k}S)}{T(q^{4k+1}S)} \tag{43}$$

then the $2n$th iterate of $(i.j)$ reads

$$S_{2n} = q^{4n}S_0 \qquad z_{2n} = z_0 \cdot \frac{\Pi(S_0)}{\Pi(S_{2n})} \tag{44}$$

i.e. the transcendent quantity $\Delta = z \cdot \Pi(S)$ is invariant by $(I \cdot J)^2$. The convergence of these Eulerian products is ensured when $a \in (-1/3, 1)$. Such values of a would produce orbits with accumulation points. If a is outside this interval, then q has unit modulus and the orbits are similar to the one shown in figure 5. The quantity $\Pi(S)$ is replaced by the standard analytic prolongation.

9. Other values of *q*

We have investigated the structure of the set of admissible patterns for other values of q.

The value $q = 4$ is an important threshold. It is known to be one for the standard scalar Potts model in statistical mechanics [5], and for the chromatic polynomials and colouring problems. What we show is that it is *the maximal value ensuring the vanishing of entropy, whatever admissible pattern we choose*.

In the case of a cyclic matrix of size q (chiral Potts model), with entries $[w_0, \ldots, w_{q-1}]$, there is a simple relation between matrix inverse and element by element inverse. Define the linear transformation C (discrete Fourier transform, or Kramers–Wannier duality)

$$C: \; w_k \longrightarrow \sum_{r=0}^{q-1} \omega^{kr} w_r$$

where ω is the qth root of unity. We then have

$$I = C^{-1} \cdot J \cdot C.$$

It is possible to analyse the singularity structure of the transformations I and J, and prove that the generating function for the degree of the iterates of φ for the chiral Potts model with q states is [13]

$$f_q(s) = \frac{(1 + s(q-1))^2}{(1-s)(1 - s(q^2 - 4q + 2) + s^2)}. \tag{45}$$

As soon as $q \geqslant 5$, the entropy becomes strictly positive for this family of admissible patterns.

For all sizes q, there exists a pattern depending on $q(q+1)/2$ homogeneous parameters: the symmetric matrix.

An open question is to decide whether or not there exist patterns depending on r parameters, with $q(q+1)/2 < r < q^2$. For $q = 3$ and $q = 4$ there is a no man's land between $q(q+1)/2$ parameters (symmetric matrix) and q^2 parameters (generic $q \times q$ matrix).

Another open question is to know if there is anything between cyclic and symmetric matrix when q is prime.

The value $q = 2$ is trivial and yields a finite group of symmetries. When $q = 3$, the same analysis can be performed completely (see below). When $q = 5$, the number of patterns is too large to be thoroughly examined, and we have restricted ourselves to patterns with only three homogeneous parameters.

Of course there exist admissible patterns with larger values of q still yielding vanishing entropy. They may have, for example, a block structure, the blocks being themselves admissible patterns of smaller sizes [20–23].

9.1. Three-state models

The analysis can be made along the same lines as for the four-state models: enumeration, calculation of the entropy, calculations of invariants, etc. It is summarized here.

The number of partitions of nine objects is 21 147 and the number of admissible patterns is only 9, when equivalences are taken into account. The list of patterns is given in appendix C. For patterns 1, 2, 3, 4 the group of **I**-symmetries is finite. Patterns 3 and 4 are related by an extended permutation similarity as defined in section (5). Patterns 5 and 6 yield the *same* generating function for the degrees of the iterates of φ

$$g_5 = g_6 = \frac{(1+2s)(1+s)^2}{(1+s+s^2)(1-s)^2}$$

and the trajectories of φ are confined to algebraic curves. Patterns 7, 8, 9 yield the generating function

$$g_7 = g_8 = g_9 = \frac{1 + 6s + 9s^2 + 2s^3 + 6s^6 + 3s^7 - 6s^8 - 3s^9}{(1+s+s^2)(1-s)^2}. \tag{46}$$

Patterns 7 and 8 (pattern 8 is the 3×3 symmetric matrix) are also related by an extended permutation similarity. A list of algebraic invariants is given for the various patterns in appendix D.

For pattern 8 (symmetric 3×3 matrices), the Boltzmann matrix is

$$W_8 = \begin{pmatrix} w_0 & w_1 & w_2 \\ w_1 & w_3 & w_4 \\ w_2 & w_4 & w_5 \end{pmatrix}. \tag{47}$$

If we normalize the entries with the condition $w_0 = 1$, the action of φ^3 reduces to the following homothetic transformation:

$$[1, w_1, w_2, w_3, w_4, w_5] \longrightarrow [1, \Delta_1 w_1, \Delta_2 w_2, \Delta_3 w_3, \Delta_4 w_4, \Delta_5 w_5]$$

with

$$\Delta_1 = \frac{\left(w_5 - {w_2}^2\right) w_3 w_4(-w_1 w_2 + w_4)}{\left(-w_3 w_5 + {w_4}^2\right) w_2(-w_2 w_3 + w_1 w_4)}$$
$$\Delta_2 = \frac{\left(w_3 - {w_1}^2\right) w_5 w_4(-w_1 w_2 + w_4)}{\left(-w_3 w_5 + {w_4}^2\right)(-w_1 w_5 + w_4 w_2) w_1}$$
$$\Delta_3 = \Delta_1^2 \qquad \Delta_4 = \Delta_1 \Delta_2 \qquad \Delta_5 = \Delta_2^2$$

where Δ_1 and Δ_2 are left invariant by the action of φ.

The appearance of the third root of unity in 46 has to do with the specificity of the third power of φ.

There is a parametrization of the general three-state model (pattern 9) which simplifies the action of **I**-symmetries. The matrix of Boltzmann weights reads

$$W = \begin{pmatrix} w_0 & w_1 & w_2 \\ w_3 & w_4 & w_5 \\ w_6 & w_7 & w_8 \end{pmatrix}.$$

If one introduces the quantities

$$\alpha = w_0^2 \qquad \beta = w_4^2 \qquad \gamma = w_8^2$$
$$\delta = w_1 w_3 \qquad \epsilon = w_2 w_6 \qquad \zeta = w_5 w_7$$
$$\lambda = \frac{w_1}{w_3} \qquad \mu = \frac{w_6}{w_2} \qquad \nu = \frac{w_5}{w_7}.$$

The array $[\alpha, \beta, \gamma, \delta, \epsilon, \zeta]$ transforms under φ^3 by multiplication of its entries by factors $[k_\alpha, k_\beta, k_\gamma, k_\delta, k_\epsilon, k_\zeta]$. One can check that all the ratios $k_\alpha/k_\beta, \ldots$ are invariants of φ^3. In a similar way, the three quantities $[\lambda, \mu, \nu]$ transform under φ^6 by multiplication by factors $[k_\lambda, k_\mu, k_\nu]$ which are all invariants of φ^6.

This yields a system of algebraic invariants of rank four of φ^3, for example:

$$\left[\frac{w_1 w_5 w_6}{w_2 w_3 w_7}, \quad \frac{w_3 w_7 w_1 w_5}{{w_4}^2 w_6 w_2}, \quad \frac{{w_0}^2 w_5 w_7}{w_1 w_2 w_3 w_6}, \quad \frac{{w_8}^2 w_3 w_1}{w_2 w_6 w_5 w_7}\right].$$

9.2. Five-state models and beyond

The number of partitions of 25 objects is over 4×10^{18}. To enumerate the configurations is beyond reach. We have restricted our analysis in the following way: if one looks for partitions with only three parts, that is to say five-state models with only three possible colours (three homogeneous parameters), then there are 'only' 141 197 991 025 $\simeq 141 \times 10^9$ patterns. We have performed their full enumeration, and found the subset of admissible ones. The interesting result is that there exists a unique admissible pattern, up to permutations of rows and columns: the symmetric Potts model, with symmetric cyclic 5×5 matrix of Boltzmann weights

$$W = \begin{pmatrix} w_0 & w_1 & w_2 & w_2 & w_1 \\ w_1 & w_0 & w_1 & w_2 & w_2 \\ w_2 & w_1 & w_0 & w_1 & w_2 \\ w_2 & w_2 & w_1 & w_0 & w_1 \\ w_1 & w_2 & w_2 & w_1 & w_0 \end{pmatrix}.$$

There are actually 72 representatives, all related by independent permutations of rows and columns, as in section 5.

The trajectories of the iterations of φ are confined to algebraic curves. The entropy vanishes and the secondary index κ has value 2 [15, 16]. The corresponding statistical mechanical model is known to be integrable, via the star–triangle relation, when these curves degenerate to rational curves and the symmetry group becomes finite [9, 10, 17, 24].

One should keep in mind that admissible patterns of size 5×5 easily yield transformations with exponential growth of the degree. For example, the cyclic matrix (chiral Potts model, see equation (45)) yields a map with non-vanishing entropy

$$\epsilon = \log\left(\frac{7 + 3\sqrt{5}}{2}\right) \tag{48}$$

and the corresponding statistical mechanical model is generically not star–triangle integrable.

We have used the arithmetical method described in section 3 to evaluate the entropy for both the symmetric pattern (symmetric 5×5 Boltzmann matrix) and the general five-state model (5×5 Boltzmann matrix with no relations between the entries)[9]. We found *the same value* for the entropy as for the cyclic matrix, given by equation (48).

In contrast, if one restricts the 5×5 Boltzmann matrix to be both cyclic and symmetric, the entropy vanishes, and the transformation φ has one algebraic invariant [25].

One may wonder if the entropy calculated for the cyclic matrix, the symmetric matrix, and the most general $q \times q$ matrix still identify for $q \geqslant 6$. One may also try to evaluate the entropy for the cyclic-symmetric pattern, and provide a closed expression valid for all values of q, and similar to (48).

10. Conclusion and perspectives

We have given a classification of four-state spin models according to the properties of the realization of **I**-symmetries.

This preparatory work provides the optimal parametrizations of the models.

If one is looking for integrability via solutions of star–triangle equations, one should stick to cases where the parameters live on invariant algebraic varieties. Those may be present for generic values of the parameters, or appear by a collapse of the realization of the infinite group to a finite group, or by the degeneration of transcendental invariants to algebraic ones.

The algebraic approach presented here may be complemented with random matrix theory, as in [19], to locate solvability of the lattice models.

One may have to examine in detail the properties of both the matrix product and the element by element product, and not just the stability by the corresponding inverses, when mixed with matrix transposition. One should note that the Boltzmann matrices of the models

[9] For a generic 5×5 matrix the coefficients got as large as $10^{720\,000}$ after six iterations.

which are known to be integrable via the star–triangle relation, i.e. $\Gamma_1, \Gamma_2, \Gamma_4, \Gamma_5$ with the notations of section 5, are (bi)-stochastic matrices (sums of elements on lines and columns are constant).

A detailed analysis of the star–triangle equations will then proceed on a case by case basis, which is beyond the scope of this paper.

Appendix A. List of the 42 admissible patterns for $q = 4$

The sixteen entries of each 4×4 matrix are presented on one line in the order $[W_{11}, W_{12}, W_{13}, W_{14}, W_{21}, W_{22}, \ldots, W_{43}, W_{44}]$. The choice of the representative is arbitrary, and should be understood modulo permutations of lines and columns. The list is arranged with increasing number of independent homogeneous parameters, and this number is indicated after the list of entries. The last item of each line is the number of patterns in the similarity class:

No 1 : $[w_0, w_0, w_0, w_1, w_0, w_0, w_1, w_0, w_0, w_1, w_0, w_0, w_1, w_0, w_0, w_0]$ 2, 3
No 2 : $[w_0, w_0, w_1, w_0, w_0, w_1, w_0, w_0, w_1, w_0, w_0, w_0, w_0, w_0, w_0, w_1]$ 2, 6
No 3 : $[w_0, w_1, w_1, w_1, w_1, w_0, w_1, w_1, w_1, w_1, w_0, w_1, w_1, w_1, w_1, w_0]$ 2, 1
No 4 : $[w_0, w_1, w_2, w_2, w_1, w_0, w_2, w_2, w_2, w_2, w_1, w_0, w_2, w_2, w_0, w_1]$ 3, 3
No 5 : $[w_0, w_1, w_2, w_0, w_1, w_0, w_0, w_2, w_2, w_0, w_0, w_1, w_0, w_2, w_1, w_0]$ 3, 3
No 6 : $[w_0, w_0, w_1, w_2, w_0, w_0, w_2, w_1, w_2, w_1, w_0, w_0, w_1, w_2, w_0, w_0]$ 3, 3
No 7 : $[w_0, w_1, w_2, w_2, w_1, w_0, w_2, w_2, w_2, w_2, w_0, w_1, w_2, w_2, w_1, w_0]$ 3, 3
No 8 : $[w_0, w_1, w_1, w_1, w_1, w_2, w_3, w_3, w_1, w_3, w_2, w_3, w_1, w_3, w_3, w_2]$ 4, 4
No 9 : $[w_0, w_1, w_1, w_1, w_1, w_2, w_3, w_2, w_1, w_3, w_2, w_2, w_1, w_2, w_2, w_3]$ 4, 12
No 10 : $[w_0, w_1, w_2, w_3, w_1, w_0, w_3, w_2, w_2, w_3, w_1, w_0, w_3, w_2, w_0, w_1]$ 4, 3
No 11 : $[w_0, w_1, w_2, w_3, w_1, w_0, w_3, w_2, w_3, w_2, w_1, w_0, w_2, w_3, w_0, w_1]$ 4, 3
No 12 : $[w_0, w_1, w_1, w_2, w_3, w_2, w_0, w_3, w_3, w_0, w_2, w_3, w_2, w_1, w_1, w_0]$ 4, 3
No 13 : $[w_0, w_1, w_0, w_2, w_2, w_3, w_1, w_3, w_0, w_2, w_0, w_1, w_1, w_3, w_2, w_3]$ 4, 3
No 14 : $[w_0, w_1, w_2, w_0, w_1, w_3, w_3, w_2, w_2, w_3, w_3, w_1, w_0, w_2, w_1, w_0]$ 4, 3
No 15 : $[w_0, w_1, w_1, w_2, w_3, w_0, w_2, w_3, w_3, w_2, w_0, w_3, w_2, w_1, w_1, w_0]$ 4, 3
No 16 : $[w_0, w_1, w_2, w_3, w_1, w_0, w_3, w_2, w_2, w_3, w_0, w_1, w_3, w_2, w_1, w_0]$ 4, 1
No 17 : $[w_0, w_1, w_2, w_3, w_1, w_0, w_3, w_2, w_3, w_2, w_0, w_1, w_2, w_3, w_1, w_0]$ 4, 3
No 18 : $[w_0, w_1, w_2, w_0, w_3, w_4, w_3, w_3, w_2, w_1, w_0, w_0, w_0, w_1, w_0, w_2]$ 5, 12
No 19 : $[w_0, w_1, w_2, w_3, w_1, w_4, w_1, w_1, w_3, w_1, w_0, w_2, w_2, w_1, w_3, w_0]$ 5, 4
No 20 : $[w_0, w_1, w_2, w_0, w_3, w_0, w_0, w_4, w_4, w_0, w_0, w_3, w_0, w_2, w_1, w_0]$ 5, 3
No 21 : $[w_0, w_1, w_2, w_1, w_3, w_2, w_4, w_0, w_2, w_1, w_0, w_1, w_4, w_0, w_3, w_2]$ 5, 6
No 22 : $[w_0, w_1, w_1, w_1, w_2, w_3, w_4, w_4, w_2, w_4, w_3, w_4, w_2, w_4, w_4, w_3]$ 5, 4
No 23 : $[w_0, w_1, w_2, w_2, w_1, w_0, w_2, w_2, w_2, w_2, w_3, w_4, w_2, w_2, w_4, w_3]$ 5, 3
No 24 : $[w_0, w_1, w_1, w_1, w_1, w_2, w_3, w_4, w_1, w_3, w_4, w_2, w_1, w_4, w_2, w_3]$ 5, 4
No 25 : $[w_0, w_1, w_2, w_3, w_4, w_0, w_4, w_2, w_2, w_3, w_0, w_1, w_4, w_2, w_4, w_0]$ 5, 6
No 26 : $[w_0, w_1, w_2, w_3, w_4, w_0, w_3, w_5, w_5, w_3, w_0, w_4, w_3, w_2, w_1, w_0]$ 6, 3
No 27 : $[w_0, w_1, w_0, w_2, w_3, w_4, w_5, w_4, w_0, w_2, w_0, w_1, w_5, w_4, w_3, w_4]$ 6, 3
No 28 : $[w_0, w_1, w_1, w_1, w_2, w_3, w_4, w_5, w_2, w_5, w_3, w_4, w_2, w_4, w_5, w_3]$ 6, 4
No 29 : $[w_0, w_1, w_2, w_1, w_3, w_4, w_3, w_5, w_2, w_1, w_0, w_1, w_3, w_5, w_3, w_4]$ 6, 3
No 30 : $[w_0, w_1, w_2, w_3, w_1, w_4, w_3, w_5, w_2, w_3, w_0, w_1, w_3, w_5, w_1, w_4]$ 6, 3
No 31 : $[w_0, w_1, w_2, w_3, w_3, w_4, w_1, w_5, w_2, w_3, w_0, w_1, w_1, w_5, w_3, w_4]$ 6, 3

No 32 : $[w_0, w_1, w_1, w_1, w_2, w_3, w_4, w_5, w_2, w_4, w_5, w_3, w_2, w_5, w_3, w_4]$	6,	4
No 33 : $[w_0, w_1, w_2, w_3, w_4, w_2, w_5, w_0, w_2, w_3, w_0, w_1, w_5, w_0, w_4, w_2]$	6,	3
No 34 : $[w_0, w_1, w_2, w_1, w_1, w_3, w_4, w_5, w_2, w_4, w_6, w_4, w_1, w_5, w_4, w_3]$	7,	6
No 35 : $[w_0, w_1, w_2, w_2, w_1, w_0, w_2, w_2, w_3, w_4, w_5, w_6, w_4, w_3, w_6, w_5]$	7,	6
No 36 : $[w_0, w_1, w_2, w_2, w_3, w_0, w_4, w_4, w_4, w_2, w_5, w_6, w_4, w_2, w_6, w_5]$	7,	6
No 37 : $[w_0, w_1, w_2, w_3, w_1, w_0, w_3, w_2, w_4, w_5, w_6, w_7, w_5, w_4, w_7, w_6]$	8,	3
No 38 : $[w_0, w_1, w_2, w_2, w_3, w_4, w_5, w_5, w_6, w_7, w_8, w_9, w_6, w_7, w_9, w_8]$	10,	6
No 39 : $[w_0, w_1, w_2, w_3, w_1, w_4, w_5, w_6, w_2, w_5, w_7, w_8, w_3, w_6, w_8, w_9]$	10,	1
No 40 : $[w_0, w_1, w_2, w_3, w_4, w_0, w_5, w_6, w_6, w_3, w_7, w_8, w_5, w_2, w_9, w_7]$	10,	3
No 41 : $[w_0, w_1, w_2, w_3, w_4, w_5, w_1, w_6, w_7, w_4, w_0, w_8, w_8, w_6, w_3, w_9]$	10,	6
No 42 : $[w_0, w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9, w_{10}, w_{11}, w_{12}, w_{13}, w_{14}, w_{15}]$	16,	1.

Appendix B. Generating functions

Pattern number	Generating function
1, 2, 3	$\frac{1}{1-s}$
4, 5, 6, 7	$\frac{(1+s)^2}{(1-s)^2}$
8, 9	$\frac{(1+s)(1+2s-s^2+s^4)}{(1-s)^2}$
10, 11, 16, 17	$\frac{(1+3s)^2}{(1-s)^3}$
12, 13, 14, 15	$\frac{1+3s}{(1-s)^2}$
18, 22	$\frac{1+4s-s^2+s^4+s^5}{(1-s)^2}$
19, 24	$\frac{(1+s)(1+6s+3s^2-s^3)}{(1-s)^3}$
20, 23	$\frac{(1+s)(1+4s-s^2)}{(1-s)^3}$
21, 25	$\frac{1+8s+12s^2+18s^3+13s^4+6s^5-2s^6}{(1+s)(1+s+s^2)(1-s)^4}$
26, 30, 31, 33	$\frac{1+11s+11s^2-7s^3}{(1-s)^4}$
27, 29	$\frac{1+5s+4s^2-3s^3+s^4}{(1-s)^3}$
28, 32	$\frac{(1+s+s^2)(1+7s-2s^2)}{(1-s)^3}$
34, 36	$\frac{1+12s+16s^2+3s^3+8s^4+4s^5-2s^6}{(1+s)(1-s)^3}$

35	$\dfrac{1+12s+24s^2+34s^3+27s^4+14s^5}{(1+s)(1+s+s^2)(1-s)^4}$
37	$\dfrac{1+17s+19s^2-5s^3}{(1-s)^4}$
38	$\dfrac{1+18s+11s^2+10s^3+8s^4+8s^5}{(1+s)(1-s)^3}$
39, 40, 41	$\dfrac{1+23s+15s^2+5s^3+4s^4}{(1-s)^4}$
42	$\dfrac{1+41s+3s^2+35s^3+16s^4}{(1-s)^4}.$

Appendix C. Admissible patterns for $q = 3$

$$
\begin{aligned}
\text{No }1 &: [w_0, w_1, w_1, w_1, w_0, w_1, w_1, w_1, w_0]\\
\text{No }2 &: [w_0, w_0, w_1, w_0, w_1, w_0, w_1, w_0, w_0]\\
\text{No }3 &: [w_0, w_1, w_2, w_2, w_0, w_1, w_1, w_2, w_0]\\
\text{No }4 &: [w_0, w_1, w_2, w_1, w_2, w_0, w_2, w_0, w_1]\\
\text{No }5 &: [w_0, w_1, w_1, w_1, w_2, w_3, w_1, w_3, w_2]\\
\text{No }6 &: [w_0, w_1, w_1, w_2, w_3, w_4, w_2, w_4, w_3]\\
\text{No }7 &: [w_0, w_1, w_2, w_2, w_3, w_4, w_1, w_5, w_3]\\
\text{No }8 &: [w_0, w_1, w_2, w_1, w_3, w_4, w_2, w_4, w_5]\\
\text{No }9 &: [w_0, w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8].
\end{aligned}
$$

Appendix D. Some algebraic invariants for $q = 3$

The lower index is the pattern number

$$\Delta_5^{(a)} = \frac{\left(w_1{}^2 w_3 - w_0 w_2{}^2\right)^2}{w_2\left(w_1{}^2 - w_0 w_2\right)\left(w_2 w_1{}^2 - w_0 w_3{}^2\right)}$$

$$\Delta_5^{(b)} = \frac{\left(w_2 w_1{}^2 - w_0 w_3{}^2\right)^2}{w_3\left(w_1{}^2 - w_0 w_3\right)\left(w_1{}^2 w_3 - w_0 w_2{}^2\right)}$$

$$\Delta_6^{(a)} = \frac{w_1 w_2}{w_2{}^2 + w_1{}^2}$$

$$\Delta_6^{(b)} = \frac{\left(w_1 w_2 w_4 - w_0 w_3{}^2\right)^2}{w_3(w_1 w_2 - w_0 w_3)\left(w_1 w_2 w_3 - w_4{}^2 w_0\right)}$$

$$\Delta_6^{(c)} = \frac{\left(w_1 w_2 w_3 - w_4{}^2 w_0\right)^2}{w_4(w_1 w_2 - w_0 w_4)\left(w_1 w_2 w_4 - w_0 w_3{}^2\right)}$$

$$\Delta_8^{(a)} = \frac{w_0\left(w_3 w_5 - w_4{}^2\right)}{(w_4 w_2 - w_1 w_5) w_1} \qquad \Delta_8^{(b)} = \frac{w_0\left(w_3 w_5 - w_4{}^2\right)}{(w_1 w_4 - w_2 w_3) w_2}$$

$$\Delta_8^{(c)} = \frac{w_2 w_3}{w_1 w_4}.$$

References

[1] Bellon M P, Maillard J-M and Viallet C-M 1991 Infinite discrete symmetry group for the Yang–Baxter equations: spin models *Phys. Lett.* A **157** 343–53

[2] Bellon M P, Maillard J-M and Viallet C-M 1991 Infinite discrete symmetry group for the Yang–Baxter equations: vertex models *Phys. Lett.* B **260** 87–100

[3] Bellon M P, Maillard J-M and Viallet C-M 1991 Rational mappings, arborescent iterations, and the symmetries of integrability *Phys. Rev. Lett.* **67** 1373–6

[4] Baxter R J 1981 *Exactly Solved Models in Statistical Mechanics* (London: Academic)

[5] Wu F Y 1982 The Potts model *Rev. Mod. Phys.* **54** 235

[6] Ashkin J and Teller E 1943 Statistics of two-dimensional lattices with four components *Phys. Rev.* **64** 178

[7] Kashiwara M and Miwa T 1986 A class of elliptic solutions to the star–triangle relation *Nucl. Phys.* B **275** 121–34

[8] Howes S, Kadanoff L P and den Nijs M 1983 Quantum model for commensurate–incommensurate transitions *Nucl. Phys.* B **215** 169

[9] Au-Yang H, Mc Coy B M, Perk J H H, Tang S and Yan M L 1987 Commuting transfer matrices in the chiral Potts models: solutions of the star–triangle equations with genus $\geqslant 1$ *Phys. Lett.* A **123** 219

[10] Baxter R J, Perk J H H and Au-Yang H 1988 New solutions of the star–triangle relations for the chiral Potts model *Phys. Lett.* A **128** 138

[11] Baxter R J 1983 On Zamolodchikov's solution of the tetrahedron equations *Commun. Math. Phys.* **88** 185–205

[12] Falqui G and Viallet C-M 1993 Singularity, complexity, and quasi-integrability of rational mappings *Commun. Math. Phys.* **154** 111–25

[13] Bellon M and Viallet C-M 1999 Algebraic entropy *Commun. Math. Phys.* **204** 425–37

[14] Gnu MultiPrecision Arbitrary precision arithmetics software, see http://www.swox.com/gmp/

[15] Gizatullin M K 1980 Rational G-surface *Izv. Akad. Nauk SSSR. Ser. Mat.* **44** 110–44
(Engl. transl. 1981 *Math. USSR, Izvestiya* **16** 103–34)

[16] Bellon M P 1999 Algebraic entropy of birational maps with invariant curves *Lett. Math. Phys.* **50** 79–90

[17] Hansel D and Maillard J M 1988 Symmetries of models with genus >1 *Phys. Lett.* A **133** 11

[18] van Hoeij M 1995 An algorithm for computing the Weierstrass normal form. *Int. Symp. on Symbolic and Algebraic Computation (ISSAC '95)* pp 90–5

[19] Anglès d'Auriac J-C, Maillard J-M and Viallet C-M 2002 Random matrix theory and higher genus integrability: the quantum chiral Potts model *J. Phys. A: Math. Gen.* **35** 4801–22 (*Preprint* cond-mat/0205101)

[20] Bellon M P, Maillard J-M and Viallet C-M 1991 Integrable coxeter groups *Phys. Lett.* A **159** 221–32

[21] Boukraa S, Maillard J-M and Rollet G 1994 Integrable mappings and polynomial growth *Physica* A **209** 162–222

[22] Meyer H, Anglès d'Auriac J C, Maillard J M and Rollet G 1994 Phase diagram of a six-state chiral Potts model *Physica* A **209** 223

[23] Meyer H and Anglès d'Auriac J-C 1995 Phase diagram of a six-state chiral Potts model *Phys. Rev.* B **51** 6379

[24] Fateev V A and Zamolodchikov A B 1982 Self-dual solutions of the star–triangle relations in $\mathbb{Z}_N$ models *Phys. Lett.* A **92** 37

[25] Bellon M P, Maillard J-M, Rollet G and Viallet C-M 1992 Deformations of dynamics associated to the chiral Potts model *Int. J. Mod. Phys.* B **6** 3575–84