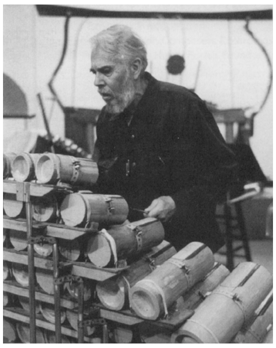
CHAPTER 6

More scales and temperaments

6.1. Harry Partch's 43 tone and other super just scales

A *super just* scale is a just scale formed with exact rational multiples for the intervals, but using primes other than the 2, 3 and 5. Most of these come from the twentieth century.



Harry Partch playing the bamboo marimba (Boo I)

Harry Partch developed a scale of 43 super just intervals which he used in a number of his compositions. The tonic for his scale is G^0. The scale is symmetric, in the sense that every interval upwards from G^0 is also an interval downwards from G^0.

The primes involved in Partch's scale are 2, 3, 5, 7 and 11. The terminology used by Partch to describe this is that his scale is based on the 11-limit, while the Pythagorean scale is based on the 3-limit and the just scales of §5.5 and §5.8 are based on the 5-limit. More generally, if p is a prime, then a p-limit scale only uses rational numbers whose denominators and numerators factor as products of prime numbers less than or equal to p (repetitions are allowed).

Harry Partch's 43 tone scale					
G^{0}	1:1	0.000		10:7	617.488
G^{+1}	81:80	21.506		16:11	648.682
	33:32	53.273	D^{-1}	40:27	680.449
	21:20	84.467	D^{0}	3:2	701.955
$\mathrm{A}\flat^{+1}$	16:15	111.713		32:21	729.219
	12:11	150.637		14:9	764.916
	11:10	165.004		11:7	782.492
A^{-1}	10:9	182.404	$\mathrm{E}\flat^{+1}$	8:5	813.686
A^{0}	9:8	203.910		18:11	852.592
	8:7	231.174	E^{-1}	5:3	884.359
	7:6	266.871	E^{0}	27:16	905.865
$\mathrm{B}\flat^{0}$	32:27	294.135		12:7	933.129
$\mathrm{B}\flat^{+1}$	6:5	315.641		7:4	968.826
	11:9	347.408	F^{0}	16:9	996.090
B^{-1}	5:4	386.314	F^{+1}	9:5	1017.596
	14:11	417.508		20:11	1034.996
	9:7	435.084		11:6	1049.363
	21:16	470.781	$\mathrm{F}\sharp^{-1}$	15:8	1088.269
C^{0}	4:3	498.045		40:21	1115.533
C^{+1}	27:20	519.551		64:33	1146.727
	11:8	551.318	G^{-1}	160:81	1178.494
	7:5	582.512	G^{0}	2:1	1200.000

Here are some other super just scales. The Chinese Lü scale by Huai-nan-dsi of the Han dynasty is the twelve tone super just scale with ratios

$$1{:}1,\ 18{:}17,\ 9{:}8,\ 6{:}5,\ 54{:}43,\ 4{:}3,\ 27{:}19,\ 3{:}2,\ 27{:}17,\ 27{:}16,\ 9{:}5,\ 36{:}19,\ (2{:}1).$$

Wendy Carlos has developed several super just scales. The "Wendy Carlos super just intonation" is the twelve tone scale with ratios

$$1{:}1,\ 17{:}16,\ 9{:}8,\ 6{:}5,\ 5{:}4,\ 4{:}3,\ 11{:}8,\ 3{:}2,\ 13{:}8,\ 5{:}3,\ 7{:}4,\ 15{:}8,\ (2{:}1).$$

The "Wendy Carlos harmonic scale" also has twelve tones, with ratios

$$1{:}1,\ 17{:}16,\ 9{:}8,\ 19{:}16,\ 5{:}4,\ 21{:}16,\ 11{:}8,\ 3{:}2,\ 13{:}8,\ 27{:}16,\ 7{:}4,\ 15{:}8,\ (2{:}1).$$

A better way of writing this might be to multiply all the entries by 16:

$$16,\ 17,\ 18,\ 19,\ 20,\ 21,\ 22,\ 24,\ 26,\ 27,\ 28,\ 30,\ (32).$$

Lou Harrison has a 16 tone super just scale with ratios

$$1{:}1,\ 16{:}15,\ 10{:}9,\ 8{:}7,\ 7{:}6,\ 6{:}5,\ 5{:}4,\ 4{:}3,\ 17{:}12,\ 3{:}2,\ 8{:}5,\ 5{:}3,\ 12{:}7,\ 7{:}4,\ 9{:}5,\ 15{:}8,\ (2{:}1).$$

Wilfrid Perret[1] has a 19-tone 7-limit super just scale with ratios

$$1{:}1,\ 21{:}20,\ 35{:}32,\ 9{:}8,\ 7{:}6,\ 6{:}5,\ 5{:}4,\ 21{:}16,\ 4{:}3,\ 7{:}5,\ 35{:}24,\ 3{:}2,\ 63{:}40,\ 8{:}5,\ 5{:}3,\ 7{:}4,\ 9{:}5,\ 15{:}8,\ 63{:}32,\ (2{:}1).$$

[1] W. Perret, *Some questions of musical theory*, W. Heffer & Sons Ltd., Cambridge, 1926.

John Chalmers also has a 19 tone 7-limit super just scale, differing from this in just two places. The ratios are

1:1, 21:20, 16:15, 9:8, 7:6, 6:5, 5:4, 21:16, 4:3, 7:5, 35:24,
3:2, 63:40, 8:5, 5:3, 7:4, 9:5, 28:15, 63:32, (2:1).

Michael Harrison has a 24 tone 7-limit super just scale with ratios

1:1, 28:27, 135:128, 16:15, 243:224, 9:8, 8:7, 7:6, 32:27, 6:5, 135:112, 5:4,
81:64, 9:7, 21:16, 4:3, 112:81, 45:32, 64:45, 81:56, 3:2, 32:21, 14:9, 128:81,
8:5, 224:135, 5:3, 27:16, 12:7, 7:4, 16:9, 15:8, 243:128, 27:14, (2:1).

Harrison writes,

> Beginning in 1986, I spent two years extensively modifying a seven-foot Schimmel grand piano to create the *Harmonic Piano*. It is the first piano tuned in Just Intonation with the flexibility to modulate to multiple key centers at the press of a pedal. With its unique pedal mechanism, the Harmonic Piano can differentiate between notes usually shared by the same piano key (for example, C-sharp and D-flat). As a result, the Harmonic Piano is capable of playing 24 notes per octave. In contrast to the three unison strings per note of the standard piano, the Harmonic Piano uses only single strings, giving it a "harp-like" timbre. Special muting systems are employed to dampen unwanted resonances and to enhance the instrument's clarity of sound.[2]

The Indian Shruti scale,[3] commonly used to play ragas, is just (i.e., 5-limit) rather than super just, with 22 tones, but has some large numerators and denominators:

1:1, 256:243, 16:15, 10:9, 9:8, 32:27, 6:5, 5:4, 81:64, 4:3, 27:20, 45:32,
729:512, 3:2, 128:81, 8:5, 5:3, 27:16, 16:9, 9:5, 15:8, 243:128, (2:1).

Various notations have been designed for describing super just scales. For example, for 7-limit scales, a three-dimensional lattice of tetrahedra and octahedra can just about be drawn on paper. Here is an example of a twelve tone 7-limit super just scale drawn three dimensionally in this way.[4]

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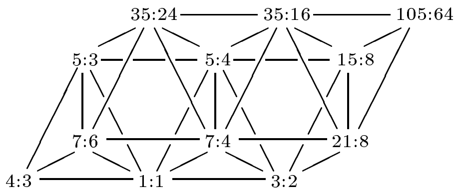
[2] From the liner notes to Harrison's *From Ancient Worlds, for Harmonic Piano*, see Appendix R.

[3] Taken from B. Chaitanya Deva, *The music of India* [**25**], Table 9.2. Note that the fractional value of note 5 given in this table should be 32/27, not 64/45, to match the other information given in this table. This also matches the value given in Tables 9.4 and 9.8 of the same work.

[4] This way of drawing the scale comes from Paul Erlich. According to Paul, the scale was probably first written down by Erv Wilson in the 1960's.



The lines indicate major and minor thirds, perfect fifths, and three different septimal consonances 7:4, 7:5 and 7:6 (notes have been normalized to lie inside the octave 1:1 to 2:1). We return to the discussion of just intonation in §6.8, where we discuss unison vectors and periodicity blocks. We put the above diagram into context in §6.9.

Exercises

1. Taking 1:1 to be C^{0}, write the Indian Shruti scale described in this section as an array using Eitz's comma notation (like the scales in §5.8).

Further reading:

Harry Partch, *Genesis of a music* [**80**].

Further listening: (See Appendix R)

Wendy Carlos, *Beauty in the Beast.*
Michael Harrison, *From Ancient Worlds.*
Harry Partch, *Bewitched.*
Robert Rich, *Rainforest*, *Gaudi.*

6.2. Continued fractions

The modern twelve tone equal tempered scale is based around the fact that

$$7/12 = 0.58333\ldots$$

is a good approximation to

$$\log_2(3/2) = 0.5849625007\ldots,$$

so that if we divide the octave into twelve equal semitones, then seven semitones is a good approximation to a perfect fifth. This suggests the following question. Can $\log_2(3/2)$ be expressed as a ratio of two integers, m/n? In other words, is $\log_2(3/2)$ a rational number? Since $\log_2(3/2)$ and $\log_2(3)$ differ by one, this is the same as asking whether $\log_2(3)$ is rational.

LEMMA 6.2.1. *The number* $\log_2(3)$ *is irrational.*

PROOF. Suppose that $\log_2(3) = m/n$ with m and n integers. Then $3 = 2^{m/n}$, or $3^n = 2^m$. This is obviously impossible, as 3^n is odd while 2^m is even. $\square$

So the best we can expect to do is to approximate $\log_2(3/2)$ by rational numbers such as $7/12$. There is a systematic theory of such rational approximations to irrational numbers, which is the theory of continued fractions.[5] A continued fraction is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

where $a_0, a_1, \dots$ are integers, usually taken to be positive for $i \geq 1$. The expression is allowed to stop at some finite stage, or it may go on for ever. For typographic convenience, we write the continued fraction in the form[6]

$$a_0 + \frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots$$

Every real number has a unique continued fraction expansion, and it stops precisely when the number is rational. The easiest way to see this is as follows. If x is a real number, then the largest integer less than or equal to x (the *integer part* of x) is written $\lfloor x \rfloor$.[7] So $\lfloor x \rfloor$ is what we take for a_0. The remainder $x - \lfloor x \rfloor$ satisfies $0 \leq x - \lfloor x \rfloor < 1$, so if it is nonzero, we now invert it to obtain a number $1/(x - \lfloor x \rfloor)$ which is strictly larger than one.

Writing $x_0 = x$, $a_0 = \lfloor x_0 \rfloor$ and $x_1 = 1/(x_0 - \lfloor x_0 \rfloor)$, we have

$$x = a_0 + \frac{1}{x_1}.$$

Now just carry on going. Let $a_1 = \lfloor x_1 \rfloor$, and $x_2 = 1/(x_1 - \lfloor x_1 \rfloor)$, so that

$$x = a_0 + \frac{1}{a_1+}\,\frac{1}{x_2}.$$

Inductively, we set $a_n = \lfloor x_n \rfloor$ and $x_{n+1} = 1/(x_n - \lfloor x_n \rfloor)$ so that

$$x = a_0 + \frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots$$

This algorithm continues provided each $x_n \neq 0$, which happens exactly when x is irrational. Otherwise, if x is rational, the algorithm terminates to give a finite continued fraction. For irrational numbers the continued fraction expansion is unique. For rational numbers, we only have uniqueness if we stipulate that the last a_n is larger than one.

As an example, let us compute the continued fraction expansion of

$$\pi = 3.14159\,26535\,89793\,23846\,26433\,83279\,50288\,41971\,69399\,37510\dots$$

In this case, we have $a_0 = 3$ and

$$x_1 = 1/(\pi - 3) = 7.062513086\dots$$

[5]The first mathematician known to have made use of continued fractions was Rafael Bombelli in 1572. The modern notation for them was introduced by P. A. Cataldi in 1613.

[6]Some authors use the notation $[a_0; a_1, a_2, a_3, \dots]$ for continued fractions.

[7]In some books, $[x]$ is used instead.

So $a_1 = 7$, and
$$x_2 = 1/(x_1 - 7) = 15.99665\ldots$$
Continuing this way, we obtain[8]
$$\pi = 3 + \frac{1}{7+}\,\frac{1}{15+}\,\frac{1}{1+}\,\frac{1}{292+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{14+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}$$
$$\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{84+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{15+}\,\frac{1}{3+}\,\frac{1}{13+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{2+}\,\frac{1}{6+}$$
$$\frac{1}{6+}\,\frac{1}{99+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{6+}\,\frac{1}{3+}\,\frac{1}{5+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{6+}\,\frac{1}{8+}\,\frac{1}{1+}\,\frac{1}{7+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}$$
$$\frac{1}{7+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{12+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{8+}\,\frac{1}{1+}\,\frac{1}{1+}\cdots$$
To get good rational approximations, we stop just before a large value of a_n. So for example, stopping just before the 15, we obtain the well known approximation $\pi \approx 22/7$.[9] Stopping just before the 292 gives us the extremely good approximation
$$\pi \approx 355/113 = 3.1415929\ldots$$
which was known to the Chinese mathematician Chao Jung-Tze (or Tsu Ch'ung-Chi, depending on how you transliterate the name) in 500 AD.

The rational approximations obtained by truncating the continued fraction expansion of a number are called the *convergents*. So the convergents for π are
$$\frac{3}{1},\ \frac{22}{7},\ \frac{333}{106},\ \frac{355}{113},\ \frac{103993}{33102},\ \frac{104348}{33215},\ldots$$
There is an extremely efficient way to calculate the convergents from the continued fraction.

THEOREM 6.2.2. *Define numbers p_n and q_n inductively as follows:*
$$p_0 = a_0, \qquad p_1 = a_1a_0 + 1, \qquad p_n = a_np_{n-1} + p_{n-2} \quad (n \ge 2) \tag{6.2.1}$$
$$q_0 = 1, \qquad q_1 = a_1, \qquad q_n = a_nq_{n-1} + q_{n-2} \quad (n \ge 2). \tag{6.2.2}$$
Then we have
$$a_0 + \frac{1}{a_1+}\,\frac{1}{a_2+}\cdots\frac{1}{a_n} = \frac{p_n}{q_n}.$$

PROOF. (see Hardy and Wright [**41**], Theorem 149, or Hua [**47**], Theorem 10.1.1).

The proof goes by induction on n. It is easy enough to check the cases $n = 0$ and $n = 1$, so we assume that $n \ge 2$ and that the theorem holds for

[8] Taken from Neil Sloane's *Handbook of integer sequences*, Academic Press, 1973, page 100; note that the values given in Hua [**47**], page 252, are erronious.

[9] According to the bible, π is equal to 3. "Also, he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits the height thereof; and a line of thirty cubits did compass it round about." I Kings 7:23.

smaller values of n. Then we have

$$a_0 + \frac{1}{a_1+}\frac{1}{a_2+}\cdots\frac{1}{a_{n-1}+}\frac{1}{a_n} = a_0 + \frac{1}{a_1+}\frac{1}{a_2+}\cdots\frac{1}{a_{n-1}+\frac{1}{a_n}}.$$

So we can use the formula given by the theorem with $n-1$ in place of n to write this as

$$\begin{aligned}\frac{(a_{n-1}+\frac{1}{a_n})p_{n-2}+p_{n-3}}{(a_{n-1}+\frac{1}{a_n})q_{n-2}+q_{n-3}} &= \frac{a_n(a_{n-1}p_{n-2}+p_{n-3})+p_{n-2}}{a_n(a_{n-1}q_{n-2}+q_{n-3})+q_{n-2}}\\ &= \frac{a_np_{n-1}+p_{n-2}}{a_nq_{n-1}+q_{n-2}} = \frac{p_n}{q_n}.\end{aligned}$$

So the theorem is true for n, and the induction is complete. □

So in the above example for π, we have $p_0 = a_0 = 3$, $q_0 = 1$, $p_1 = a_1a_0 + 1 = 22$, $q_1 = a_1 = 7$, we get

$$\frac{p_2}{q_2} = \frac{p_0 + 15p_1}{q_0 + 15q_1} = \frac{333}{106}$$

so that $p_2 = 333$, $q_2 = 106$,

$$\frac{p_3}{q_3} = \frac{p_1 + p_2}{q_1 + q_2} = \frac{355}{113}$$

so that $p_3 = 355$, $q_3 = 113$, and so on.

Examining the value of x_2 in the case $x = \pi$ above, it may look as though it would be of advantage to allow negative as well as positive values for a_n. However, this doesn't really help, because if x_n is very slightly less than $a_n + 1$ then a_{n+1} will be equal to one, and from there on the sequence as it would have been. In other words, the rational approximations obtained this way are no better. A related observation is that if $a_{n+1} = 2$ then it is worth examining the approximation given by replacing a_n by $a_n + 1$ and stopping there.

The continued fraction expansion for the base of natural logarithms

$$\begin{aligned}e &= 2.71828\,18284\,59045\,23536\,02874\,71352\,66249\,77572\,47093\ldots\\ &= 2 + \frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{1+}\frac{1}{4+}\frac{1}{1+}\frac{1}{1+}\frac{1}{6+}\frac{1}{1+}\frac{1}{1+}\frac{1}{8+}\frac{1}{1+}\frac{1}{1+}\cdots\end{aligned}$$

follows an easily described pattern, as was discovered by Leonhard Euler. The continued fraction expansion of the golden ratio is even easier to describe:

$$\tau = \tfrac{1}{2}(1+\sqrt{5}) = 1 + \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\cdots$$

Although the continued fraction expansion of π is not regular in this way, there is a closely related formula (Brouncker)

$$\frac{\pi}{4} = \frac{1}{1+}\frac{1}{3+}\frac{4}{5+}\frac{9}{7+}\frac{16}{9+}\cdots$$

which is a special case of the arctan formula

$$\tan^{-1} z = \frac{z}{1+}\,\frac{z^2}{3+}\,\frac{4z^2}{5+}\,\frac{9z^2}{7+}\,\frac{16z^2}{9+}\cdots$$

The tan formula

$$\tan z = \frac{z}{1+}\,\frac{-z^2}{3+}\,\frac{-z^2}{5+}\,\frac{-z^2}{7+}\cdots$$

can be used to show that π is irrational (Pringsheim).

How good are the rational approximations obtained from continued fractions? This is answered by the following theorems. Recall that $x_n = p_n/q_n$ denotes the nth convergent. In other words,

$$\frac{p_n}{q_n} = a_0 + \frac{1}{a_1+}\,\frac{1}{a_2+}\cdots\frac{1}{a_{n-1}+}\,\frac{1}{a_n}.$$

THEOREM 6.2.3. *The error in the nth convergent of the continued fraction expansion of a real number x is bounded by*

$$\left|\frac{p_n}{q_n} - x\right| < \frac{1}{q_n^2}.$$

PROOF. (see Hardy and Wright [**41**], Theorem 171, or Hua [**47**], Theorem 10.2.6).

First, we notice that $p_{n-1}q_n - p_nq_{n-1} = (-1)^n$. This is easiest to see by induction. For $n = 1$, we have $p_0 = a_0$, $q_0 = 1$, $p_1 = a_0a_1 + 1$, $q_1 = a_1$, so $p_0a_1 - p_1a_0 = -1$. For $n > 1$, using equations (6.2.1) and (6.2.2) we have

$$\begin{aligned} p_{n-1}q_n - p_nq_{n-1} &= p_{n-1}(q_{n-2} + a_nq_{n-1}) - (p_{n-2} + a_np_{n-1})q_{n-1} \\ &= p_{n-1}q_{n-2} - p_{n-2}q_{n-1} \\ &= -(p_{n-2}q_{n-1} - p_{n-1}q_{n-2}) \\ &= -(-1)^{n-1} = (-1)^n. \end{aligned}$$

Now we use the fact that x lies between

$$\frac{p_{n-2} + a_np_{n-1}}{q_{n-2} + a_nq_{n-1}} \qquad \text{and} \qquad \frac{p_{n-2} + (a_n+1)p_{n-1}}{q_{n-2} + (a_n+1)q_{n-1}}$$

or in other words between $\dfrac{p_n}{q_n}$ and $\dfrac{p_n + p_{n-1}}{q_n + q_{n-1}}$. The distance between these two numbers is

$$\begin{aligned} \left|\frac{p_n + p_{n-1}}{q_n + q_{n-1}} - \frac{p_n}{q_n}\right| &= \left|\frac{(p_n + p_{n-1})q_n - p_n(q_n + q_{n-1})}{(q_n + q_{n+1})q_n}\right| \\ &= \left|\frac{p_{n-1}q_n - p_nq_{n-1}}{q_n^2 + q_nq_{n-1}}\right| = \left|\frac{(-1)^n}{q_n^2 + q_nq_{n-1}}\right| < \frac{1}{q_n^2}. \end{aligned}$$ □

Notice that if we choose a denominator q at random, then the intervals between the rational numbers of the form p/q are of size $1/q$. So by choosing p to minimize the error, we get $|p/q - x| \le 1/2q$. So the point of the above theorem is that the convergents in the continued fraction expansion are considerably better than random denominators. In fact, more is true.

THEOREM 6.2.4. *Among the fractions p/q with $q \le q_n$, the closest to x is p_n/q_n.*

PROOF. See Hardy and Wright [**41**], Theorem 181. □

It is not true that if p/q is a rational number satisfying $|p/q - x| < 1/q^2$ then p/q is a convergent in the continued fraction expansion of x. However, a theorem of Hurwitz (see Hua [**47**], Theorem 10.4.1) says that of any two consecutive convergents to x, at least one of them satisfies $|p/q - x| < 1/2q^2$. Moreover, if a rational number p/q satisfies this inequality then it is a convergent in the continued fraction expansion of x (see Hua [**47**], Theorem 10.7.2).

Distribution of the a_n

If we perform continued fractions on a transcendental number x, given an integer k, how likely is it that $a_n = k$? It seems plausible that $a_n = 1$ is the most likely, and that the probabilities decrease rapidly as k increases, but what is the exact distribution of probabilities?

Gauss answered this question in a letter addressed to Laplace, although he never published a proof.[10] Writing $\mu\{-\}$ for the measure of a set $\{-\}$, what he proved is the following. Given any t in the range $(0, 1)$, the measure of the set of numbers x in the interval $(0, 1)$ for which $x_n - \lfloor x_n \rfloor$ is at most t is given by[11]

$$\lim_{n\to\infty} \mu\{\, x \in (0,1) \mid x_n - \lfloor x_n \rfloor \le t \,\} = \log_2(1+t).$$

The continued fraction process says that we should then invert $x_n - \lfloor x_n \rfloor$. Writing u for $1/t$, we obtain

$$\lim_{n\to\infty} \mu\{\, x \in (0,1) \mid \frac{1}{x_n - \lfloor x_n \rfloor} \ge u \,\} = \log_2(1 + 1/u).$$

Now we need to take the integer part of $1/(x_n - \lfloor x_n \rfloor)$ to obtain a_{n+1}. So if k is an integer with $k \ge 1$ then

$$\begin{aligned}\lim_{n\to\infty} \mu\{\, x \in (0,1) \mid a_n = k \,\} &= \log_2(1 + \frac{1}{k}) - \log_2(1 + \frac{1}{k+1}) \\ &= \log_2\left(\frac{(k+1)^2}{k(k+2)}\right) = \log_2\left(1 + \frac{1}{k(k+2)}\right).\end{aligned}$$

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[10] According to A. Ya. Khinchin, *Continued Fractions*, Dover 1964, page 72, the first published proof was by Kuz'min in 1928.

[11] If you don't know what measure means in this context, think of this as giving the probability that a randomly chosen number in the given interval satisfies the hypothesis.

We now tabulate the probabilities given by this formula.

Value of k	Limiting probability that $a_n = k$ as $n \to \infty$
1	0.4150375
2	0.2223924
3	0.0931094
4	0.0588937
5	0.0406420
6	0.0297473
7	0.0227201
8	0.0179219
9	0.0144996
10	0.0119726

For large k, this decreases like $1/k^2$.

Multiple continued fractions

It is sometimes necessary to make simultaneous rational approximations for more than one irrational number. For example, in the equal tempered scale, not only do seven semitones approximate a perfect fifth with ratio 3:2, but also four semitones approximates a major third with ratio 5:4. So we have

$$\log_2(3/2) \approx 7/12; \qquad \log_2(5/4) \approx 4/12.$$

A theorem of Dirichlet tells us how closely we should expect to be able to approximate a set of k real numbers simultaneously.

THEOREM 6.2.5. *If $\alpha_1, \alpha_2, \ldots \alpha_k$ are real numbers, and at least one of them is irrational, then the there exist an infinite number of ways of choosing a denominator q and numerators $p_1, p_2, \ldots, p_k$ in such a way that the approximations*

$$p_1/q \approx \alpha_1; \quad p_2/q \approx \alpha_2; \quad \ldots \quad p_k/q \approx \alpha_k$$

have the property that the errors are all less than $1/q^{1+\frac{1}{k}}$.

PROOF. See Hardy and Wright [**41**], Theorem 200. □

The case $k = 1$ of this theorem is just Theorem 6.2.3. There is no known method when $k \geq 2$ analogous to the method of continued fractions for obtaining the approximations whose existence is guaranteed by this theorem. Of course, we can just work through the possibilities for q one at a time, but this is much more tedious than one would like.

The power of q in the denominator in the above theorem (i.e., $1 + \frac{1}{k}$) is known to be the best possible. Notice that the error term remains better than the error term $1/2q$ which would result by choosing q randomly. But the extent to which it is better diminishes to insignificant as k grows large.

Exercises

1. Investigate the convergents for the continued fraction expansion of the golden ratio $\tau = \frac{1}{2}(1+\sqrt{5})$. What do these convergents have to do with the Fibonacci series?

Coupled oscillators have a tendency to seek frequency ratios which can be expressed as rational numbers with small numerators and denominators. For example, Mercury rotates on its axis exactly three times for every two rotations around the sun, so that one Mercurial day lasts two Mercurial years. In a similar way, the orbital times of Jupiter and the minor planet Pallas around the sun are locked in a ratio of 18 to 7 (Gauss calculated in 1812 that this would be true, and observation has confirmed it). This is also why the moon rotates once around its axis for each rotation around the earth, so that it always shows us the same face.

Among small frequency ratios for coupled oscillators, the golden ratio is the least likely to lock in to a nearby rational number. Why?

"This must be Fibonacci's."

2. Find the continued fraction expansion of $\sqrt{2}$. Show that if a number has a periodic continued fraction expansion then it satisfies a quadratic equation with integer coefficients. In fact, the converse is also true: if a number satisfies a quadratic equation with integer coefficients then it has a periodic continued fraction expansion. See for example Hardy and Wright [**41**], §10.12.

3. (Hua [**47**]) The synodic month is the period of time between two new moons, and is 29.5306 days. When projected onto the star sphere, the path of the moon intersects the ecliptic (the path of the sun) at the ascending and the descending nodes. A draconic month is the period of time for the moon to return to the same node, and is 27.2123 days. Show that the solar and lunar eclipses occur in cycles with a period of 18 years 10 days.

4. In this problem, you will prove that π is not equal to $\frac{22}{7}$. This problem is not really relevant to the text, but it is interesting anyway.

Use partial fractions (actually, just the long division part of the algorithm) to prove that

$$\int_0^1 \frac{x^4(1-x)^4\,dx}{1+x^2} = \tfrac{22}{7} - \pi.$$

Deduce that $\pi < \frac{22}{7}$. Show that

$$\int_0^1 x^4(1-x)^4\,dx = \tfrac{1}{630},$$

and use this to deduce that

$$\tfrac{1}{1260} < \tfrac{22}{7} - \pi < \tfrac{1}{630}.$$

What would this sentence be like if π were 3?

> If π were equal to 3, this sentence
> would look something like this.

(Scott Kim/Harold Cooper, quoted from Douglas Hofstadter's *Metamagical Themas*, Basic Books, 1985).

5. Show that if a and b have no common factor then $\log_a(b)$ is irrational. Show that if no pair among a, b and c has a common factor then $\log_a(b)$ and $\log_a(c)$ are rationally independent. In other words, there cannot exist nonzero integers n_1, n_2 and n_3 such that $n_1 \log_a(b) + n_2 \log_a(c) = n_3$.

6. Find the continued fraction expansion for the rational number 531441/524288 which represents the frequency ratio for the Pythagorean comma. Explain in terms of this example the relationship between the continued fraction expansion of a rational number and Euclid's algorithm for finding highest common factors.

7. The *Gaussian integers* are the complex numbers of the form $a + bi$ where a and b are in the rational integers $\mathbb{Z}$. Develop a theory of continued fractions for simultaneously approximating two real numbers α and β, by considering the complex number $\alpha + \beta i$. Explain why this method favors denominators which can be expressed as a sum of two squares, so that it does not always find the best approximations.

8. A certain number is known to be a ratio of two 3-digit integers. Its decimal expansion, to nine significant figures, is 0.137637028. What are the integers?

Further reading:

Hardy and Wright, *Number theory* [**41**], chapter X.

Hua, *Introduction to number theory* [**47**], chapter 10.

Hubert Stanley Wall, *Analytic theory of continued fractions.* Chelsea, New York, 1948. ISBN 0828402078.

J. Murray Barbour, *Music and ternary continued fractions*, American Mathematical Monthly 55 (9) (1948), 545–555.

Viggo Brun, *Music and ternary continued fractions*, Norske Vid. Selsk. Forh., Trondheim 23 (1950), 38–40. This article is a response to the above article of Murray Barbour.

Viggo Brun, *Music and Euclidean algorithms*, Nordisk Mat. Tidskr. 9 (1961), 29–36, 95.

J. B. Rosser, *Generalized ternary continued fractions*, American Mathematical Monthly 57 (8) (1950), 528–535. This is another response to Murray Barbour's article.

6.3. Fifty-three tone scale

The first continued fraction expansion of interest to us is the one for $\log_2(3/2)$. The first few terms are

$$\log_2(3/2) = \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{5+}\,\frac{1}{2+}\,\frac{1}{23+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{1+}\cdots$$

The sequence of convergents for the continued fraction expansion of $\log_2(3/2)$ is

$$1, \frac{1}{2}, \frac{3}{5}, \frac{7}{12}, \frac{24}{41}, \frac{31}{53}, \frac{179}{306}, \frac{389}{665}, \frac{9126}{15601}, \ldots$$

The bottoms of these fractions tell us how many equal notes to divide an octave into, and the tops tell us how many of these notes make up one approximate fifth. The fourth of the above approximations give us our western scale. The next obvious places to stop are at $31/53$ and $389/665$, just before large denominators.

The fifty-three tone equally tempered scale is interesting enough to warrant some discussion. In 1876, Robert Bosanquet made a "generalized keyboard harmonium" with fifty-three notes to an octave.[12] A photograph of this

[12]Described in Bosanquet, *Musical intervals and temperaments*, Macmillan and Co., London, 1876. Reprinted with commentary by Rudolph Rasch, Diapason Press, Utrecht, 1986.

Bosanquet's harmonium

instrument can be found on page 160. A discussion of this harmonium can be found in the translator's appendix XX.F.8 (pages 479–481) in Helmholtz [**43**]. One way of thinking of the fifty-three note scale is that it is based around the approximation which makes the Pythagorean comma equal to one fifty-third of an octave, or $1200/53 = 22.642$ cents, rather than the true value of 23.460 cents. So if we go around a complete circle of fifths, we get from C to a note which we may call B$\sharp$ 22.642 cents higher. This corresponds to the equation

$$12 \times 31 - 7 \times 53 = 1,$$

which can be interpreted as saying that twelve 53-tone equal temperament fifths minus seven octaves equals one step in the 53-tone scale.

The following table shows the fifty-three tone equivalents of the notes on the Pythagorean scale:

note	C	B♯	D♭	C♯	D	E♭	D♯	E	F	G♭
degree	0	1	4	5	9	13	14	18	22	26

note	F♯	G	A♭	G♯	A	B♭	A♯	C♭	B	C
degree	27	31	35	36	40	44	45	48	49	53

Thus the fifty-three tone scale is made up of five whole tones each of nine scale degrees and two semitones each of four scale degrees, $5\times 9+2\times 4=53$. Flattening or sharpening a note changes it by five scale degrees. The perfect fifth is extremely closely approximated in this scale by the thirty-first degree, which is

$$\frac{31}{53}\times 1200 = 701.887$$

cents rather than the true value of 701.955.

The just major third is also closely approximated in this scale by the seventeenth degree, which is

$$\frac{17}{53}\times 1200 = 384.906$$

cents rather than the true value of 386.314 cents. In effect, what is happening is that we are approximating both the Pythagorean comma and the syntonic comma by a single scale degree in the 53 note scale, which is roughly half way between them. So in Eitz's notation, we are identifying the note $\mathrm{G}\sharp^{0}$ with $\mathrm{A}\flat^{+1}$, whose difference is one schisma. Similarly, we are identifying the note B^{-1} with $\mathrm{C}\flat^{0}$, $\mathrm{B}\sharp^{-1}$ with C^{0}, and so on. We are also identifying the note G^{+2} with $\mathrm{A}\flat^{-2}$, whose difference is a diesis minus four commas, or

$$\frac{256}{243}\left(\frac{80}{81}\right)^4 = \frac{2^{24}5^4}{3^{21}} = \frac{10485760000}{10460353203},$$

or about 4.200 cents. The effect of this is that the array notation introduced in §5.7 becomes periodic in both directions, so that we obtain the diagram on page 162. In this diagram, the top and bottom row are identified with each other, and the left and right walls are identified with each other. The resulting geometric figure is called a torus, and it looks like a bagel, or a tire.

It appears that the Pythagoreans were aware of the 53 tone equally tempered scale. Philolaus, a disciple of Pythagoras, thought of the tone as being divided into two minor semitones and a Pythagorean comma, and took each minor semitone to be four commas. This makes nine commas to the whole tone and four commas to the minor semitone, for a total of 53 commas to the octave. The Chinese theorist King Fâng of the third century B.C. also seems to have been aware that the 54th note in the Pythagorean system is almost identical to the first.

After 53, the next good denominator in the continued fraction expansion of $\log_2(3/2)$ is 665. The extra advantages obtained by going to an equally tempered 665 tone scale, which is gives a remarkably good approximation to the perfect fifth, are far outweighed by the fact that adjacent tones are

22	0	31	9	40	18	49	27	5	
5	36	14	45	23	1	32	10	41	19
19	50	28	6	37	15	46	24	2	33
33	11	42	20	51	29	7	38	16	47
47	25	3	34	12	43	21	52	30	8
8	39	17	48	26	4	35	13	44	22
22	0	31	9	40	18	49	27	5	

F^{0}	C^{0}	G^{0}	D^{0}	A^{0}	E^{0}	B^{0}	$F\sharp^{0}$	$C\sharp^{0}$	
$C\sharp^{0}$	$A\flat^{+1}$	$E\flat^{+1}$	$B\flat^{+1}$	F^{+1}	C^{+1}	G^{+1}	D^{+1}	A^{+1}	E^{+1}
E^{+1}	B^{+1}	$F\sharp^{+1}$	$C\sharp^{+1}$	$A\flat^{+2}$	$E\flat^{+2}$	$B\flat^{+2}$	F^{+2}	C^{+2}	G^{+2}
G^{+2}	D^{+2}	A^{+2}	E^{+2}	C^{-2}	G^{-2}	D^{-2}	A^{-2}	E^{-2}	B^{-2}
B^{-2}	$F\sharp^{-2}$	$C\sharp^{-2}$	$A\flat^{-1}$	$E\flat^{-1}$	$B\flat^{-1}$	F^{-1}	C^{-1}	G^{-1}	D^{-1}
D^{-1}	A^{-1}	E^{-1}	B^{-1}	$F\sharp^{-1}$	$C\sharp^{-1}$	$A\flat^{0}$	$E\flat^{0}$	$B\flat^{0}$	F^{0}
F^{0}	C^{0}	G^{0}	D^{0}	A^{0}	E^{0}	B^{0}	$F\sharp^{0}$	$C\sharp^{0}$	

Torus of thirds and fifths in 53 tone equal temperament

so close together (1.805 cents) as to be almost indistinguishable. If 53 tone equal temperament is thought of as the scale of commas, then 665 tone equal temperament might be thought of as a scale of schismas.

6.4. Other equal tempered scales

Other divisions of the octave into equal intervals which have been used for experimental tunings have included 19, 24, 31 and 43. The 19 tone scale has the advantage of excellent approximations to the 6:5 minor third and the 5:3 major sixth as well as reasonable approximations to the 5:4 major third and the 8:5 minor sixth. The eleventh degree gives an approximation to the 3:2 perfect fifth which is somewhat worse than in twelve tone equal temperament, but still acceptable.

name	ratio	cents	19-tone degree	cents
fundamental	1:1	0.000	0	0.000
minor third	6:5	315.641	5	315.789
major third	5:4	386.314	6	378.947
perfect fifth	3:2	701.955	11	694.737
minor sixth	8:5	813.687	13	821.053
major sixth	5:3	884.359	14	884.211
octave	2:1	1200.000	19	1200.000

Christiaan Huygens, in the late 17th century, seems to have been the first to use the equally tempered 19 tone scale as a way of approximating just intonation in a way that allowed for modulation into other keys. Yasser[13] was an important twentieth century proponent. The properties of 19 tone equal temperament with respect to formation of a diatonic scale are very similar to those for 12 tones. But accidentals and chromatic scales behave very differently.

The main purpose I can see for the equally tempered 24 tone scale, usually referred to as the *quarter-tone scale*, is that it increases the number of tones available without throwing out the familiar twelve tones. It contains no better approximations to the ratios 3:2 and 5:4 than the twelve tone scale, but has a marginally better approximation to 7:4 and a significantly better approximation to 11:8. The two sets of twelve notes formed by taking every other note from the 24 tone scale can be alternated with interesting effect, but using notes from both sets of twelve at once has a strong tendency to make discords. Examples of works using the quarter-tone scale include the German composer Richard Stein's *Zwei Konzertstücke* op. 26, 1906 for cello and piano and Alois Hába's Suite for String Orchestra, 1917.[14] Twentieth century American composers such as Howard Hanson and Charles Ives have composed music designed for two pianos tuned a quarter tone apart.

Appendix E contains a table of various equal tempered scales, quantifying how well they approximate perfect fifths, just major thirds, and seventh harmonics. An examination of this table reveals that the 31 tone scale is unusually good at approximating all three at once. We examine this scale in the next section.

Further reading:

Jim Aikin, *Discover 19-tone equal temperament*, Keyboard, March 1988, p. 74–80.

[13]Joseph Yasser, *A theory of evolving tonality*, American Library of Musicology, New York, 1932.

[14]It is said that Hába practised to the point where he could accurately sing five divisions to a semitone, or sixty to an octave.

M. Yunik and G. W. Swift, *Tempered music scales for sound synthesis*, Computer Music Journal 4 (4) (1980), 60–65.

Further listening: (See Appendix R)

Easley Blackwood *Microtonal Compositions.* This is a recording of a set of microtonal compositions in each of the equally tempered scales from 13 tone to 24 tone,

Clarence Barlow's "OTOdeBLU" is in 17 tone equal temperament, played on two pianos.

William Sethares, *Xentonality*, Music in 10-, 17- and 19-tet.

6.5. Thirty-one tone scale

The 31 tone equal tempered scale was first investigated by Nicola Vicentino[15] and also later by Christiaan Huygens.[16] It gives a better approximation to the perfect fifth than the 19 tone scale, but it is still worse than the 12 tone scale.

name	ratio	cents	31-tone degree	cents
fundamental	1:1	0.000	0	0.000
major third	5:4	386.314	10	387.097
perfect fifth	3:2	701.955	18	696.774
minor sixth	8:5	813.687	21	812.903
seventh harmonic	7:4	968.826	25	967.742

It also contains good approximations to the major third and minor sixth, as well as the seventh harmonic.

The main reason for interest in 31-tone equal temperament is that note 18 of this scale is an unexpectedly good approximation to the meantone fifth (696.579) rather than the perfect fifth. So the entire meantone scale can be approximated as shown in the table below. Fokker[17] was an important twentieth century proponent of the 31 tone scale.

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[15]Nicola Vicentino, *L'antica musica ridotta alla moderna pratica*, Rome, 1555. Translated as *Ancient music adapted to modern practice*, Yale University Press, 1996.

[16]Christiaan Huygens, *Lettre touchant le cycle harmonique*, Letter to the editor of the journal *Histoire des Ouvrage de Sçavans*, Rotterdam 1691. Reprinted with English and Dutch translation (ed. Rudolph Rasch), Diapason Press, Utrecht, 1986.

[17]See for example A. D. Fokker, *The qualities of the equal temperament by 31 fifths of a tone in the octave*, Report of the Fifth Congress of the International Society for Musical Research, Utrecht, 3–7 July 1952, Vereniging voor Nederlandse Muziekgeschiedenis, Amsterdam (1953), 191–192; *Equal temperament with 31 notes*, Organ Institute Quarterly 5 (1955), 41; *Equal temperament and the thirty-one-keyed organ*, Scientific Monthly 81 (1955), 161–166. Also M. Joel Mandelbaum, *31-Tone Temperament: The Dutch Legacy*, Ear Magazine East, New York, 1982/1983; Henk Badings, *A. D. Fokker: new music with 31 notes*, Zeitschrift für Musiktheorie 7 (1976), 46–48.

note	meantone	31-tone	
C	0.000	0	0.000
C♯	76.049	2	77.419
D	193.157	5	193.548
E♭	310.265	8	309.677
E	386.314	10	387.097
F	503.422	13	503.226
F♯	579.471	15	580.645
G	696.579	18	696.774
A♭	813.686	21	812.903
A	889.735	23	890.323
B♭	1006.843	26	1006.452
B	1082.892	28	1083.871
C	1200.000	31	1200.000

The picture above shows a 31 tone equal tempered instrument, made by Vitus Trasuntinis in 1606. This instrument is currently in the State Museum in Bologna. Each octave has seven keys as usual where the white keys would normally go, and five sets of four keys where the five black keys would normally go. Then there are two keys each between the white keys that would normally not be separated by black keys, for a total of $7+4\times 5+2\times 2=31$.

Let us examine the relationship between the meantone scale and 31 tone equal temperament in terms of continued fractions. Since the meantone scale is generated by the meantone fifth, which represents a ratio of $\sqrt[4]{5}:1$, we should look at the continued fraction for $\log_2(\sqrt[4]{5})$. We obtain

$$\begin{aligned}\log_2(\sqrt[4]{5}) &= \tfrac{1}{4}\log_2(5) = 0.580482024\ldots\\ &= \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{5+}\,\frac{1}{1+}\cdots\end{aligned}$$

with convergents

$$\frac{0}{1},\ \frac{1}{1},\ \frac{1}{2},\ \frac{3}{5},\ \frac{4}{7},\ \frac{7}{12},\ \frac{11}{19},\ \frac{18}{31},\ \frac{101}{174},\ \frac{119}{205},\ldots$$

Cutting off just before the denominator 5 gives the approximation 18/31, which gives rise to the 31 tone equal tempered scale described above.

Exercises

1. Draw a torus of thirds and fifths, analogous to the one on page 162, for the 31 tone equal tempered scale, regarded as an approximation to meantone tuning.

2. In the text, the 31 tone equal tempered scale was compared with the usual (quarter comma) meantone scale, using the observation that taking multiples of the fifth generates a meantone scale, and then applying the theory of continued fractions to approximate the fifth. Carry out the same process to make the following comparisons.

(i) Compare the 19 tone equal tempered scale with Salinas' $\frac{1}{3}$ comma meantone scale.

(ii) Compare the 43 tone equal tempered scale with the $\frac{1}{5}$ comma meantone scale of Verheijen and Rossi.

(iii) Compare the 50 tone equal tempered scale with Zarlino's $\frac{2}{7}$ comma meantone scale.

(iv) Compare the 55 tone equal tempered scale with Silbermann's $\frac{1}{6}$ comma meantone scale.

Appendix J has a diagram which is relevant to this question.

6.6. The scales of Wendy Carlos

The idea behind the alpha, beta and gamma scales of Wendy Carlos is to ignore the requirement that there are a whole number of notes to an octave, and try to find equal tempered scales which give good approximations to the just intervals 3:2 and 5:4 (perfect fifth and major third). Since $6/5 = 3/2 \div 5/4$, this automatically gives good approximations to the 6:5 minor third. This means that we need $\log_2(3/2)$ and $\log_2(5/4)$ to be close to integer multiples of the scale degree. So we must find rational approximations to the ratio of these quantities.

We investigate the continued fraction expansion of the ratio:

$$\frac{\log_2(3/2)}{\log_2(5/4)} = \frac{\ln(3/2)}{\ln(5/4)} = 1 + \frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{2+}\,\frac{1}{6+}\,\frac{1}{1+}\,\frac{1}{10+}\,\frac{1}{135+}\cdots$$

The sequence of convergents obtained by truncating this continued fraction is:

$$1,\ 2,\ \frac{9}{5},\ \frac{20}{11},\ \frac{129}{71},\ \frac{149}{82},\ldots$$

Carlos' α (alpha) scale arises from the approximation $9/5$ for the above ratio. This means taking a value for the scale degree so that nine of them approximate a 3:2 perfect fifth, five of them approximate a 5:4 major third, and four of them approximate a 6:5 minor third. In order to make the approximation as good as possible we minimize the mean square deviation. So if x denotes the scale degree (taking the octave as unit) then we must minimize

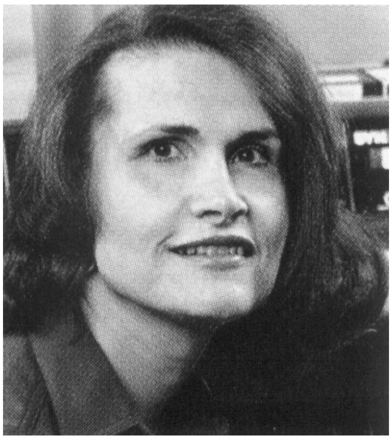
$$(9x - \log_2(3/2))^2 + (5x - \log_2(5/4))^2 + (4x - \log_2(6/5))^2.$$

Setting the derivative with respect to x of this quantity equal to zero, we obtain the equation

$$x = \frac{9\log_2(3/2) + 5\log_2(5/4) + 4\log_2(6/5)}{9^2 + 5^2 + 4^2} \approx 0.06497082462$$

Multiplying by 1200, we obtain a scale degree of 77.965 cents, and there are 15.3915 of them to the octave.[18]

[18] This actually differs very slightly from Carlos' figure of 15.385 α-scale degrees to the octave. This is obtained by approximating the scale degree to 78.0 cents.



Wendy Carlos

Carlos also considers the scale α' obtained by doubling the number of notes in the octave. This gives the same approximations as before for the ratios 3:2, 5:4 and 6:5, but the twenty-fifth degree of the new scale (974.562 cents) is a good approximation to the seventh harmonic in the form of the ratio 7:4 (968.826 cents).

If instead we use the approximation

$$1 + \frac{1}{1+}\,\frac{1}{5} = \frac{11}{6}$$

we obtain Carlos' β (beta) scale. We choose a value of the scale degree so that eleven of them approximate a 3:2 perfect fifth, six of them approximate a 5:4 major third, and five of them approximate a 4:3 minor third. Proceeding as before, we see that the proportion of an octave occupied by each scale degree is

$$\frac{11\log_2(3/2) + 6\log_2(5/4) + 5\log_2(6/5)}{11^2 + 6^2 + 5^2} \approx 0.05319411048.$$

Multiplying by 1200, we obtain a scale degree of 63.833 cents, and there are 18.7991 of them to the octave.[19] One advantage of the beta scale over the alpha scale is that the fifteenth scale degree (957.494 cents) is a reasonable approximation to the seventh harmonic in the form of the ratio 7:4 (968.826 cents). Indeed, it may be preferable to include this approximation into the above least squares calculation to get a scale in which the proportion of an octave occupied by each scale degree is

$$\frac{15\log_2(7/4) + 11\log_2(3/2) + 6\log_2(5/4) + 5\log_2(6/5)}{15^2 + 11^2 + 6^2 + 5^2} \approx 0.05354214235.$$

This gives a scale degree of 64.251 cents, and there are 18.677 of them to the octave. The fifteenth scale degree is then 963.759 cents.

Going one stage further, and using the approximation 20/11, we obtain Carlos' γ (gamma) scale. We choose a value of the scale degree so that twenty of them approximate a 3:2 perfect fifth, nine of them approximate a 5:4 major third, and eleven of them approximate a 4:3 minor third. The proportion of an octave occupied by each scale degree is

$$\frac{20\log_2(3/2) + 11\log_2(5/4) + 9\log_2(6/5)}{20^2 + 11^2 + 9^2} \approx 0.02924878523.$$

Multiplying by 1200, we obtain a scale degree of 35.099 cents, and there are 34.1895 of them to the octave.[20] This scale contains almost pure perfect fifths

[19] Carlos has 18.809 β-scale degrees to the octave, corresponding to a scale degree of 63.8 cents.

[20] Carlos has 34.188 γ-scale degrees to the octave, corresponding to a scale degree of 35.1 cents.

and major thirds, but it does not contain a good approximation to the ratio 7:4.

name	ratio	cents	α	cents	β	cents	γ	cents
fundamental	1:1	0.000	0	0.000	0	0.000	0	0.000
minor third	6:5	315.641	4	311.860	5	319.165	9	315.887
major third	5:4	386.314	5	389.825	6	382.998	11	386.084
perfect fifth	3:2	701.955	9	701.685	11	702.162	20	701.971
seventh harmonic	7:4	968.826	$12\frac{1}{2}$	974.562	15	957.494	–	— —

6.7. The Bohlen–Pierce scale

Jaja, unlike Stravinsky, has never been guilty of composing harmony in all his life. Jaja is pure absolute twelve tone. Never tempted, like some of the French composers, to write with thirteen tones. Oh no. This, says Jaja, is the baker's dozen, the "Nadir of Boulanger."

From Gerard Hoffnung's Interplanetary Music Festival, analysis by two "distinguished teutonic musicologists" of the work of a fictitious twelve tone composer, Bruno Heinz Jaja.

The Bohlen–Pierce scale is the thirteen tone scale described in the article of Mathews and Pierce, forming Chapter 13 of [**66**]. Like the scales of Wendy Carlos, it is not based around the octave as the basic interval. But whereas Carlos uses 3:2 and 5:4, Bohlen and Pierce replace the octave by an octave and a perfect fifth (a ratio of 3:1). In the equal tempered version, this is divided into thirteen equal parts. This gives a good approximation to a "major" chord with ratios 3:5:7. The idea is that only odd multiples of frequencies are used. Music written using this scale works best if played on an instrument such as the clarinet, which involves predominantly odd harmonics, or using specially created synthetic voices with the same property. We shall prefix all words associated with the Bohlen–Pierce scale with the letters BP to save confusion with the corresponding notions based around the octave.

The basic interval of an octave and a perfect fifth, which is a ratio of exactly 3:1 or an interval of 1901.955 cents, is called a *BP-tritave*. In the equal tempered 13 tone scale, each scale degree is one thirteenth of this, or 146.304 cents. It may be felt that the scale of cents is inappropriate for calculations with reference to this scale, but we shall stick with it nonetheless for comparison with intervals in scales based around the octave.

The Pythagorean approach to the division of the tritave begins with a ratio of 7:3 as the analog of the fifth. We shall call this interval the perfect BP-tenth, since it will correspond to note ten in the BP-scale. The corresponding continued fraction is

$$\log_3(7/3) = \frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{4+}\,\frac{1}{22+}\,\frac{1}{32+}\cdots,$$

whose convergents are

$$\frac{0}{1}, \frac{1}{1}, \frac{3}{4}, \frac{7}{9}, \frac{10}{13}, \frac{27}{35}, \frac{118}{153}, \ldots$$

If we perform the same calculation for the 5:3 ratio, we obtain the continued fraction

$$\log_3(5/3) = \frac{1}{2+}\,\frac{1}{6+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{7+}\cdots$$

with convergents

$$\frac{0}{1}, \frac{1}{2}, \frac{6}{13}, \frac{7}{15}, \frac{13}{28}, \frac{20}{43}, \frac{73}{157}, \ldots.$$

Comparing these continued fractions, it looks like a good idea to divide the tritave into 13 equal intervals, with note 10 approximating the ratio 7:3, and note 6 approximating the ratio 5:3.

note	degree	7/3-Pythag	Just
C	0	1:1	1:1
D	2	19683:16807	25:21
E	3	9:7	9:7
F	4	343:243	7:5
G	6	81:49	5:3
H	7	49:27	9:5
J	9	729:343	15:7
A	10	7:3	7:3
B	12	6561:2401	25:9
C	13	3:1	3:1

Basing a BP-Pythagorean scale around the ratio 7:3, we obtain a scale of 13 notes in which the circle of BP-tenths has a BP 7/3-comma given by a ratio of

$$\frac{7^{13}}{3^{23}} = \frac{96889010407}{94143178827}$$

or about 49.772 cents.

Using perfect BP-tenths to form a diatonic BP-Pythagorean scale, we obtain the third column of the table to the left. Following Bohlen, we name the notes of the scale using the letters A–H and J. Note that our choice of the second degree of the diatonic scale differs from the choice made by Mathews and Pierce, and gives what Bohlen calls the Lambda scale.

To obtain a major 3:5:7 triad, we introduce a just major BP-sixth with a ratio of 5:3. This is very close to the BP 7/3-Pythagorean G, which gives rise to an interval called the BP-minor diesis, expressing the difference between these two versions of G. This interval, namely the difference between 5:3 and 81:49, is a ratio of 245:243 or about 14.191 cents.

The BP version of Eitz's notation works in a similar way to the octave version. We start with the BP 7/3-Pythagorean values for the notes and then adjust by a number of BP-minor dieses indicated by a superscript. So G^0 denotes the 81:49 version of G, while G^{+1} denotes the 5:3 version. The just scale given in the table above is then described by the following array:

$$\begin{array}{ccccc} & D^{+2} & & B^{+2} & \\ J^{+1} & & G^{+1} & & \\ E^0 & & C^0 & & A^0 \\ & H^{-1} & & F^{-1} & \end{array}$$

A reasonable way to fill this in to a thirteen tone just scale is as follows:

BP Monochord

$$\begin{array}{ccccc} F\sharp^{+2} & & D^{+2} & & B^{+2} \\ & J^{+1} & & G^{+1} & \\ E^0 & & C^0 & & A^0 \\ & H^{-1} & & F^{-1} & \\ D\flat^{-2} & & B\flat^{-2} & & J\flat^{-2} \end{array}$$

For comparison, here is a table of the scales discussed above, in cents to three decimal places, and also in the BP version of Eitz's notation. The column marked "discrepancy" gives the difference between the equal and just versions.

	BP 7/3-Pythag		BP-just		BP-equal	discrepancy
C	0.000	0	0.000	0	0.000	0.000
D♭	161.619	0	133.238	−2	146.304	+13.066
D	273.465	0	301.847	+2	292.608	−9.239
E	435.084	0	435.084	0	438.913	+3.829
F	596.703	0	582.512	−1	585.217	+2.705
F♯	708.550	0	736.931	+2	731.521	−5.410
G	870.168	0	884.359	+1	877.825	−6.534
H	1031.787	0	1017.596	−1	1024.130	+6.534
J♭	1193.405	0	1165.024	−2	1170.434	+5.410
J	1305.252	0	1319.443	+1	1316.738	−2.705
A	1466.871	0	1466.871	0	1463.042	−3.829
B♭	1628.490	0	1600.108	−2	1609.347	+9.239
B	1740.336	0	1768.717	+2	1755.651	−13.066
C	1901.955	0	1901.955	0	1901.955	0.000

A number of the intervals in the BP scale approximate intervals in the usual octave based scale, and some of these approximations are just far enough off to be disturbing to trained musicians. It is plausible that proper appreciation of music written in the BP scale would involve learning to "forget" the accumulated experience of the perpetual bombardment by octave based music which we receive from the world around us, even if we are not musicians. For this reason, it seems unlikely that such music will become popular. On the other hand, according to John Pierce (chapter 1 of [**24**]), Maureen Chowning, a coloratura soprano, has learned to sing in the BP scale, Richard Boulanger has composed a "considerable piece" using it, and two CDs by Charles Carpenter are available which make extensive use of scale (see below).

Exercises

1. (Paul Erlich) Investigate the refinement of the Bohlen–Pierce scale in which there are 39 tones to the BP-tritave. What relevant ratios are approximated by scale degrees 5, 7, 11, 13, 16, 22, 28 and 34?

Further reading:

H. Bohlen, *13 Tonstufen in der Duodezeme.* Acustica 39 (1978), 76–86.

M. V. Mathews and J. R. Pierce, *The Bohlen–Pierce scale.* Chapter 13 of [**66**].

M. V. Mathews, J. R. Pierce, A. Reeves and L. A. Roberts, *Theoretical and experimental explorations of the Bohlen–Pierce scale.* J. Acoust. Soc. Amer. 84 (1988), 1214–1222.

M. V. Mathews, L. A. Roberts and J. R. Pierce, *Four new scales based on non-successive-integer-ratio chords.* J. Acoust. Soc. Amer. 75 (1984), S10(A).

L. A. Roberts and M. V. Mathews, *Intonation sensitivity for traditional and non-traditional chords.* J. Acoust. Soc. Amer. 75 (1984), 952–959.

Further listening: (See Appendix R)

Charles Carpenter, *Frog à la Pêche* and *Splat* are composed using the Bohlen–Pierce scale, and played in a progressive rock/jazz style.

On the CD of examples accompanying Cook [**15**], track 62 demonstrates the Bohlen–Pierce scale.

On the CD of examples accompanying Mathews and Pierce [**66**], tracks 71–74 demonstrate the Bohlen–Pierce scale.

6.8. Unison vectors and periodicity blocks

In this section, we return to just intonation, and we describe Fokker's periodicity blocks and unison vectors. The periodicity block corresponds to what a mathematician would call a *set of coset representatives*, or a *fundamental domain.* The starting point is octave equivalence; notes differing by a whole number of octaves are considered to be equivalent.

The Pythagorean scale is the one dimensional version of the theory. We place the notes of the Pythagorean scale along a one dimensional lattice, with the origin at C^0. We have labeled the vertices in three notations, namely note names, ratios and cents, for comparison. Because of octave equivalence, the value in cents is reduced or augmented by a multiple of 1200 as necessary to put it in the interval between zero and 1200.

792.180	294.135	996.090	498.045	0.000	701.955	203.910	905.865	407.820	1109.775	611.731	113.685
128/81	32/27	16/9	4/3	1/1	3/2	9/8	27/16	81/64	243/128	729/512	2187/2048
$\mathrm{A}\flat^0$	$\mathrm{E}\flat^0$	$\mathrm{B}\flat^0$	F^0	C^0	G^0	D^0	A^0	E^0	B^0	$\mathrm{F}\sharp^0$	$\mathrm{C}\sharp^0$

We write $\mathbb{Z}^1$ for this one dimensional lattice, to emphasize the fact that the points in the lattice may be indexed using the integers. In fact, given the choice of C^0 as the origin, there are two sensible ways to index with integers. One makes G^0 correspond to 1, D^0 to 2, F^0 to -1, etc., and the other makes F^0 correspond to 1, $\mathrm{B}\flat^0$ to 2, G^0 to -1, etc. For the sake of definiteness, we choose the former.

The twelve tone Pythagorean scale comes from observing that in this system, C^0 and $\mathrm{B}\sharp^0$ are close enough together in pitch that we may not want both of them in our scale. Since $\mathrm{B}\sharp^0$ is the twelfth note, we say that (12) is a unison vector. A periodicity block would then consist of a choice of 12 consecutive points on this lattice, to constitute a scale. Other choices of unison vector would include (53) and (665) (cf. §6.3).

Just intonation, as we introduced it in §5.5, is really a 2-dimensional lattice, which we write as $\mathbb{Z}^2$. In Eitz's notation (see §5.7), here is a small part of the lattice with the origin circled.

$$\begin{array}{cccccc}
 & \mathrm{F}\sharp^{-2} & \mathrm{C}\sharp^{-2} & \mathrm{G}\sharp^{-2} & \mathrm{D}\sharp^{-2} & \\
 & \mathrm{D}^{-1} & \mathrm{A}^{-1} & \mathrm{E}^{-1} & \mathrm{B}^{-1} & \mathrm{F}\sharp^{-1} \\
\mathrm{B}\flat^{0} & \mathrm{F}^{0} & \mathrm{C}^{0} & \mathrm{G}^{0} & \mathrm{D}^{0} & \mathrm{A}^{0} \\
\mathrm{D}\flat^{+1} & \mathrm{A}\flat^{+1} & \mathrm{E}\flat^{+1} & \mathrm{B}\flat^{+1} & \mathrm{F}^{+1} & \mathrm{C}^{+1} \\
 & \mathrm{F}\flat^{+2} & \mathrm{C}\flat^{+2} & \mathrm{G}\flat^{+2} & \mathrm{D}\flat^{+2} & \mathrm{A}\flat^{+2}
\end{array}$$

The same in ratio notation is as follows.

$$\begin{array}{cccccc}
 & \frac{25}{18} & \frac{25}{24} & \frac{25}{16} & \frac{75}{64} & \\
 & \frac{10}{9} & \frac{5}{3} & \frac{5}{4} & \frac{15}{8} & \frac{45}{32} \\
\frac{16}{9} & \frac{4}{3} & \frac{1}{1} & \frac{3}{2} & \frac{9}{8} & \frac{27}{16} \\
\frac{16}{15} & \frac{8}{5} & \frac{6}{5} & \frac{9}{5} & \frac{27}{20} & \frac{81}{80} \\
 & \frac{32}{25} & \frac{48}{25} & \frac{36}{25} & \frac{27}{25} & \frac{81}{50}
\end{array}$$

We can choose a basis for this lattice, and write everything in terms of vectors with respect to this basis. This is the two dimensional version of our choice from two different ways of indexing the Pythagorean scale by the integers, but this time there are an infinite number of choices of basis.

For example, if our basis consists of G^0 and E^{-1} (i.e., $\frac{3}{2}$ and $\frac{5}{4}$) then here is the same part of the lattice in vector notation.

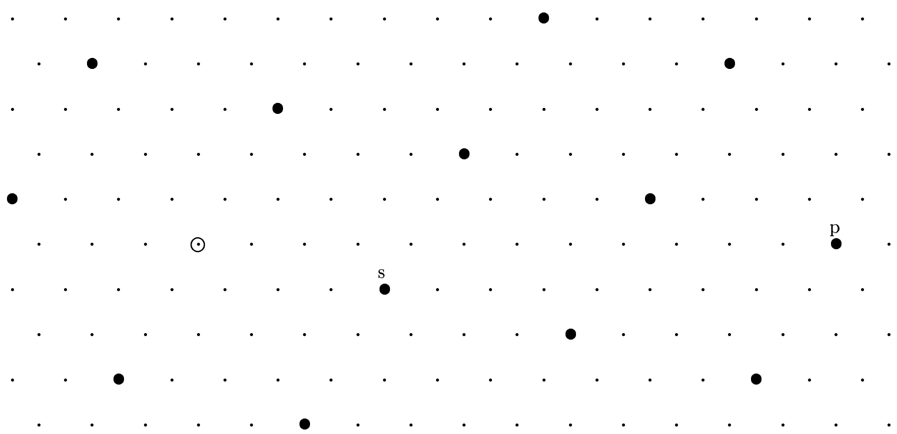
$$
\begin{array}{cccccc}
(-2,\ 2) & (-1,\ 2) & (\ 0,\ 2) & (\ 1,\ 2) & & \\
(-2,\ 1) & (-1,\ 1) & (\ 0,\ 1) & (\ 1,\ 1) & (\ 2,\ 1) & \\
(-2,\ 0) & (-1,\ 0) & (\ 0,\ 0) & (\ 1,\ 0) & (\ 2,\ 0) & (\ 3,\ 0) \\
(-1,-1) & (\ 0,-1) & (\ 1,-1) & (\ 2,-1) & (\ 3,-1) & (\ 4,-1) \\
(\ 0,-2) & (\ 1,-2) & (\ 2,-2) & (\ 3,-2) & (\ 4,-2) &
\end{array}
$$

The defining property of a basis is that every vector in the lattice has a unique expression as an integer combination of the basis vectors. The number of vectors in a basis is the dimension of the lattice.

Now we need to choose our unison vectors. The classical choice here is $(4,-1)$ and $(12,0)$, corresponding to the syntonic comma and the Pythagorean comma. The sublattice *generated* by these unison vectors consists of all linear combinations

$$m(4,-1) + n(12,0) = (4m + 12n, -4m)$$

with $m, n \in \mathbb{Z}$. This is called the *unison sublattice.*



In this diagram, the syntonic comma and Pythagorean comma are marked with s and p respectively. Each vector (a, b) in the lattice may then be thought of as *equivalent* to the vectors

$$(a, b) + m(4,-1) + n(12,0) = (a + 4m + 12n, b - 4m)$$

with $m, n \in \mathbb{Z}$, differing from it by vectors in the unison sublattice. So for example, taking $m = -3$ and $n = 1$, we see that the vector $(0, 3)$ is in the unison sublattice. This corresponds to the fact that three just major thirds approximately make one octave.

There are many ways of choosing unison vectors generating a given sublattice. In the above example, $(4, -1)$ and $(0, 3)$ generate the same sublattice.

The set of vectors (or pitches) equivalent to a given vector is called a *coset.* The number of cosets is called the *index* of the unison sublattice in the lattice. It can be calculated by taking the *determinant* of the matrix formed from the unison vectors. So in our example, the index of the unison sublattice is

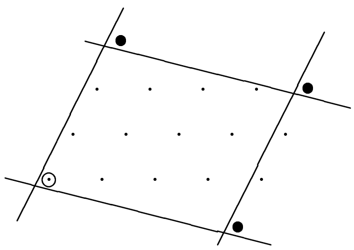
$$\begin{vmatrix} 4 & -1 \\ 12 & 0 \end{vmatrix} = 12.$$

The formula for the determinant of a 2×2 matrix is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

If the determinant comes out negative, the index is the corresponding positive quantity. If two rows of a matrix are swapped, then the determinant changes sign, so the sign of the determinant is irrelevant to the index. It has to do with *orientation*, and will not be discussed here.

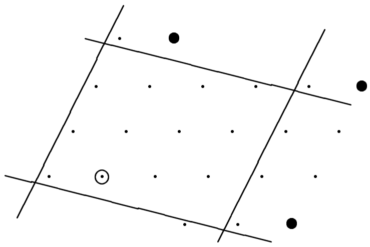
A *periodicity block* consists of a choice of one vector from each coset. In other words, we find a finite set of vectors with the property that each vector in the whole lattice is equivalent to a unique vector from the periodicity block. One way to do this is to draw a parallelogram using the unison vectors. We can then tile the plane using copies of this parallelogram, translated along unison vectors. In the above example, if we use the unison vectors $(4, -1)$ and $(0, 3)$ to generate the unison sublattice, then the parallelogram looks like this.



This choice of periodicity block leads to the following just scale with twelve tones.

$$\begin{array}{cccccc} & & \mathrm{G}\sharp^{-2} & \mathrm{D}\sharp^{-2} & \mathrm{A}\sharp^{-2} & \mathrm{E}\sharp^{-2} \\ & \mathrm{E}^{-1} & \mathrm{B}^{-1} & \mathrm{F}\sharp^{-1} & \mathrm{C}\sharp^{-1} & \\ \mathrm{C}^{0} & \mathrm{G}^{0} & \mathrm{D}^{0} & \mathrm{A}^{0} & & \end{array}$$

Of course, there are many other choices of periodicity block. For example, shifting this parallelogram one place to the left gives rise to Euler's monochord, described on page 124.



Periodicity blocks do not have to be parallelograms. For example, we can chop off a corner of the parallelogram, translate it through a unison vector, and stick it back on somewhere else to get a hexagon. Each of the just intonation scales in §5.8 may be interpreted as a periodicity block for the above choice of unison sublattice.

Of course, there are other choices of unison sublattices. If we choose the unison vectors $(4, -1)$ and $(-1, 5)$ for example, then we get a scale of

$$\begin{vmatrix} 4 & -1 \\ -1 & 5 \end{vmatrix} = 19$$

tones. This gives rise to just scales approximating the equal tempered scale described at the beginning of §6.4. The choice of $(4, 2)$ and $(-1, 5)$ gives the Indian scale of 22 Srutis described in §6.1, corresponding to the calculation

$$\begin{vmatrix} 4 & 2 \\ -1 & 5 \end{vmatrix} = 22.$$

Taking the unison vectors $(4, -1)$ and $(3, 7)$ gives rise to 31 tone scales approximating 31 tone equal temperament, whose relationship with meantone is described in §6.4. This corresponds to the calculation

$$\begin{vmatrix} 4 & -1 \\ 3 & 7 \end{vmatrix} = 31$$

The vectors $(8, 1)$ and $(-5, 6)$ correspond in the same way to just scales approximating the 53 tone equal tempered scale described in §6.3, corresponding to the calculation

$$\begin{vmatrix} 8 & 1 \\ -5 & 6 \end{vmatrix} = 53.$$

An example of a periodicity block for this choice of unison vectors can be found on page 162.

When we come to study groups and normal subgroups in §8.11, we shall make some more comments on how to interpret unison vectors and periodicity blocks in group theoretical language.

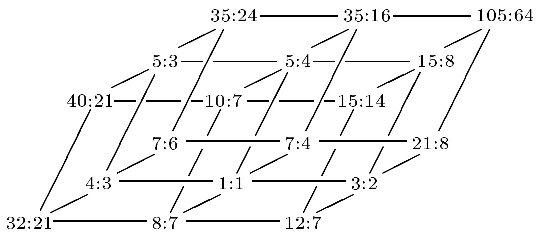
Further reading:

Paul Erlich, http://www.ixpres.com/interval/td/erlich/intropblock1.htm
Much of the material in this and the next section expresses ideas from Paul's online article, together with the work of Fokker.

A. D. Fokker, *Selections from the harmonic lattice of perfect fifths and major thirds containing 12, 19, 22, 31, 41 or 53 notes*, Proc. Koninkl. Nederl. Akad. Wetenschappen, Series B, 71 (1968), 251–266.

6.9. Septimal harmony

Septimal harmony refers to 7-limit just intonation; in other words, just intonation involving the primes 2, 3, 5 and 7. Taking octave equivalence into account, this means that we need three dimensions, or $\mathbb{Z}^3$ to represent the septimal version of just intonation, to take account of the primes 3, 5 and 7. It is harder to draw a three dimensional lattice, but it can be done. In ratio notation, it will then look as follows.



We take as our basis vectors the ratios $\frac{3}{2}$, $\frac{5}{4}$ and $\frac{7}{4}$. So the vector (a, b, c) represents the ratio $3^a.5^b.7^c$, multiplied if necessary by a power of 2 so that it is between 1 and 2 (octave equivalence). The septimal comma introduced in §5.6 is a ratio of $64 : 63$, which corresponds to the vector $(-2, 0, -1)$. So it would be reasonable to use the three commas $(4, -1, 0)$ (syntonic), $(12, 0, 0)$ (Pythagorean), and $(-2, 0, -1)$ (septimal) as unison vectors.

The determinant of a 3×3 matrix

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

is given by the formula

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - bdi - afh.$$

This can be visualized as three leading diagonals minus three trailing diagonals. If you have trouble visualizing these diagonals, it may help to think of the matrix as wrapped around a cylinder. So you should write the first two columns of the matrix again to the right of the matrix, and then the leading and trailing diagonals really look diagonal.

With the three commas as unison vectors, the determinant is

$$\begin{vmatrix} 4 & -1 & 0 \\ 12 & 0 & 0 \\ -2 & 0 & -1 \end{vmatrix} = 0 + 0 + 0 - 0 - 12 - 0 = -12.$$

vector			ratio	cents
(−7	4	1)	$\frac{4375}{4374}$	0.40
(−1	−2	4)	$\frac{2401}{2400}$	0.72
(−8	2	5)	$\frac{420175}{419904}$	1.12
(9	3	−4)	$\frac{2460375}{2458624}$	1.23
(8	1	0)	$\frac{32805}{32764}$	1.95
(1	5	1)	$\frac{65625}{65536}$	2.35
(0	3	5)	$\frac{2100875}{2097152}$	3.07
(−8	−6	2)	$\frac{102760448}{102515625}$	4.13
(1	−3	−2)	$\frac{6144}{6125}$	5.36
(0	−5	2)	$\frac{3136}{3125}$	6.08
(−7	−1	3)	$\frac{10976}{10935}$	6.48
(2	2	−1)	$\frac{225}{224}$	7.71
(8	−4	2)	$\frac{321489}{320000}$	8.04
(−5	6	0)	$\frac{15625}{15552}$	8.11
(1	0	3)	$\frac{1029}{1024}$	8.43
(3	7	0)	$\frac{2109375}{2083725}$	10.06
(−5	−2	−3)	$\frac{2097152}{2083725}$	11.12
(3	−1	−3)	$\frac{1728}{1715}$	13.07
(−4	3	−2)	$\frac{4000}{3969}$	13.47
(2	−3	1)	$\frac{126}{125}$	13.79
(−5	1	2)	$\frac{245}{243}$	14.19
(10	−2	1)	$\frac{413343}{409600}$	15.75
(3	2	2)	$\frac{33075}{32768}$	16.14
(−3	0	−4)	$\frac{65536}{64827}$	18.81
(3	−6	−1)	$\frac{110592}{109375}$	19.16
(−4	−2	0)	$\frac{2048}{2025}$	19.55
(5	1	−4)	$\frac{2430}{2401}$	20.79
(4	−1	0)	$\frac{81}{80}$	21.51
(−3	3	1)	$\frac{875}{864}$	21.90
(12	0	0)	$\frac{531441}{524288}$	23.46
(5	4	1)	$\frac{1063125}{1048576}$	23.86
(4	2	5)	$\frac{34034175}{33554432}$	24.58
(−3	−5	−2)	$\frac{4194302}{4134375}$	24.91
(−10	−1	−1)	$\frac{2097152}{2066715}$	25.31
(5	−4	−2)	$\frac{31104}{30625}$	26.87

vector			ratio	cents
(−2	0	−1)	$\frac{64}{63}$	27.26
(−3	−2	3)	$\frac{686}{675}$	27.99
(−1	5	0)	$\frac{3125}{3072}$	29.61
(−2	3	4)	$\frac{303125}{294912}$	30.33
(−1	−3	−3)	$\frac{131072}{128625}$	32.63
(−8	1	−2)	$\frac{327680}{321989}$	33.02
(−9	−1	2)	$\frac{100352}{98415}$	33.74
(0	2	−2)	$\frac{50}{49}$	34.98
(−1	0	2)	$\frac{49}{48}$	35.70
(1	7	−1)	$\frac{234375}{229376}$	37.33
(7	1	2)	$\frac{535815}{524288}$	37.65
(0	5	3)	$\frac{1071875}{1048576}$	38.05
(1	−1	−4)	$\frac{12278}{12005}$	40.33
(0	−3	0)	$\frac{128}{125}$	41.06
(−7	1	1)	$\frac{2240}{2187}$	41.45
(2	4	−1)	$\frac{5625}{5488}$	42.69
(1	2	1)	$\frac{525}{512}$	43.41
(0	0	5)	$\frac{16807}{16384}$	44.13
(1	−6	−2)	$\frac{786432}{765625}$	46.42
(−6	−2	−1)	$\frac{131072}{127575}$	46.81
(2	−1	−1)	$\frac{36}{35}$	48.77
(−6	1	4)	$\frac{12005}{11664}$	49.89
(2	2	4)	$\frac{540225}{524288}$	51.84
(2	−6	1)	$\frac{16128}{15625}$	54.85
(−5	−2	2)	$\frac{6272}{6075}$	55.25
(4	1	−2)	$\frac{405}{392}$	56.48
(3	−1	2)	$\frac{1323}{1280}$	57.20
(−4	3	3)	$\frac{42875}{42472}$	57.60
(4	−4	0)	$\frac{648}{625}$	62.57
(−3	0	1)	$\frac{28}{27}$	62.96
(−1	2	0)	$\frac{25}{24}$	70.72
(1	−1	1)	$\frac{21}{20}$	84.42
(3	1	0)	$\frac{135}{128}$	92.23
(−3	3	1)	$\frac{3584}{3575}$	104.02
(−1	4	−2)	$\frac{625}{588}$	105.65

Ignoring signs as usual, this tells us that we should expect the periodicity block to have 12 elements. One choice of periodicity block gives the 7-limit just intonation diagram on page 150.

There are many choices of unison vector in 7-limit just intonation. The table on page 177, adapted from Fokker, gives some of the most useful ones. Fokker also develops an elaborate system of notation for 7-limit just intonation, in which he ends up with notes such as $\diagdown f \downharpoonleft \downharpoonleft$.

Further reading:

A. D. Fokker, *Unison vectors and periodicity blocks in the three-dimensional (3-5-7-) harmonic lattice of notes*, Proc. Koninkl. Nederl. Akad. Wetenschappen, Series B, 72 (1969), 153–168.