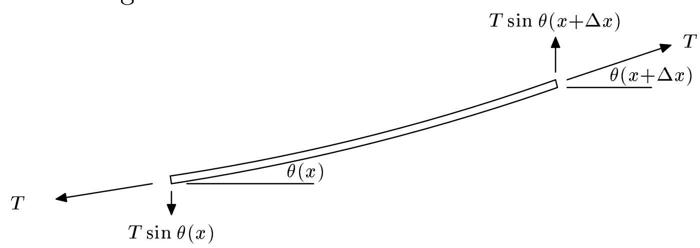
CHAPTER 3

A mathematician's guide to the orchestra

3.1. The wave equation for strings

Now let us return to the subject of §1.6, and consider the relevance of Fourier series to the vibration of a string held at both ends. To make a more accurate analysis, we need to regard the displacement y as a function both of time t and position x along the string. Since y is being regarded as a function of two variables, the appropriate equations are written in terms of *partial derivatives*, and Appendix P gives a brief summary of partial derivatives. The equation describing the vibration of a string is called the *wave equation* in one dimension, which we now develop. This equation supposes that the displacement of the string is such that its slope at any point along its length at any time is small. For large displacements, the analysis is harder. Note that we are only concerned here with *transverse waves*, namely motion perpendicular to the string. Motion parallel to the string is called *longitudinal waves*, and will be ignored here.



Write T for the tension on the string (in newtons $=$ kg m/s^2), and ρ for the linear density of the string (in kg/m). Then at position x along the string, the angle $\theta(x)$ between the string and the horizontal will satisfy $\tan\theta(x) = \dfrac{\partial y}{\partial x}$. On a small segment of string from x to $x + \Delta x$, the vertical component of force at the left end will be $-T\sin\theta(x)$, and at the right end will be $T\sin\theta(x + \Delta x)$.

Provided that $\theta(x)$ is small, $\sin\theta(x)$ and $\tan\theta(x)$ are approximately equal. So the difference in vertical components of force between the two ends of the segment will be approximately

$$T\tan\theta(x+\Delta x) - T\tan\theta(x) = T\left(\frac{\partial y(x+\Delta x)}{\partial x} - \frac{\partial y(x)}{\partial x}\right)$$

$$= T\Delta x \frac{\dfrac{\partial y(x+\Delta x)}{\partial x} - \dfrac{\partial y(x)}{\partial x}}{\Delta x}$$

$$\approx T\Delta x \frac{\partial^2 y}{\partial x^2}. \tag{3.1.1}$$

The mass of the segment of string will be approximately $\rho\Delta x$. So Newton's law ($F = ma$) for the acceleration $a = \dfrac{\partial^2 y}{\partial t^2}$ gives

$$T\Delta x \frac{\partial^2 y}{\partial x^2} \approx (\rho\Delta x)\frac{\partial^2 y}{\partial t^2}.$$

Cancelling a factor of Δx on both sides gives

$$T\frac{\partial^2 y}{\partial x^2} \approx \rho\frac{\partial^2 y}{\partial t^2}.$$

In other words, as long as $\theta(x)$ never gets large, the motion of the string is essentially determined by the wave equation

$$\boxed{\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}} \tag{3.1.2}$$

where $c = \sqrt{T/\rho}$.

D'Alembert[1] discovered a strikingly simple method for finding the general solution to equation (3.1.2). Roughly speaking, his idea is to factorize the differential operator

$$\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}$$

as

$$\Big(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\Big)\Big(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\Big).$$

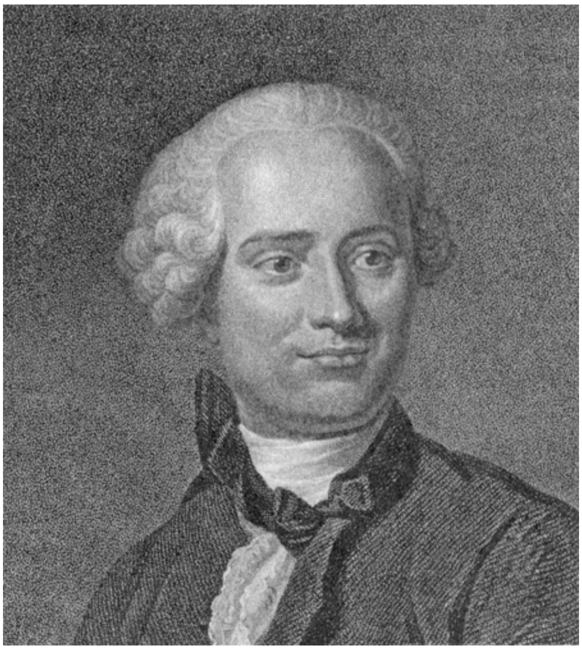
More precisely, we make a change of variables

$$u = x + ct, \qquad v = x - ct.$$

Then by the multivariable form of the chain rule, we have

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial t} = c\frac{\partial y}{\partial u} - c\frac{\partial y}{\partial v}.$$

[1]Jean-le-Rond d'Alembert was born in Paris on November 16, 1717, and died there on October 29, 1783. He was the illegitimate son of a chevalier by the name of Destouches, and was abandoned by his mother on the steps of a small church called St. Jean-le-Rond, from which his first name is taken. He grew up in the family of a glazier and his wife, and lived with his adoptive mother until she died in 1757. But his father paid for his education, which allowed him to be exposed to mathematics. Two essays written in 1738 and 1740 drew attention to his mathematical abilities, and he was elected to the French Academy in 1740. Most of his mathematical works were written there in the years 1743–1754, and his solution of the wave equation appeared in his paper: *Recherches sur la courbe que forme une corde tendue mise en vibration*, Hist. Acad. Sci. Berlin 3 (1747), 214–219.



Jean-le-Rond d'Alembert (1717–1783)

Differentiating again, we have

$$\begin{aligned}\frac{\partial^2 y}{\partial t^2} &= \frac{\partial}{\partial u}\Big(\frac{\partial y}{\partial t}\Big)\frac{\partial u}{\partial t} + \frac{\partial}{\partial v}\Big(\frac{\partial y}{\partial t}\Big)\frac{\partial v}{\partial t} \\ &= c\Big(c\frac{\partial^2 y}{\partial u^2} - c\frac{\partial^2 y}{\partial u \partial v}\Big) - c\Big(c\frac{\partial^2 y}{\partial v \partial u} - c\frac{\partial^2 y}{\partial v^2}\Big) \\ &= c^2\left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right).\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial y}{\partial x} &= \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial y}{\partial u} + \frac{\partial u}{\partial x}, \\ \frac{\partial^2 y}{\partial x^2} &= \frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}.\end{aligned}$$

Then equation (3.1.2) becomes

$$c^2\left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right) = c^2\left(\frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right)$$

or

$$\boxed{\frac{\partial^2 y}{\partial u \partial v} = 0.}$$

This equation may be integrated directly to see that the general solution is given by $y = f(u) + g(v)$ for suitably chosen functions f and g. Substituting back, we obtain

$$\boxed{y = f(x+ct) + g(x-ct).}$$

This represents a superposition of two waves, one traveling to the right and one traveling to the left, each with velocity c.

Now the boundary conditions tell us that the left and right ends of the string are fixed, so that when $x = 0$ or $x = l$ (the length of the string), we have $y = 0$ (independent of t). The condition with $x = 0$ gives

$$0 = f(ct) + g(-ct)$$

for all t, so that

$$g(\lambda) = -f(-\lambda) \tag{3.1.3}$$

for any value of λ. Thus

$$y = f(x+ct) - f(ct-x).$$

Physically, this means that the wave traveling to the left hits the end of the string and returns inverted as a wave traveling to the right. This is called the "principle of reflection".

Substituting the other boundary condition $x = l$, $y = 0$ gives $f(l+ct) = f(ct-l)$ for all t, so that

$$f(\lambda) = f(\lambda + 2l) \tag{3.1.4}$$

for all values of λ. We summarise all the above information in the following theorem.

THEOREM 3.1.1 (d'Alembert). *The general solution of the wave equation*

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

is given by

$$y = f(x+ct) + g(x-ct).$$

The solutions satisfying the boundary conditions $y = 0$ for $x = 0$ and for $x = l$, for all values of t, are of the form

$$y = f(x+ct) - f(-x+ct)$$

where f satisfies $f(\lambda) = f(\lambda + 2l)$ for all values of λ.

One interesting feature of d'Alembert's solution to the wave equation is worth emphasizing. Although the wave equation only makes sense for functions with second order partial derivatives, the solutions make sense for any *continuous* periodic function f. (Discontinuous functions cannot represent displacement of an unbroken string!) This allows us, for example, to make sense of the plucked string, where the initial displacement is continuous, but not even once differentiable. This is a common phenomenon when solving partial differential equations. A technique which is very often used is to rewrite the equation as an integral equation, meaning an equation involving integrals rather than derivatives. Integrable functions are much more general than differentiable functions, so one should expect a more general class of solutions.

Equation (3.1.4) means that the function f appearing in d'Alembert's solution is periodic with period $2l$, so that f has a Fourier series expansion. So for example if only the fundamental frequency is present, then the function $f(x)$ is a sine wave $f(x) = C\sin((\pi x/l) + \phi)$. If only the nth harmonic is present, then we have $f(x) = C\sin((n\pi x/l) + \phi)$,

$$y = C\sin\left(\frac{n\pi(x+ct)}{l} + \phi\right) - C\sin\left(\frac{n\pi(ct-x)}{l} + \phi\right).$$

The theory of Fourier series allows us to write the general solution as a combination of the above harmonics, as long as we take care of the details of what sort of functions are allowed and what sort of convergence is intended.

Marin Mersenne
(1588–1648)

Using equation (1.7.12), we can rewrite the nth harmonic solution (3.1) as

$$y = 2C\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l} + \phi\right).$$

Thus the frequency of the nth harmonic is given by $2\pi\nu = n\pi c/l$, or replacing c by its value $\sqrt{T/\rho}$,

$$\boxed{\nu = (n/2l)\sqrt{T/\rho}.}$$

This formula for frequency was essentially discovered by Marin Mersenne[2] as his "laws of stretched strings". These say that the frequency of a stretched string is inversely proportional to its length, directly proportional to the square root of its tension, and inversely proportional to the square root of the linear density.

Exercises

1. Piano wire is manufactured from steel of density approximately 5,900 kg/m^3. The manufacturers recommend a stress of approximately 1.1×10^9 Newtons/m^2. What is the speed of propagation of waves along the wire? Does it depend on cross-sectional area? How long does the string need to be to sound middle C (262 Hz)?

2. By what factor should the tension on a string be increased, to raise its pitch by a perfect fifth? Assume that the length and linear density remain constant.
[A perfect fifth represents a frequency ratio of 3:2]

[2]Marin Mersenne, *Harmonie Universelle*, Sebastien Cramoisy, Paris, 1636–37. Translated by R. E. Chapman as *Harmonie Universelle: The Books on Instruments*, Martinus Nijhoff, The Hague, 1957. Also republished in French by the CNRS in 1975 from a copy annotated by Mersenne.

3.2. Initial conditions

In this section, we see that in the analysis of the wave equation (3.1.2) described in the last section, specifying the initial position and velocity of each point on the string uniquely determines the subsequent motion.

Let $s_0(x)$ and $v_0(x)$ be the initial vertical and velocity of the string as functions of the horizontal coordinate x, for $0 \le x \le l$. These must satisfy $s_0(0) = s_0(l) = 0$ and $v_0(0) = v_0(l) = 0$ to fit with the boundary conditions at the two ends of the string.

The first step is to extend the definitions of s_0 and v_0 to all values of x using the reflection principle. If we specify that $s_0(-x) = -s_0(x)$ and $v_0(-x) = -v_0(x)$, so that s_0 and v_0 are odd functions of x, this extends the domain of definition to the values $-l \le x \le l$. The values match up at $-l$ and l, so we can extend to all values of x by specifying periodicity with period $2l$; namely that $s_0(x+2l) = s_0(x)$ and $v_0(x+2l) = v_0(x)$.

Now we simply substitute into the solution given by d'Alembert's theorem. Namely, we know that

$$y = f(x+ct) - f(-x+ct) \tag{3.2.1}$$

where f is periodic with period $2l$. Differentiating with respect to t gives the formula for velocity

$$\frac{\partial y}{\partial t} = cf'(x+ct) - cf'(x-ct).$$

Substituting $t = 0$ in both the equation and its derivative gives the following equations

$$f(x) - f(-x) = s_0(x) \tag{3.2.2}$$

$$cf'(x) - cf'(-x) = v_0(x). \tag{3.2.3}$$

Integrating equation (3.2.3) and noting that $v_0(0) = 0$, we obtain

$$cf(x) + cf(-x) = \int_0^x v_0(u)\,du.$$

We divide this equation by c to obtain a formula for $f(x) + f(-x)$. So we can then add equation (3.2.2) and divide by two to obtain $f(x)$. This gives

$$f(x) = \tfrac{1}{2}s_0(x) + \frac{1}{2c}\int_0^x v_0(u)\,du.$$

Putting this back into equation (3.2.1) gives

$$y = \tfrac{1}{2}(s_0(x+ct) - s_0(-x+ct)) + \frac{1}{2c}\left(\int_0^{x+ct} v_0(u)\,du - \int_0^{-x+ct} v_0(u)\,du\right).$$

Using the fact that v_0 is an odd function, we have

$$\int_{x-ct}^{-x+ct} v_0(u)\,du = 0.$$

So we can rewrite the solution as

$$\boxed{y = \tfrac{1}{2}(s_0(x+ct) - s_0(-x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(u)\,du.}$$

It is now easy to check that this is the unique solution satisfying both the initial conditions and the boundary conditions.

So for example, if the initial velocity is zero, as is the case for a plucked string, then the solution is given by

$$y = \tfrac{1}{2}(s_0(x+ct) - s_0(-x+ct)).$$

In other words, the initial displacement moves both ways along the string, with velocity c, and the displacement at time t is the average of the two traveling waves.

Exercises

1. (Effect of errors in initial conditions) Consider two sets of initial conditions for the wave equation (3.1.2), $s_0(x)$ and $v_0(x)$, $s_0'(x)$ and $v_0'(x)$, and let y and y' be the corresponding solutions. If we have bounds (not depending on x) on the distance between these initial conditions,

$$|s_0(x) - s_0'(x)| < \varepsilon_s, \qquad |v_0(x) - v_0'(x)| < \varepsilon_v,$$

show that the distance between y and y' satisfies

$$|y - y'| < \varepsilon_s + \frac{L\varepsilon_v}{2c}$$

(independently of x and t). This means, in particular, that the solution to the wave equation (3.1.2) depends *continuously* on the initial conditions.

3.3. Wind instruments

To understand the vibration of air in a tube or pipe, we introduce two variables, displacement and acoustic pressure. Both of these will end up satisfying the wave equation, but with different phases.

We consider the air in the tube to have a rest position, and the wave motion is expressed in terms of displacement from that position. So let x denote position along the tube, and let $\xi(x,t)$ denote the displacement of the air at position x at time t. The pressure also has a rest value, namely the ambient air pressure ρ. We measure the *acoustic pressure* $p(x,t)$ by subtracting ρ from the absolute pressure $P(x,t)$, so that

$$p(x,t) = P(x,t) - \rho.$$

Hooke's law in this situation states that

$$p = -B\frac{\partial\xi}{\partial x}$$

where B is the *bulk modulus* of air. Newton's second law of motion implies that

$$\frac{\partial p}{\partial x} = -\rho\frac{\partial^2\xi}{\partial t^2}.$$

Combining these equations, we obtain the equations

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \xi}{\partial t^2} \tag{3.3.1}$$

and

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}. \tag{3.3.2}$$

where $c = \sqrt{B/\rho}$. These equations are the wave equation for displacement and acoustic pressure respectively.

The boundary conditions depend upon whether the end of the tube is open or closed. For a closed end of a tube, the displacement ξ is forced to be zero for all values of t. For an open end of a tube, the acoustic pressure p is zero for all values of t. Actually, for an open end, this is really only an approximation, because the volume of air just outside of the tube is not infinite. A good way to adjust to make a more accurate representation of an actual tube is to work in terms of an *effective* length, and consider the tube to end a little beyond where it really does.

If both ends are open, the boundary conditions for the differential equation for p are exactly the same as for a string in Section 3.1. So in this case, as with a string, the solutions can all be expressed in terms of integer multiples of a fundamental frequency of vibration. Pictures can be found in Section 1.6.

3.4. The horn

The horn can be regarded as a hard walled tube of varying cross-section. Fortunately, the cross-section matters more than the exact shape and curvature of the tube.

If $A(x)$ represents the cross-section as a function of position x along the tube, then assuming that the wavefronts are approximately planar and propagate along the direction of the horn, equation (3.3.2) can be modified to *Webster's horn equation*

$$\frac{1}{A(x)}\frac{\partial}{\partial x}\left(A(x)\frac{\partial p}{\partial x}\right) = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2},$$

or equivalently

$$\frac{\partial^2 p}{\partial x^2} + \frac{1}{A}\frac{dA}{dx}\frac{\partial p}{\partial x} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}.$$

Solutions of this equation can be described using the theory of *Sturm–Liouville equations*. The theory of Sturm–Liouville equations is described in many standard texts on partial differential equations.

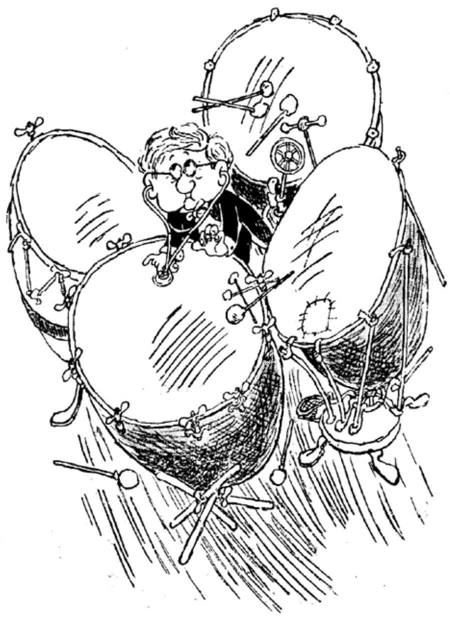
Further reading:

Fletcher and Rossing, *The physics of musical instruments* [**30**], §8.6.

A. G. Webster, *Acoustical impedance, and the theory of horns and of the phonograph*, Proc. Nat. Acad. Sci. (US) 5 (1919), 275–282.

3.5. The drum

Consider a circular drum whose skin has area density (mass per unit area) ρ. If the boundary is under uniform tension T, this ensures that the entire surface is under the same uniform tension. The tension is measured in force per unit distance (newtons per meter).



The Timpani (Hoffnung)

To understand the wave equation in two dimensions, for a membrane such as the surface of a drum, the argument is analogous to the one dimensional case. We parametrize the surface with two variables x and y, and we use z to denote the displacement perpendicular to the surface. Consider a rectangular element of surface of width Δx and length Δy. Then the tension on the left and right sides is $T\Delta y$, and the argument which gave equation (3.1.1) in the one dimensional case shows in this case that the difference in vertical components is approximately

$$(T\Delta y)\left(\Delta x\frac{\partial^2 z}{\partial x^2}\right).$$

Similarly, the difference in vertical components between the front and back of the rectangular element is approximately

$$(T\Delta x)\left(\Delta y\frac{\partial^2 z}{\partial y^2}\right).$$

So the total upward force on the element of surface is approximately

$$T\Delta x\Delta y\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right).$$

The mass of the element of surface is approximately $\rho\Delta x\Delta y$, so Newton's second law of motion gives

$$T\Delta x\Delta y\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)\approx(\rho\Delta x\Delta y)\frac{\partial^2 z}{\partial t^2}.$$

Dividing by $\Delta x\Delta y$, we obtain the wave equation in two dimensions, namely the partial differential equation

$$\rho\frac{\partial^2 z}{\partial t^2}=T\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right).$$

As in the one dimensional case, we set $c = \sqrt{T/\rho}$, which will play the role of the speed of the waves on the membrane. So the wave equation becomes

$$\frac{\partial^2 z}{\partial t^2} = c^2 \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right).$$

Converting to polar coordinates (r, θ) using equation (P.4), we obtain

$$\frac{\partial^2 z}{\partial t^2} = c^2 \left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} \right). \tag{3.5.1}$$

We look for *separable* solutions of this equation, namely solutions of the form

$$z = f(r)g(\theta)h(t).$$

The reason for looking for separable solutions will be explained further in the next section. Substituting this into the wave equation, we obtain

$$f(r)g(\theta)h''(t) = c^2 \left(f''(r)g(\theta)h(t) + \frac{1}{r}f'(r)g(\theta)h(t) + \frac{1}{r^2}f(r)g''(\theta)h(t) \right).$$

Dividing by $f(r)g(\theta)h''(t)$ gives

$$\frac{h''(t)}{h(t)} = c^2 \left(\frac{f''(r)}{f(r)} + \frac{1}{r}\frac{f'(r)}{f(r)} + \frac{1}{r^2}\frac{g''(\theta)}{g(\theta)} \right).$$

In this equation, the left hand side only depends on t, and is independent of r and θ, while the right hand side only depends on r and θ, and is independent of t. Since t, r and θ are three independent variables, this implies that the common value of the two sides is independent of t, r and θ, so that it has to be a constant. We shall see in the next section that this constant has to be a negative real number, so we shall write it as $-\omega^2$. So we obtain two equations,

$$h''(t) = -\omega^2 h(t), \tag{3.5.2}$$

$$\frac{f''(r)}{f(r)} + \frac{1}{r}\frac{f'(r)}{f(r)} + \frac{1}{r^2}\frac{g''(\theta)}{g(\theta)} = -\frac{\omega^2}{c^2}. \tag{3.5.3}$$

The general solution to equation (3.5.2) is a multiple of the solution

$$h(t) = \sin(\omega t + \phi),$$

where ϕ is a constant determined by the initial temporal phase. Multiplying equation (3.5.3) by r^2 and rearranging, we obtain

$$r^2 \frac{f''(r)}{f(r)} + r\frac{f'(r)}{f(r)} + \frac{\omega^2}{c^2}r^2 = -\frac{g''(\theta)}{g(\theta)}.$$

The left hand side depends only on r, while the right hand side depends only on θ, so their common value is again a constant. This makes $g(\theta)$ either a sine function or an exponential function, depending on the sign of the constant. But the function $g(\theta)$ has to be periodic of period 2π since it is a function of angle. So the common value of the constant must be the square of an integer n, so that

$$g''(\theta) = -n^2 g(\theta)$$

and $g(\theta)$ is a multiple of $\sin(n\theta+\psi)$. Here, ψ is another constant representing spatial phase. So we obtain

$$r^2\frac{f''(r)}{f(r)}+r\frac{f'(r)}{f(r)}+\frac{\omega^2}{c^2}r^2=n^2.$$

Multiplying by $f(r)$, dividing by r^2 and rearranging, this becomes

$$f''(r)+\frac{1}{r}f'(r)+\left(\frac{\omega^2}{c^2}-\frac{n^2}{r^2}\right)=0.$$

Now Exercise 2 in §2.10 shows that the general solution to this equation is a linear combination of $J_n(\omega r/c)$ and $Y_n(\omega r/c)$. But the function $Y_n(\omega r/c)$ tends to $-\infty$ as r tends to zero, so this would introduce a singularity at the center of the membrane. So the only physically relevant solutions to the above equation are multiples of $J_n(\omega r/c)$. So we have shown that the functions

$$\boxed{z=AJ_n(\omega r/c)\sin(\omega t+\phi)\sin(n\theta+\psi)}$$

are solutions to the wave equation.

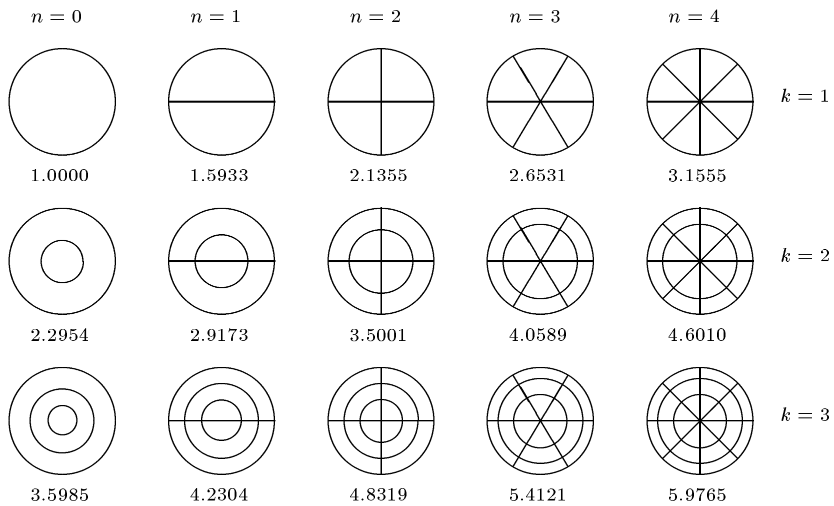
If the radius of the drum is a, then the boundary condition which we must satisfy is that $z=0$ when $r=a$, for all values of t and θ. So it follows that $J_n(\omega a/c)=0$. This is a constraint on the value of ω. The function J_n takes the value zero for a discrete infinite set of values of its argument. So ω is also constrained to an infinite discrete set of values.

It turns out that linear combinations of functions of the above form uniformly approximate the general, twice continuously differentiable solution of (3.5.1) as closely as desired, so that these form the drum equivalent of the sine and cosine functions of Fourier series.

Here is a table of the first few zeros of the Bessel functions. For more, see Appendix B.

k	J_0	J_1	J_2	J_3	J_4
1	2.40483	3.83171	5.13562	6.38016	7.58834
2	5.52008	7.01559	8.41724	9.76102	11.06471
3	8.65373	10.17347	11.61984	13.01520	14.37254

We have seen that to choose a vibrational mode, we must choose a nonnegative integer n and we must choose a zero of $J_n(z)$. Denoting the kth zero of J_n by $j_{n,k}$, the corresponding vibrational mode has frequency $(cj_{n,k}/a)$, which is $j_{n,k}/j_{0,1}$ times the fundamental frequency. The stationary points have the following pictures. Underneath each picture, we have recorded the value of $j_{n,k}/j_{0,1}$ for the relative frequency.



In practice, for a drum in which the air is confined (such as a kettledrum) the fundamental mode of the drum is heavily damped, because it involves compression and expansion of the air enclosed in the drum. So what is heard as the fundamental is really the mode with $n = 1$, $k = 1$, namely the second entry in the top row in the above diagram. The higher modes mostly involve moving the air from side to side. The inertia of the air has the effect of raising the frequency of the modes with $n = 0$, especially the fundamental, while the modes with $n > 0$ are lowered in frequency in such a way as to widen the frequency gaps. For an open drum, on the other hand, all the vibrational frequencies are lowered by the inertia of the air, but the ones of lower frequency are lowered the most.

The design of the orchestral kettledrum carefully utilises the inertia of the air to arrange for the modes with $n = 1$, $k = 1$ and $n = 2$, $k = 1$ to have frequency ratio approximating 3:2, so that what is perceived is a missing fundamental at half the actual fundamental frequency. Furthermore, the modes with $n = 3$, 4 and 5 (still with $k = 1$) are arranged to approximate frequency ratios of 4:2, 5:2 and 6:2 with the $n = 1$, $k = 1$ mode, thus accentuating the perception of the missing fundamental. The frequency of the $n = 1$, $k = 1$ mode is called the *nominal frequency* of the drum.

It is not true that the air in the kettle of a kettledrum acts as a resonator. A kettledrum can be retuned by a little more than a perfect fourth, whereas if the air were acting as a resonator, it could only do so for a small part of the frequency range. In fact, the resonances of the body of air are usually much higher in pitch, and do not have much effect on the overall sound. A more important effect is that the underside of the drum skin is prevented from radiating sound, and this makes the radiation of sound from the upper side more efficient.

Exercises

1. Find the separable solutions (i.e., the ones of the form $z = f(x)g(y)h(t)$) to the wave equation for a square drum. Write the answer in the form of an essay, with title: "What does a square drum sound like?". Try to integrate the words with the mathematics. Explain what you're doing at each step, and don't forget to answer the title question (i.e., describe the frequency spectrum).

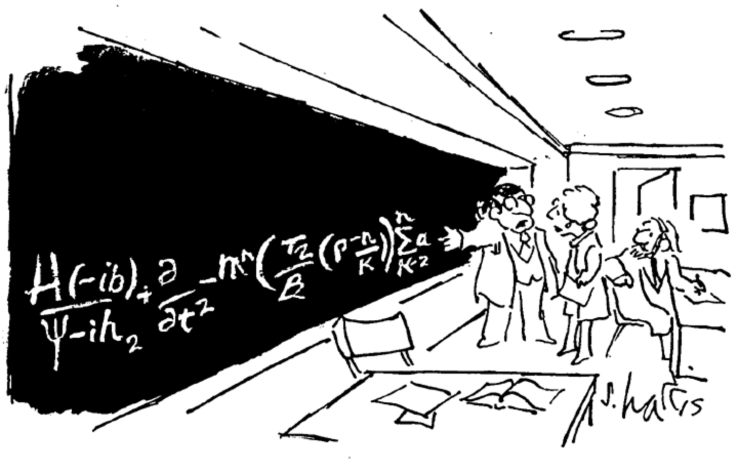
Further reading:

Campbell and Greated, *The musician's guide to acoustics* [**11**], chapter 10.

Elmore and Heald, *Physics of waves* [**28**], chapter 2.

Rossing, *Science of percussion instruments* [**98**].

3.6. Eigenvalues of the Laplace operator



"But this *is* the simplified version for the general public."

In this section, we put the discussion of the vibrational modes of the drum into a broader context. Namely, we explain the relationship between the shape of a drum and its frequency spectrum, in terms of the eigenvalues of the Laplace operator. This discussion explains the connection between the uses of the word "spectrum" in linear algebra, where it refers to the eigenvalues of an operator, and in music, where it refers to the distribution of frequency components. Parts of this discussion assume that the reader is familiar with elementary vector calculus and the divergence theorem.

We write ∇^2 for the operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. This is known as the *Laplace operator* (in three dimensions the Laplace operator ∇^2 denotes $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$; the analogous operator makes sense for any number of variables). In this notation, the wave equation becomes

$$\frac{\partial^2 z}{\partial t^2} = c^2 \nabla^2 z.$$

We consider the solutions to this equation on a closed and bounded region Ω. So for the drum of the last section, Ω was a disc in two dimensions.

A *separable solution* to the wave equation is one of the form

$$z = f(x, y)h(t).$$

Substituting into the wave equation, we obtain

$$f(x, y)h''(t) = c^2\nabla^2 f(x, y)\, h(t)$$

or

$$\frac{h''(t)}{h(t)} = c^2\frac{\nabla^2 f(x, y)}{f(x, y)}.$$

The left hand side is independent of x and y, while the right hand side is independent of t, so their common value is a constant. We write this constant as $-\omega^2$, because it will transpire that it has to be negative. Then we have

$$g''(t) = -\omega^2 g(t), \tag{3.6.1}$$

$$\nabla^2 f(x, y) = -\frac{\omega^2}{c^2} f(x, y). \tag{3.6.2}$$

The first of these equations is just the equation for simple harmonic motion with angular frequency ω, so the general solution is

$$g(t) = A\sin(\omega t + \phi).$$

A nonzero, twice differentiable function $f(x, y)$ satisfying the second equation is called an *eigenfunction* of the Laplace operator ∇^2, with *eigenvalue*[3]

$$\lambda = \omega^2/c^2. \tag{3.6.3}$$

There are two important kinds of eigenfunctions and eigenvalues. The *Dirichlet spectrum* is the set of eigenvalues for eigenfunctions which vanish on the boundary of the region Ω. The *Neumann spectrum* is the set of eigenvalues for eigenfunctions with vanishing derivative normal (i.e., perpendicular) to the boundary. The latter functions are important when studying the wave equation for sound waves, where the dependent variable is acoustic pressure (i.e., pressure minus the average ambient pressure).

For the benefit of the reader who knows vector calculus, in Appendix W we give the proof that the eigenvalues of ∇^2 (i.e., the values of λ for which $\nabla^2 z = -\lambda z$ has a nonzero solution) are positive and real, along with some other standard facts about the wave equation.

Furthermore, the Dirichlet spectrum, namely the set of eigenvalues λ for eigenfunctions which vanish on the boundary, is known to be discrete—there is no positive real number with an accumulation of Dirichlet eigenvalues around it. So the eigenvalues can be ordered:

$$0 < \lambda_1 < \lambda_2 < \dots$$

[3]In linear algebra, it would be more usual to say that $-\omega^2/c^2$ is the eigenvalue, but the usage here is more usual in the theory of partial differential equations.

For each eigenvalue, there can only be a finite number of linearly independent Dirichlet eigenfunctions. The eigenvalue λ determines the frequency of the corresponding vibration via (3.6.3):

$$\omega = c\sqrt{\lambda}. \tag{3.6.4}$$

The crucial property of the eigenvalues of the Laplace operator is *completeness.* This states that every twice continuously differentiable function $f(x, y)$ on a closed bounded region Ω can be written as the sum of an absolutely and uniformly convergent series of the form $f(x, y) = \sum_\lambda a_\lambda f_\lambda(x, y)$. Here, the sum runs over Dirichlet eigenvalues, and each f_λ is a Dirichlet eigenfunction on Ω with eigenvalue λ.

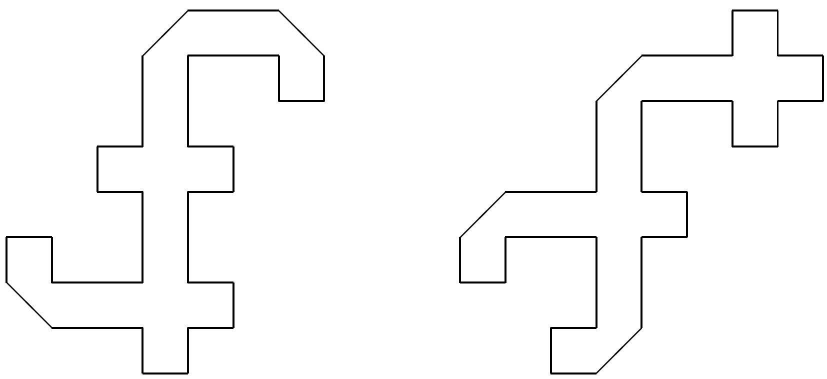
Initial conditions for the wave equation on Ω are specified by stipulating the values of z and $\frac{\partial z}{\partial t}$ for (x, y) in Ω, at $t = 0$. To solve the wave equation subject to these initial conditions, use completeness to write $z = \sum_\lambda a_\lambda f_\lambda(x, y)$ and $\frac{\partial z}{\partial t} = \sum_\lambda b_\lambda g_\lambda(x, y)$ at $t = 0$. Then the unique solution is given by

$$z = \sum_\lambda \left(a_\lambda f_\lambda(x, y) \cos(c\sqrt{\lambda}\, t) + \frac{b_\lambda}{c\sqrt{\lambda}} g_\lambda(x, y) \sin(c\sqrt{\lambda}\, t) \right).$$

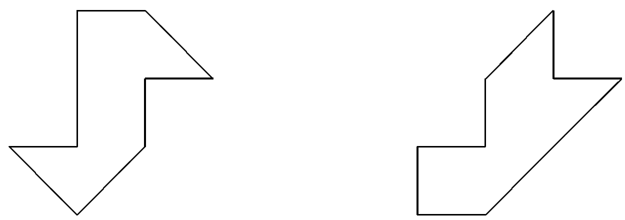
The angular frequency of $c\sqrt{\lambda}$ comes from equation (3.6.4).

We have phrased the above discussion in terms of the two dimensional wave equation, but the same arguments work in any number of dimensions. For example, in one dimension it corresponds to the vibrational modes of a string, and we recover the theory of Fourier series, but with more stringent differentiability conditions.

An interesting problem, which was posed by Mark Kac in 1965 and solved by Gordon, Webb and Wolpert in 1991, is *whether one can hear the shape of a drum.* In other words, can one tell the shape of a simply connected closed region in two dimensions from its Dirichlet spectrum? Simply connected just means there are no holes in the region. Based on a method developed by Sunada a few years previously, Gordon, Webb and Wolpert found examples of pairs of regions with the same Dirichlet spectrum. The example which appears in their paper is the following.



Admittedly, it had probably not occurred to anyone to make drums using vibrating surfaces of these shapes, prior to this investigation. Many other pairs of regions with the same Dirichlet spectrum have been found more recently, including the following much simpler example which was investigated by Tobin Driscoll.



Many more can be found in a paper of Buser, Conway, Doyle and Semmler, but it is still not known whether there are any *convex* examples.

Further reading:

David Colton, *Partial differential equations, an introduction* [**14**], contains a proof of eigenvalue completeness for the Laplace operator on a compact domain; a C^2 boundary is assumed.

Tobin Driscoll, *Eigenmodes of isospectral drums.* SIAM Rev. 39 (1997), 1-17.

Carolyn Gordon, David L. Webb, and Scott Wolpert, *One cannot hear the shape of a drum*, Bulletin of the Amer. Math. Soc. 27 (1992), 134–138.

Carolyn Gordon, David L. Webb, and Scott Wolpert, *Isospectral plane domains and surfaces via Riemannian orbifolds*, Invent. Math. 110 (1992), 1–22.

Mark Kac, *Can one hear the shape of a drum?*, Amer. Math. Monthly 73, (1966), 1–23.

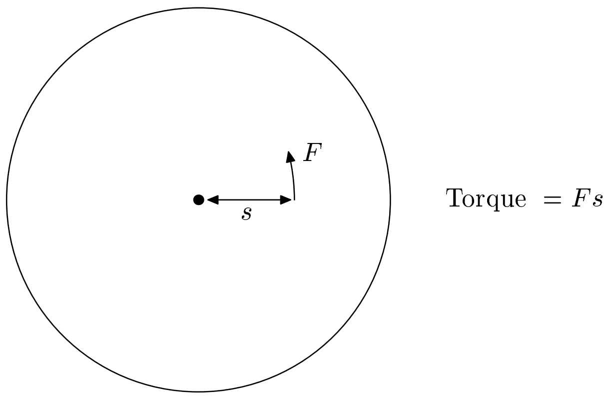
M. H. Protter, *Can one hear the shape of a drum? Revisited.* SIAM Rev. 29 (1987), 185–197.

T. Sunada, *Riemannian coverings and isospectral manifolds*, Ann. of Math. 121 (1985), 169–186.

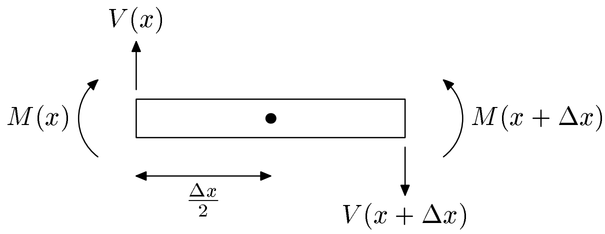
3.7. Xylophones and tubular bells

In this section we examine the theory of transverse waves in a slender stiff rod. This theory applies to instruments such as the xylophone and the tubular bells. We shall see that in this case, just as in the case of the drum, the vibrational modes do not consist of integer multiples of a fundamental frequency. Our goal will be to derive and solve the differential equation (3.7.2).

As well as the assumptions made in §3.1 about small angles, the basic assumption we shall make in order to obtain the appropriate differential equation is that terms coming from the resistance to motion caused by the rotational inertia of a segment of the rod are very small compared with terms coming from (vertical) linear inertia. This is only realistic for a slender rod. The upshot of this assumption is that the total *torque* on a segment of rod can be taken to be zero. Recall that if we try to twist an object about an axis, by applying a force F at distance s from the axis, then the torque applied is defined to be Fs. This is reasonable because the effect of such a turning force is proportional to the distance from the axis, as well as to the magnitude of the force.



Consider a segment of rod of length Δx, and let $V(x)$ be the vertical force (or *shearing force*) applied by the left end of the segment on the right end of the adjacent segment.



The torque on the segment due to this shearing force is

$$-V(x)\left(\frac{\Delta x}{2}\right) - V(x+\Delta x)\left(\frac{\Delta x}{2}\right) \approx -V(x)\Delta x$$

(the minus sign is because we regard counterclockwise as the positive direction for torque). Since we are regarding rotational inertia as negligible, this means that the torque, or *bending moment*, $M(x)$ applied by the segment on the adjacent segment satisfies

$$M(x+\Delta x) - M(x) - V(x)\Delta x \approx 0,$$

or

$$V(x) \approx \frac{M(x+\Delta x) - M(x)}{\Delta x}.$$

Taking limits as $\Delta x \to 0$, we obtain

$$V(x) = \frac{dM(x)}{dx}.$$

The upward force on the segment can now be calculated as

$$V(x) - V(x+\Delta x) \approx -\Delta x \frac{dV(x)}{dx} \approx -\Delta x \frac{d^2M(x)}{dx^2}.$$

Now the functions $V(x)$, $M(x)$, etc. are really functions of both x and t; we have suppressed the dependence on t in the above discussion. So we really need to write the total upwards force on the segment as

$$-\Delta x \frac{\partial^2 M(x,t)}{\partial x^2}.$$

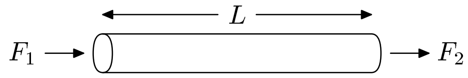
If the linear density of the rod is ρ (measured in kg/m) then the mass of the segment is $\rho\Delta x$. Writing y for the vertical displacement, Newton's second law of motion gives

$$-\Delta x \frac{\partial^2 M}{\partial x^2} = \rho \Delta x \frac{\partial^2 y}{\partial t^2},$$

or

$$\frac{\partial^2 y}{\partial t^2} + \frac{1}{\rho}\frac{\partial^2 M}{\partial x^2} = 0. \tag{3.7.1}$$

Now the bending moment M causes the rod to bend, and so there is a close relationship between M and $\partial^2 y/\partial x^2$. To understand this relationship, we must begin by introducing the concepts of stress, strain and Young's modulus. If a force $F = F_2 - F_1$ stretches or compresses a stiff slender rod of length L and cross-sectional area A,



then the length will increase by an amount ΔL. The *tension stress* (or just the *tension*) is defined to be

$$f = F/A.$$

The *tension strain* (or *extension*) is defined to be the proportional increase in length,

$$\epsilon = \Delta L/L.$$

Hooke's law for a stiff rod states that the extension is proportional to the tension,

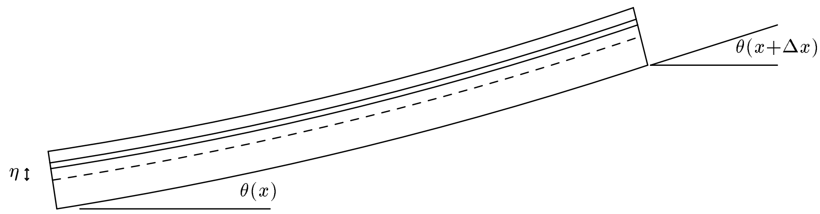
$$f = E\epsilon.$$

The constant of proportionality E is called the *Young's modulus*[4] (or *longitudinal elasticity*). Values for the Young's modulus for various materials at room temperature (18°C) are given in the following table.

[4]Named after the British physicist and physician Thomas Young (1773–1829).

Material	Young's modulus (N/m^2)
Aluminum	7.05×10^{10}
Brass	9.7–10.4×10^{10}
Copper	12.98×10^{10}
Gold	7.8×10^{10}
Iron	21.2×10^{10}
Lead	1.62×10^{10}
Silver	8.27×10^{10}
Steel	21.0×10^{10}
Zinc	9.0×10^{10}

Now we are ready to examine the segment of rod in more detail as it bends. There is a *neutral surface* in the middle of the rod, which is neither compressed nor stretched. It is represented by the dotted line in the diagram below. One side of this surface the horizontal filaments of rod are compressed, the other side they are stretched. Denote by η the distance from the neutral surface to the filament.



Write R for the radius of curvature of the neutral surface, so that the length of the segment at the neutral surface is $R\Delta\theta$. The length of the filament is $(R-\eta)\Delta\theta$, so the tension strain is $-(\eta\Delta\theta)/(R\Delta\theta) = -\eta/R$. So by Hooke's law, the tension stress on the filament is $-E\eta\Delta A/R$, where ΔA is the cross-sectional area of the filament.

Since the total horizontal force is supposed to be zero, we have

$$-\frac{E}{R}\int \eta\, dA = 0$$

so that $\int \eta\, dA = 0$. This says that the neutral surface passes through the *centroid* of the cross-sectional area. The total bending moment is obtained by multiplying by $-\eta$ and integrating:[5]

$$M = \frac{E}{R}\int \eta^2\, dA.$$

[5]The minus sign comes from the fact that counterclockwise moment is positive.

The quantity $I = \int \eta^2 \, dA$ is called the *sectional moment* of the cross-section of the rod. So we obtain $M = -EI/R$. Now the formula for radius of curvature is $R = \left(1 + (\frac{dy}{dx})^2\right)^{\frac{3}{2}} / \frac{d^2y}{dx^2}$. Assuming that $\frac{dy}{dx}$ is small, this can be approximated by the formula $1/R = \frac{d^2y}{dx^2}$, so that

$$M(x,t) = EI\frac{\partial^2 y}{\partial x^2}.$$

Combining this with equation (3.7.1) gives

$$\boxed{\frac{\partial^2 y}{\partial t^2} + \frac{EI}{\rho}\frac{\partial^4 y}{\partial x^4} = 0.} \tag{3.7.2}$$

This is the differential equation which governs the transverse waves on the rod. It is known as the Euler–Bernoulli beam equation.

We look for separable solutions to equation (3.7.2). Setting

$$y = f(x)g(t)$$

we obtain

$$f(x)g''(t) + \frac{EI}{\rho}f^{(4)}(x)g(t) = 0$$

or

$$\frac{g''(t)}{g(t)} = -\frac{EI}{\rho}\frac{f^{(4)}(x)}{f(x)}.$$

Since the left hand side does not depend on x and the right hand side does not depend on t, both sides are constant. So

$$g''(t) = -\omega^2 g(t) \tag{3.7.3}$$

$$f^{(4)}(x) = \frac{\omega^2\rho}{EI}f(x). \tag{3.7.4}$$

Equation (3.7.3) says that $g(t)$ is a multiple of $\sin(\omega t + \phi)$, while equation (3.7.4) has solutions

$$f(x) = A\sin\kappa x + B\cos\kappa x + C\sinh\kappa x + D\cosh\kappa x$$

where

$$\kappa = \sqrt[4]{\frac{\omega^2\rho}{EI}} \tag{3.7.5}$$

(see Appendix C for the hyperbolic functions sinh and cosh). The general solution then decomposes as a sum of the normal modes

$$y = (A\sin\kappa x + B\cos\kappa x + C\sinh\kappa x + D\cosh\kappa x)\sin(\omega t + \phi).$$

The boundary conditions depend on what happens at the end of the rod. It is these boundary conditions which constrain ω to a discrete set of values. If an end of the rod is free, then the quantities $V(x,t)$ and $M(x,t)$ have to vanish for all t, at the value of x corresponding to the end of the rod. So $\partial^2 y/\partial x^2 = 0$ and $\partial^3 y/\partial x^3 = 0$. If an end of the rod is clamped, then the displacement and slope vanish, so $y = 0$ and $\partial y/\partial x = 0$ for all t at the value of x corresponding to the end of the rod.

We calculate

$$\begin{aligned}
\partial y/\partial x &= \kappa(A\cos\kappa x - B\sin\kappa x + C\cosh\kappa x + D\sinh\kappa x)\\
\partial^2 y/\partial x^2 &= \kappa^2(-A\sin\kappa x - B\cos\kappa x + C\sinh\kappa x + D\cosh\kappa x)\\
\partial^3 y/\partial x^3 &= \kappa^3(-A\cos\kappa x + B\sin\kappa x + C\cosh\kappa x + D\sinh\kappa x).
\end{aligned}$$

In the case of the xylophone or tubular bell, both ends are free. We take the two ends to be at $x = 0$ and $x = \ell$. The conditions $\partial^2 y/\partial x^2 = 0$ and $\partial^3 y/\partial x^3 = 0$ at $x = 0$ give $B = D$ and $A = C$. These conditions at $x = \ell$ give

$$\begin{aligned}
A(\sinh\kappa\ell - \sin\kappa\ell) + B(\cosh\kappa\ell - \cos\kappa\ell) &= 0\\
A(\cosh\kappa\ell - \cos\kappa\ell) + B(\sinh\kappa\ell + \sin\kappa\ell) &= 0.
\end{aligned}$$

These equations admit a nonzero solution in A and B exactly when the determinant

$$(\sinh\kappa\ell - \sin\kappa\ell)(\sinh\kappa\ell + \sin\kappa\ell) - (\cosh\kappa\ell - \cos\kappa\ell)^2$$

vanishes. Using the relations $\cosh^2\kappa\ell - \sinh^2\kappa\ell = 1$ and $\sin^2\kappa\ell + \cos^2\kappa\ell = 1$, this condition becomes

$$\cosh\kappa\ell\cos\kappa\ell = 1.$$

The values of $\kappa\ell$ for which this equation holds determine the allowed frequencies via the formula (3.7.5).

Set $\lambda = \kappa\ell$, so that λ has to be a solution of the equation

$$\cosh\lambda\cos\lambda = 1. \tag{3.7.6}$$

Then equation (3.7.5) shows that the angular frequency and the frequency are given by

$$\omega = \sqrt{\frac{EI}{\rho}}\,\frac{\lambda^2}{\ell^2}; \qquad \nu = \frac{\omega}{2\pi} = \sqrt{\frac{EI}{\rho}}\,\frac{\lambda^2}{2\pi\ell^2}. \tag{3.7.7}$$

Numerical computations for the positive solutions to equation (3.7.6) (using Newton's method) give the following values, with more accuracy than is strictly necessary.

$$\begin{aligned}
\lambda_1 &= \ \ 4.7300407448627040260240481\\
\lambda_2 &= \ \ 7.8532046240958375564770667\\
\lambda_3 &= 10.9956078380016709066690325\\
\lambda_4 &= 14.1371654912574641771059179
\end{aligned}$$

As n increases, $\cosh\lambda_n$ increases exponentially, and so $\cos\lambda_n$ has to be very small and positive. So λ_n is close to $(n+\frac{1}{2})\pi$, the nth zero of the cosine function. For $n \geq 5$, the approximation

$$\lambda_n \approx (n+\tfrac{1}{2})\pi + \frac{(-1)^{n+1}}{\cosh(n+\frac{1}{2})\pi} \tag{3.7.8}$$

holds to at least ten decimal places.

Using equation (3.7.7), we find that the frequency ratios as multiples of the fundamental are given by the quantities λ_n^2/λ_1^2:

n	λ_n^2/λ_1^2
1	1.00000
2	2.75654
3	5.40392
4	8.93295

The resulting set of frequencies is certainly inharmonic, just as in the case of the drum. But as n increases, equation (3.7.8) shows that the higher partials have ratios approximating those of the squares of odd integers.

Further reading:

Elmore and Heald, *Physics of waves* [**28**], Chapter 3.

Rossing, *Science of percussion instruments* [**98**], Chapters 5–7.

3.8. The gong

As a first approximation, the gong can be thought of as a circular flat stiff metal plate. In practise, the gong is slightly curved, but for the moment we shall ignore this. The stiff metal plate behaves like a mixture of the drum and the stiff rod. So the partial differential equation governing its motion is fourth order, as in the case of the stiff rod, but there are two directions in which to take partial derivatives, as in the case of the drum. If z represents displacement, and x and y represent Cartesian coordinates on the gong, then the equation is

$$\frac{\partial^2 z}{\partial t^2} + \frac{Eh^2}{12\rho(1-s^2)}\nabla^4 z = 0. \tag{3.8.1}$$

This equation first appears (without the explicit value of the constant in front of the second term) in a paper of Sophie Germain.[6] In this equation, h is the thickness of the plate, and an easy calculation shows that $\frac{h^2}{12} = \frac{1}{h}\int_{-h/2}^{+h/2} z^2\,dz$ is the corresponding sectional moment in the one thickness direction (in the case of the stiff rod, there were two dimensions for the cross-section, so the case of the stiff plate is easier in this regard). The quantity E is the Young's modulus as before, ρ is area density, and s is *Poisson's ratio*. This is a measure of the ratio of sideways spreading to the compression. The extra factor of $(1-s^2)$ in the denominator on the right hand of the above equation does

[6]Sophie Germain's paper, "Recherches sur la théorie des surfaces élastiques," written in 1815 and published in 1821, won her a prize of a kilogram of gold from the French Academy of Sciences in 1816. The paper contained some significant errors, but became the basis for work on the subject by Lagrange, Poisson, Kirchoff, Navier and others.

Sophie Germain is probably better known for having made one of the first significant breakthroughs in the study of Fermat's last theorem. She proved that if x, y and z are integers satisfying $x^5+y^5=z^5$, then at least one of x, y and z has to be divisible by 5. More generally, she showed that the same was true when 5 is replaced by any prime p such that $2p+1$ is also a prime.

not correspond to any term in equation (3.7.2). It arises from the fact that when the plate is bent downwards in one direction, it causes it to curl up in the perpendicular direction along the plate.

The term $\nabla^4 z$ denotes

$$\nabla^2\nabla^2 z = \frac{\partial^4 z}{\partial x^4} + 2\frac{\partial^4 z}{\partial x^2 \partial y^2} + \frac{\partial^4 z}{\partial y^4}.$$

Observe the cross terms carefully. Without them, a rotational change of coordinates would not preserve this operation.

In the case of the stiff rod, we had to use the hyperbolic functions as well as the trigonometric functions. In this case, we are going to need to use the *hyperbolic Bessel functions.* These are defined by

$$I_n(z) = i^{-n} J_n(iz).$$

Looking for separable solutions $z = Z(x,y)h(t) = f(r)g(\theta)h(t)$ to equation (3.8.1), we arrive at the equations

$$\nabla^4 Z = \kappa^4 Z \tag{3.8.2}$$

and

$$\frac{\partial^2 h}{\partial t^2} = -\omega^2 h \tag{3.8.3}$$

where ω and κ are related by

$$\kappa^4 = \frac{12\rho(1-s^2)\omega^2}{Eh^2}.$$

We factor equation (3.8.2) as

$$(\nabla^2 - \kappa^2)(\nabla^2 + \kappa^2)z = 0. \tag{3.8.4}$$

So any solution to either the equation

$$\nabla^2 z = \kappa^2 z \tag{3.8.5}$$

or to the equation

$$\nabla^2 z = -\kappa^2 z \tag{3.8.6}$$

is also a solution to (3.8.2).

LEMMA 3.8.1. *Every solution z to equation* (3.8.2) *can be written uniquely as $z_1 + z_2$ where z_1 satisfies equation* (3.8.5) *and z_2 satisfies equation* (3.8.6).

PROOF. We use a variation of the even and odd function method. If $\nabla^4 z = \kappa^4 z$, we set

$$z_1 = \tfrac{1}{2}(z + \kappa^{-2}\nabla^2 z), \qquad z_2 = \tfrac{1}{2}(z - \kappa^{-2}\nabla^2 z).$$

Then

$$\nabla^2 z_1 = \tfrac{1}{2}(\nabla^2 z + \kappa^{-2}\nabla^4 z) = \tfrac{1}{2}(\nabla^2 z + \kappa^2 z) = \kappa^2 z_1,$$
$$\nabla^2 z_2 = \tfrac{1}{2}(\nabla^2 z - \kappa^{-2}\nabla^4 z) = \tfrac{1}{2}(\nabla^2 z - \kappa^2 z) = -\kappa^2 z_2.$$

and $z_1 + z_2 = z$.

For the uniqueness, if z_1' and z_2' constitute another choice, then rearranging the equation $z_1 + z_2 = z_1' + z_2'$, we have $z_1 - z_1' = z_2' - z_2$. The common value z_3 of $z_1 - z_1'$ and $z_2' - z_2$ satisfies both equations (3.8.5) and (3.8.6). So $z_3 = \kappa^{-2}\nabla^2 z_3 = -z_3$, and hence $z_3 = 0$. It follows that $z_1 = z_1'$ and $z_2 = z_2'$. □

Solving equation 3.8.5 is just the same as in the case of the drum, and the solutions are given as trigonometric functions of θ multiplied by Bessel functions of r. Equation 3.8.6 is similar, except that we must use the hyperbolic Bessel functions instead of the Bessel functions. We then have to combine the two classes of solutions in order to satisfy the boundary conditions, just as we did with the trigonometric and hyperbolic functions for the stiff rod. This leads us to solutions of the form

$$z = (AJ_n(\kappa r) + BI_n(\kappa r))\sin(\omega t + \phi)\sin(n\theta + \psi).$$

The boundary conditions for the gong require considerable care, and the first correct analysis was given by Kirchoff in 1850. His boundary conditions can be stated for any region with smooth boundary. Choosing coordinates in such a way that the element of boundary is a small segment of the y axis going through the origin, they are as follows.

$$\frac{\partial^2 z}{\partial x^2} + s\frac{\partial^2 z}{\partial y^2} = 0$$

$$\frac{\partial^3 z}{\partial x^3} + (2-s)\frac{\partial^3 w}{\partial x \partial y^2} = 0.$$

Further reading:

Fletcher and Rossing, *The physics of musical instruments* [**30**], §§3.5–3.6.

Graff, *Wave motion in elastic solids* [**37**].

Morse and Ingard, *Theoretical acoustics* [**74**], §5.3.

Rossing, *Science of percussion instruments* [**98**], Chapters 8 and 9.

T. D. Rossing and N. H. Fletcher, *Nonlinear vibrations in plates and gongs*, J. Acoust. Soc. Am. 73 (1983), 345–351.

M. D. Waller, *Vibrations of free circular plates. Part I: Normal modes*, Proc. Phys. Soc. 50 (1938), 70–76.