CHAPTER 2

Fourier theory

To be sung to the tune of Gilbert and Sullivan's *Modern Major General*:

I am the very model of a genius mathematical,
  For I can do mechanics, both dynamical and statical,
Or integrate a function round a contour in the complex plane,
  Yes, even if it goes off to infinity and back again;
Oh, I know when a detailed proof's required and when a guess'll do
  I know about the functions of Laguerre and those of Bessel too,
I've finished every tripos question back to 1948;
  There ain't a function you can name that I can't differentiate!
  There ain't a function you can name that he can't differentiate [Tris]

I've read the text books and I can extremely quickly tell you where
  To look to find Green's Theorem or the Principal of d'Alembert
Or I can work out Bayes' rule when the loss is not Quadratical
  In short I am the model of a genius mathematical!
For he can work out Bayes' rule when the loss is not Quadratical
  In short he is the model of a genius mathematical!

Oh, I can tell in seconds if a graph is Hamiltonian,
  And I can tell you if a proof of $4CC$'s a phoney 'un
I read up all the journals and I'm ready with the latest news,
  And very good advice about the Part II lectures you should choose.
Oh, I can do numerical analysis without a pause,
  Or comment on the far-reaching significance of Newton's laws
I know when polynomials are soluble by radicals,
  And I can reel off simple groups, especially sporadicals.
For he can reel off simple groups, especially sporadicals [Tris]

Oh, I like relativity and know about fast moving clocks
  And I know what you have to do to get round Russel's paradox
In short, I think you'll find concerning all things problematical
  I am the very model of a genius mathematical!
In short we think you'll find concerning all things problematical
  He is the very model of a genius mathematical!

Oh, I know when a matrix will be diagonalisable
  And I can draw Greek letters so that they are recognizable
And I can find the inverse of a non-zero quaternion
  I've made a model of a rhombicosidodecahedron;
Oh, I can quote the theorem of the separating hyperplane
  I've read MacLane and Birkoff not to mention Birkoff and MacLane
My understanding of vorticity is not a hazy 'un
  And I know why you should (and why you shouldn't) be a Bayesian!
  For he knows why you should (and why you shouldn't) be a Bayesian! [Tris]

I'm not deterred by residues and really I am quite at ease
  When dealing with essential isolated singularities,
In fact as everyone agrees (and most are quite emphatical)
  I am the very model of a genius mathematical!
In fact as everyone agrees (and most are quite emphatical)
  He is the very model of a genius mathematical!

—from CUYHA songbook, Cambridge (privately distributed) 1976.

2.1. Introduction

How can a string vibrate with a number of different frequencies at the same time? This problem occupied the minds of many of the great mathematicians and musicians of the seventeenth and eighteenth century. Among the people whose work contributed to the solution of this problem are Marin Mersenne, Daniel Bernoulli, the Bach family, Jean-le-Rond d'Alembert, Leonhard Euler, and Jean Baptiste Joseph Fourier. In this chapter, we discuss Fourier's theory of harmonic analysis. This is the decomposition of a periodic wave into a (usually infinite) sum of sines and cosines. The frequencies involved are the integer multiples of the fundamental frequency of the periodic wave, and each has an amplitude which can be determined as an integral. A superb book on Fourier series and their continuous frequency spectrum counterpart, Fourier integrals, is Tom Körner [**54**]. The reader should be warned, however, that the level of sophistication of Körner's book is much greater than the level of these notes.

We also discuss d'Alembert's solution of the wave equation for strings, and the role of Bessel functions in the harmonic series for a drum.

2.2. Fourier coefficients

Engraving of Jean Baptiste Joseph Fourier
(1768–1850) by Boilly (1823)
Académie des Sciences, Paris

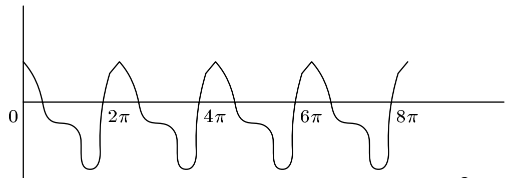
Fourier introduced the idea that periodic functions can be analyzed by using trigonometric series as follows.[1] The functions $\cos\theta$ and $\sin\theta$ are *periodic* with period 2π, in the sense that they satisfy

$$\cos(\theta + 2\pi) = \cos\theta$$
$$\sin(\theta + 2\pi) = \sin\theta.$$

In other words, translating by 2π along the θ axis leaves these functions unaffected. There are many other functions $f(\theta)$ which are periodic of period 2π, meaning that they satisfy the equation

$$f(\theta + 2\pi) = f(\theta).$$

We need only specify the function f on the half-open interval $[0, 2\pi)$ in any way we please, and then the above equation determines the value at all other values of θ.



A periodic function with period 2π

Other examples of such functions are the constant functions, and the functions $\cos(n\theta)$ and $\sin(n\theta)$ for any positive integer n. Negative values of n give us no more, since

$$\cos(-n\theta) = \cos(n\theta),$$
$$\sin(-n\theta) = -\sin(n\theta).$$

More generally, we can write down any series of the form

$$f(\theta) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos(n\theta) + b_n\sin(n\theta)). \tag{2.2.1}$$

Here, $\frac{1}{2}a_0$ is just a constant—the reason for the factor of $\frac{1}{2}$ will be explained in due course. Such a series is called a *trigonometric series*. Provided that there are no convergence problems, such a series will always define a function satisfying $f(\theta + 2\pi) = f(\theta)$.

[1]The basic ideas behind Fourier series were introduced in Jean Baptiste Joseph Fourier, *La théorie analytique de la chaleur*, F. Didot, Paris, 1822. Fourier was born in Auxerre, France in 1768 as the son of a taylor. He was orphaned in childhood and was educated by a school run by the Benedictine order. He was politically active during the French Revolution, and was almost executed. After the revolution, he studied in the then new Ecole Normale in Paris with teachers such as Lagrange, Monge and Laplace. In 1822, with the publication of the work mentioned above, he was elected *secretaire perpetuel* of the Académie des Sciences in Paris. Following this, his role seems principally to have been to encourage younger mathematicians such as Dirichlet, Liouville and Sturm, until his death in 1830.

The question which naturally arises at this stage is, to what extent can we find a trigonometric series whose sum is equal to a given periodic function? To begin to answer this question, we first ask: given a function defined by a trigonometric series, how can the coefficients a_n and b_n be recovered?

The answer lies in the formulae (for $m \geq 0$ and $n \geq 0$)

$$\int_0^{2\pi} \cos(m\theta)\sin(n\theta)\,dt = 0 \tag{2.2.2}$$

$$\int_0^{2\pi} \cos(m\theta)\cos(n\theta)\,dt = \begin{cases} 2\pi & \text{if } m = n = 0 \\ \pi & \text{if } m = n > 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.2.3}$$

$$\int_0^{2\pi} \sin(m\theta)\sin(n\theta)\,dt = \begin{cases} \pi & \text{if } m = n \\ 0 & \text{otherwise} \end{cases} \tag{2.2.4}$$

These equations can be proved by using equations (1.7.7)–(1.7.11) to rewrite the product of trigonometric functions inside the integral as a sum before integrating.[2] The extra factor of two in (2.2.3) for $m = n = 0$ will explain the factor of $\frac{1}{2}$ in front of a_0 in (2.2.1).

This suggests that in order to find the coefficent a_m, we multiply $f(\theta)$ by $\cos(m\theta)$ and integrate. Let us see what happens when we apply this process to equation (2.2.1). Provided we can pass the integral through the infinite sum, only one term gives a nonzero contribution. So for $m > 0$ we have

$$\int_0^{2\pi} \cos(m\theta)f(\theta)\,d\theta = \int_0^{2\pi} \cos(m\theta)\Big(\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos(n\theta) + b_n\sin(n\theta))\Big)\,d\theta$$

$$= \tfrac{1}{2}a_0\int_0^{2\pi}\cos(m\theta)\,d\theta + \sum_{n=1}^{\infty}\Big(a_n\int_0^{2\pi}\cos(m\theta)\cos(n\theta)\,d\theta + b_n\int_0^{2\pi}\cos(m\theta)\sin(n\theta)\,d\theta\Big)$$

$$= \pi a_m.$$

Thus we obtain, for $m > 0$,

$$\boxed{a_m = \frac{1}{\pi}\int_0^{2\pi}\cos(m\theta)f(\theta)\,d\theta.} \tag{2.2.5}$$

A standard theorem of analysis says that provided the sum converges absolutely (in other words, if the sum of the absolute values converges) then the integral can be passed through the infinite sum in this way. Under the same conditions, we obtain for $m > 0$

$$\boxed{b_m = \frac{1}{\pi}\int_0^{2\pi}\sin(m\theta)f(\theta)\,d\theta.} \tag{2.2.6}$$

[2]The relations (2.2.2)–(2.2.4) are sometimes called *orthogonality relations.* The idea is that the integrable periodic functions form an infinite dimensional vector space with an inner product given by $\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)g(\theta)\,d\theta$. With respect to this inner product, the functions $\sin(m\theta)$ ($m > 0$) and $\cos(m\theta)$ ($m \geq 0$) are *orthogonal*, or perpendicular.

The functions a_m and b_m defined by equations (2.2.5) and (2.2.6) are called the *Fourier coefficients* of the function $f(\theta)$.

We can now explain the appearance of the coefficient of one half in front of the a_0 in equation (2.2.1). Namely, since π is one half of 2π and $\cos(0\theta) = 1$ we have

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} \cos(0\theta) f(\theta)\, d\theta \tag{2.2.7}$$

which means that the formula (2.2.5) for the coefficient a_m holds for all $m \geq 0$.

It would be nice to think that when we use equations (2.2.5), (2.2.6) and (2.2.7) to define a_m and b_m, the right hand side of equation (2.2.1) always converges to $f(\theta)$. This is true for nice enough functions f, but unfortunately, not for all functions f. In Section 2.4, we investigate conditions on f which ensure that this happens.

Of course, any interval of length 2π, representing one complete period, may be used instead of integrating from 0 to 2π. It is sometimes more convenient, for example, to integrate from $-\pi$ to π:

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(m\theta) f(\theta)\, d\theta$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(m\theta) f(\theta)\, d\theta.$$

In practise, the variable θ will not quite correspond to time, because the period is not necessarily 2π seconds. If the fundamental frequency (the reciprocal of the period) is ν then the correct substitution is $\theta = 2\pi\nu t$. Setting $F(t) = f(2\pi\nu t) = f(\theta)$ and substituting gives a Fourier series of the form

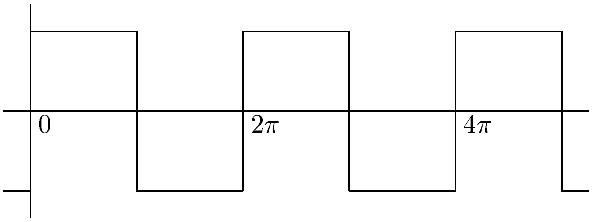
$$F(t) = \tfrac{1}{2}a_0 + \sum_{n=0}^{\infty}(a_n \cos(2\pi n\nu t) + b_n \sin(2\pi n\nu t)),$$

and the following formula for Fourier coefficients.

$$a_m = 2\nu\int_0^{1/\nu} \cos(2\pi m\nu t) F(t)\, dt,$$

$$b_m = 2\nu\int_0^{1/\nu} \sin(2\pi m\nu t) F(t)\, dt.$$

Example. The square wave sounds vaguely like the waveform produced by a clarinet, where odd harmonics dominate. It is the function $f(\theta)$ defined by $f(\theta) = 1$ for $0 \leq \theta < \pi$ and $f(\theta) = -1$ for $\pi \leq \theta < 2\pi$ (and then extend to all values of θ by making it periodic, $f(\theta + 2\pi) = f(\theta)$).



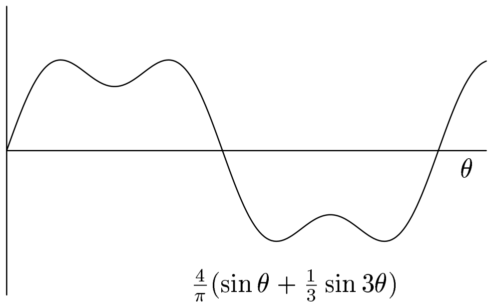
This function has Fourier coefficients

$$\begin{aligned}
a_m &= \frac{1}{\pi}\left(\int_0^{\pi}\cos(m\theta)\,d\theta - \int_{\pi}^{2\pi}\cos(m\theta)\,d\theta\right)\\
&= \frac{1}{\pi}\left(\left[\frac{\sin(m\theta)}{m}\right]_0^{\pi} - \left[\frac{\sin(m\theta)}{m}\right]_{\pi}^{2\pi}\right) = 0\\
b_m &= \frac{1}{\pi}\left(\int_0^{\pi}\sin(m\theta)\,d\theta - \int_{\pi}^{2\pi}\sin(m\theta)\,d\theta\right)\\
&= \frac{1}{\pi}\left(\left[-\frac{\cos(m\theta)}{m}\right]_0^{\pi} - \left[-\frac{\cos(m\theta)}{m}\right]_{\pi}^{2\pi}\right)\\
&= \frac{1}{\pi}\left(-\frac{(-1)^m}{m} + \frac{1}{m} + \frac{1}{m} - \frac{(-1)^m}{m}\right)\\
&= \begin{cases} 4/m\pi & (m \text{ odd})\\ 0 & (m \text{ even})\end{cases}
\end{aligned}$$

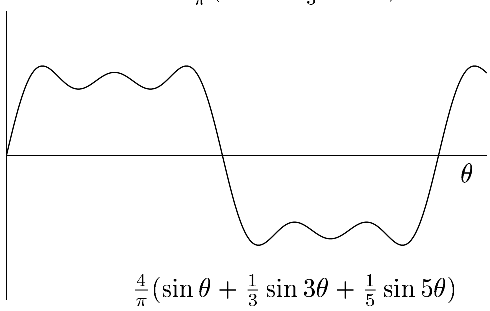
Thus the Fourier series for this square wave is

$$\tfrac{4}{\pi}(\sin\theta + \tfrac{1}{3}\sin 3\theta + \tfrac{1}{5}\sin 5\theta + \dots). \tag{2.2.8}$$

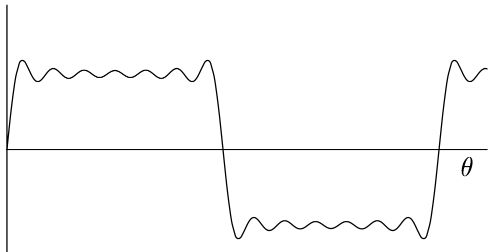
Let us examine the first few terms in this series:



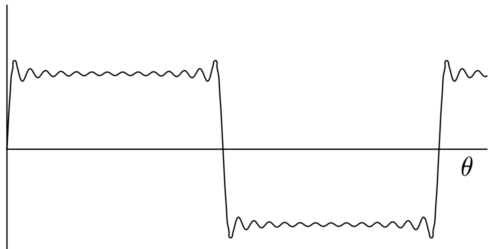
$\frac{4}{\pi}(\sin\theta + \frac{1}{3}\sin 3\theta)$



$\frac{4}{\pi}(\sin\theta + \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta)$



$\frac{4}{\pi}(\sin\theta + \frac{1}{3}\sin 3\theta + \cdots + \frac{1}{13}\sin 13\theta)$



$\frac{4}{\pi}(\sin\theta + \frac{1}{3}\sin 3\theta + \cdots + \frac{1}{27}\sin 27\theta)$

Some features of this example are worth noticing. The first observation is that these graphs seem to be converging to a square wave. But they seem to be converging quite slowly, and getting more and more bumpy in the process. Next, observe what happens at the point of discontinuity of the original function. The Fourier coefficients did not depend on what value we assigned to the function at the discontinuity, so we do not expect to recover that information. Instead, the series is converging to a value which is equal to the average of the higher and the lower values of the function. This is a general phenomenon, which we shall discuss later.

Finally, there is a very interesting phenomenon which is happening right near the discontinuity. There is an overshoot, which never seems to get any smaller.

Does this mean that the series is not converging properly? Well, not quite. At each given value of θ, the series is converging just fine. It's just when we look at values of θ closer and closer to the discontinuity that we find problems. This is because of a lack of *uniform* convergence. This overshoot is called the Gibbs phenomenon, and we shall discuss it in more detail in §2.5.

Exercises

1. Prove equations (2.2.2)–(2.2.4) by rewriting the products of trigonometric functions inside the integral as sums before integrating.

2. Are the following functions of θ periodic? If so, determine the smallest period, and which multiples of the fundamental frequency are present. If not, explain why not.

(i) $\sin\theta + \sin\frac{5}{4}\theta$.

(ii) $\sin\theta + \sin\sqrt{2}\,\theta$.

(iii) $\sin^2\theta$.

(iv) $\sin(\theta^2)$.

(v) $\sin\theta + \sin(\theta + \frac{\pi}{3})$.

3. Draw graphs of the functions $\sin(220\pi t) + \sin(440\pi t)$ and $\sin(220\pi t) + \cos(440\pi t)$. Explain why these sound the same, even though the graphs look quite different.

2.3. Even and odd functions

A function $f(\theta)$ is said to be *even* if $f(-\theta) = f(\theta)$, and it is said to be *odd* if $f(-\theta) = -f(\theta)$. For example, $\cos\theta$ is even, while $\sin\theta$ is odd.

Given any function $f(\theta)$, we can obtain an even function by taking the average of $f(\theta)$ and $f(-\theta)$, i.e., $\frac{1}{2}(f(\theta) + f(-\theta))$. Similarly, $\frac{1}{2}(f(\theta) - f(-\theta))$ is an odd function. These add up to give the original function $f(\theta)$, so we have written $f(\theta)$ as a sum of its *even part* and its *odd part*,

$$f(\theta) = \tfrac{1}{2}(f(\theta) + f(-\theta)) + \tfrac{1}{2}(f(\theta) - f(-\theta)).$$

Multiplication of even and odd functions works as you might expect: even times even or odd times odd gives even, and even times odd or odd times even gives odd.

Now for any odd function $f(\theta)$, and for any $a > 0$, we have

$$\int_{-a}^{0} f(\theta)\, d\theta = -\int_{0}^{a} f(\theta)\, d\theta$$

so that

$$\int_{-a}^{a} f(\theta) = 0.$$

So for example, if $f(\theta)$ is even and periodic with period 2π, then $\sin(m\theta)f(\theta)$ is odd, and so the Fourier coefficients b_m are zero, since

$$b_m = \frac{1}{\pi}\int_{0}^{2\pi} \sin(m\theta)f(\theta)\, d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(m\theta)f(\theta)\, d\theta = 0.$$

Similarly, if $f(\theta)$ is odd and periodic with period 2π, then $\cos(m\theta)f(\theta)$ is odd, and so the Fourier coefficients a_m are zero, since

$$a_m = \frac{1}{\pi}\int_{0}^{2\pi} \cos(m\theta)f(\theta)\, d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(m\theta)f(\theta)\, d\theta = 0.$$

This explains, for example, why $a_m = 0$ in the example on page 33. The square wave is not quite an even function, because $f(\pi) \neq f(-\pi)$, but changing the value of a function at a finite set of points in the interval of integration never affects the value of an integral, so we just replace $f(\pi)$ and $f(-\pi)$ by zero.

There is a similar explanation for why $b_{2m} = 0$ in the same example, using a different symmetry. The discussion of even and odd functions depended on the symmetry $\theta \mapsto -\theta$ of order two. For periodic functions of period 2π, there is another symmetry of order two, namely $\theta \mapsto \theta + \pi$. The functions

$f(\theta)$ satisfying $f(\theta+\pi) = f(\theta)$ are *half-period symmetric*, while functions satisfying $f(\theta + \pi) = -f(\theta)$ are *half-period antisymmetric*. Any function $f(\theta)$ can be decomposed into half-period symmetric and antisymmetric parts:

$$f(\theta) = \tfrac{1}{2}(f(\theta) + f(\theta + \pi)) + \tfrac{1}{2}(f(\theta) - f(\theta + \pi)).$$

Multiplying half-period symmetric and antisymmetric functions works in the same way as for even and odd functions.

If $f(\theta)$ is half-period antisymmetric, then

$$\int_{\pi}^{2\pi} f(\theta)\, d\theta = -\int_0^{\pi} f(\theta)\, d\theta$$

and so

$$\int_0^{2\pi} f(\theta)\, d\theta = 0.$$

Now the functions $\sin(m\theta)$ and $\cos(m\theta)$ are both half-period symmetric if m is even, and half-period antisymmetric if m is odd. So we deduce that if $f(\theta)$ is half-period symmetric, $f(\theta + \pi) = f(\theta)$, then the Fourier coefficients with odd indices (a_{2m+1} and b_{2m+1}) are zero, while if $f(\theta)$ is antisymmetric, $f(\theta + \pi) = -f(\theta)$, then the Fourier coefficients with even indices a_{2m} and b_{2m} are zero (check that this holds for a_0 too!). This corresponds to the fact that half-period symmetry is really the same thing as being periodic with half the period, so that the frequency components have to be even multiples of the defining frequency; while half-period antisymmetric functions only have frequency components at odd multiples of the defining frequency.

In the example on page 33, the function is half-period antisymmetric, and so the coefficients a_{2m} and b_{2m} are zero.

Exercises

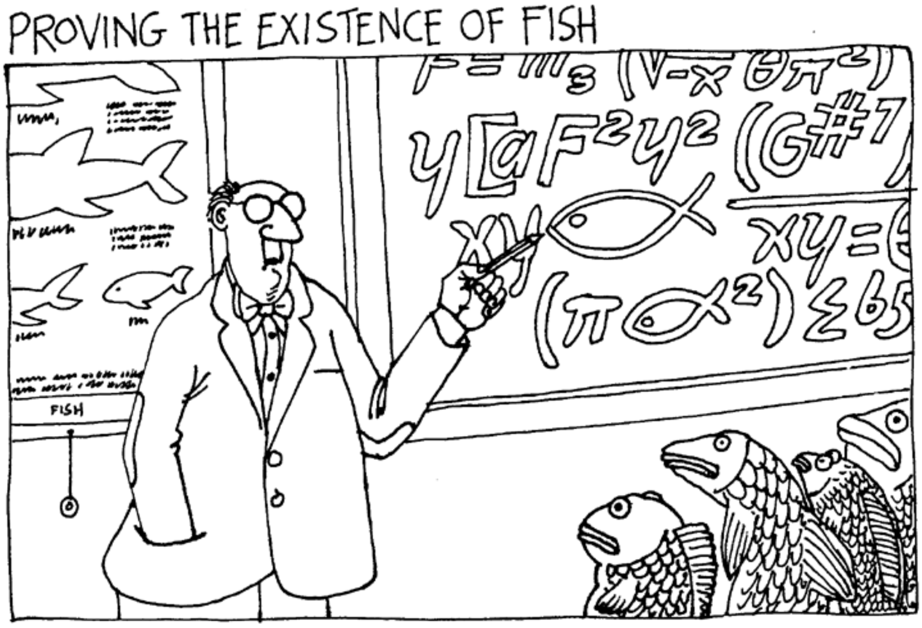
1. Evaluate $\displaystyle\int_0^{2\pi} \sin(\sin\theta)\sin(2\theta)\, d\theta$.

2. Using the theory of even and odd functions, and the theory of half-period symmetric and antisymmetric functions, which Fourier coefficients of $\tan\theta$ have to be zero? Find the first nonzero coefficient.

3. Which Fourier coefficients vanish for a periodic function $f(\theta)$ of period 2π satisfying $f(\theta) = f(\pi - \theta)$? What about $f(\theta) = -f(\pi - \theta)$?

[Hint: Consider the symmetry $\theta \mapsto \pi - \theta$, and compare $\displaystyle\int_{-\pi/2}^{\pi/2} f(\theta)\, d\theta$ with $\displaystyle\int_{\pi/2}^{3\pi/2} f(\theta)\, d\theta$ for antisymmetric functions with respect to this symmetry.]

2.4. Conditions for convergence



B. Kliban

Unfortunately, it is not true that if we start with a periodic function $f(\theta)$, form the Fourier coefficients a_m and b_m according to equations (2.2.5) and (2.2.6) and then form the sum (2.2.1), then we recover the original function $f(\theta)$. The most obvious problem is that if two functions differ just at a single value of θ then the Fourier coefficients will be identical. So we cannot possibly recover the function from its Fourier coefficients without some further conditions. However, if the function is nice enough, it can be recovered in the manner indicated. The following is a consequence of the work of Dirichlet.

THEOREM 2.4.1. *Suppose that $f(\theta)$ is periodic with period 2π, and that it is continuous and has a bounded continuous derivative except at a finite number of points in the interval $[0, 2\pi]$. If a_m and b_m are defined by equations* (2.2.5) *and* (2.2.6) *then the series defined by equation* (2.2.1) *converges to $f(\theta)$ at all points where $f(\theta)$ is continuous.*

PROOF. See Körner [**54**], Theorem 1 and Chapters 15 and 16. □

An important special case of the above theorem is the following. A C^1 function is defined to be a function which is differentiable with continuous derivative. If $f(\theta)$ is a periodic C^1 function with period 2π, then $f'(\theta)$ is continuous on the closed interval $[0, 2\pi]$, and hence bounded there. So $f(\theta)$ satisfies the conditions of the above theorem.

It is important to note that continuity, or even differentiability of $f(\theta)$ is not sufficient for the Fourier series for $f(\theta)$ to converge to $f(\theta)$. Paul DuBois Reymond constructed an example of a continuous function for which the coefficients a_m and b_m are not bounded. The construction is by no means easy and we shall not give it here. The reader may form the impression at this stage that the only purpose for the existence of such functions is to beset theorems such as the above with conditions, and that in real life, all functions are just as differentiable as we would like them to be. This point of view is refuted by the observation that many phenomena in real life are governed by some form of Brownian motion. Functions describing these phenomena will tend to be everywhere continuous but nowhere differentiable.[3] In music, noise is an example of the same phenomenon. Many of the functions employed in musical synthesis are not even continuous. Sawtooth functions and square waves are typical examples.

However, the question of convergence of the Fourier series is not the same as the question of whether the function $f(\theta)$ can be reconstructed from its Fourier coefficients a_n and b_n. At the age of 19, Fejér proved the remarkable theorem that any continuous function $f(\theta)$ can be reconstructed from its Fourier coefficients. His idea was that if the partial sums s_m defined by

$$s_m = \tfrac{1}{2}a_0 + \sum_{n=1}^{m}(a_n \cos(n\theta) + b_n \sin(n\theta)) \tag{2.4.1}$$

converge, then their averages

$$\sigma_m = \frac{s_0 + \cdots + s_m}{m+1}$$

converge to the same limit. But it is conceivable that the σ_m could converge without the s_m converging. This idea for smoothing out the convergence had already been around for some time when Fejér approached the problem. It had been used by Euler and extensively studied by Cesàro, and goes by the name of Cesàro summability.

THEOREM 2.4.2 (Fejér). *If $f(\theta)$ is a Riemann integrable periodic function then the Cesàro sums σ_m converge to $f(\theta)$ as m tends to infinity at every value of θ where $f(\theta)$ is continuous.*[4]

PROOF. We shall prove this theorem in Section 2.7. See also Körner [**54**], Chapter 2. □

---

[3]The first examples of functions which are everywhere continuous but nowhere differentiable were constructed by Weierstrass, *Abhandlungen aus der Functionenlehre*, Springer (1886), p. 97. He showed that if $0 < b < 1$, a is an odd integer, and $ab > 1 + \frac{3\pi}{2}$ then $f(t) = \sum_{n=1}^{\infty} b^n \cos a^n(2\pi\nu)t$ is a uniformly convergent sum, and that $f(t)$ is everywhere continuous but nowhere differentiable. G. H. Hardy, *Weierstrass's non-differentiable function*, Trans. Amer. Math. Soc. 17 (1916), 301–325, showed that the same holds if the bound on ab is replaced by $ab > 1$. Manfred Schroeder, *Fractals, chaos and power laws*, W. H. Freeman and Co., 1991, p. 96, points out that functions of this form can be thought of as *fractal* waveforms. For example, if we set $a = 2^{13/12}$, then doubling the speed of this function will result in a tone which sounds similar to the original, but lowered by a semitone and softer by a factor of b. This sort of self-similarity is characteristic of fractals. It is ironic that Weierstrass, in contrast with the vast majority of mathematicians, held a dislike for music.

[4]Continuous functions are Riemann integrable, so Fejér's theorem applies to all continuous periodic functions.

We shall interpret this theorem as saying that every continuous function may be reconstructed from its Fourier coefficients. But the reader should bear in mind that if the function does not satisfy the hypotheses of Theorem 2.4.1 then the reconstruction of the function is done via Cesàro sums, and not simply as the sum of the Fourier series.

There are other senses in which we could ask for a Fourier series to converge. One of the most important ones is *mean square convergence.*

THEOREM 2.4.3. *Let $f(\theta)$ be a continuous periodic function with period 2π. Then among all the functions $g(\theta)$ which are linear combinations of $\cos(n\theta)$ and $\sin(n\theta)$ with $0 \le n \le m$, the partial sum s_m defined in equation* (2.4.1) *minimizes the mean square error of $g(\theta)$ as an approximation to $f(\theta)$,*

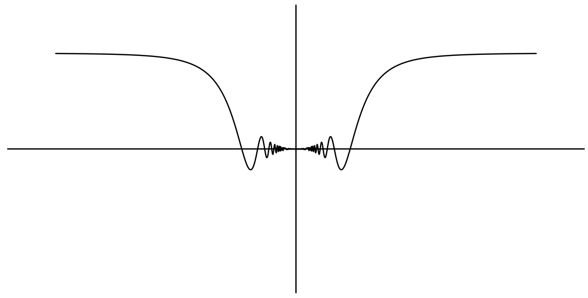
$$\frac{1}{2\pi}\int_0^{2\pi} |f(\theta) - g(\theta)|^2 \, d\theta.$$

Furthermore, in the limit as m tends to infinity, the mean square error of s_m as an approximation to $f(\theta)$ tends to zero.

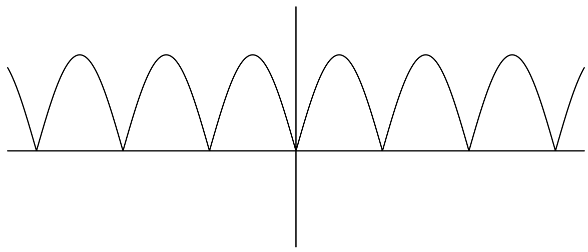
PROOF. See Körner [**54**], Chapters 32–34. □

Exercises

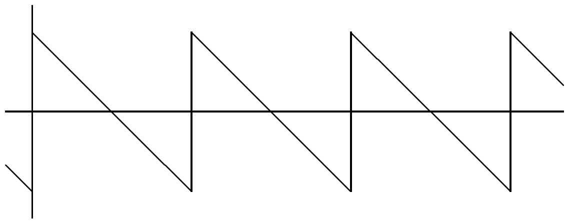
1. Show that the function $f(x) = x^2 \sin(1/x^2)$ is differentiable for all values of x, but its derivative is unbounded around $x = 0$.



2. Find the Fourier series for the periodic function $f(\theta) = |\sin\theta|$ (the absolute value of $\sin\theta$). In other words, find the coefficients a_m and b_m using equations (2.2.5) and (2.2.6). You will need to divide the interval from 0 to 2π into two subintervals in order to evaluate the integral.



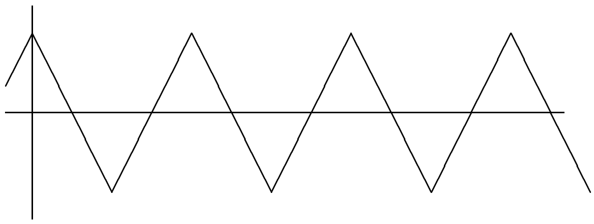
3. Let $\phi(\theta)$ be the periodic sawtooth function with period 2π defined by $\phi(\theta) = (\pi - \theta)/2$ for $0 < \theta < 2\pi$ and $\phi(0) = \phi(2\pi) = 0$. Find the Fourier series for $\phi(\theta)$.[5]



4. Find the Fourier series of the continuous periodic triangular wave function defined by

$$f(\theta) = \begin{cases} \frac{\pi}{2} - \theta & 0 \le \theta \le \pi \\ \theta - \frac{3\pi}{2} & \pi \le \theta \le 2\pi \end{cases}$$

and $f(\theta + 2\pi) = f(\theta)$.



5. (a) Show that if $f(\theta)$ is a bounded (and Riemann integrable) periodic function with period 2π then the Fourier coefficients a_m and b_m defined by (2.2.5)–(2.2.7) are bounded.

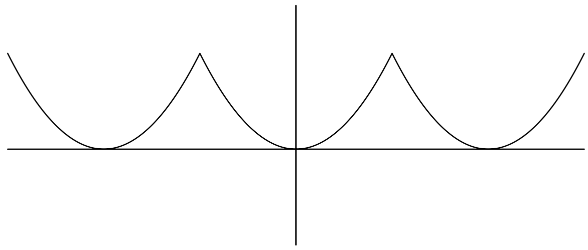
(b) If $f(\theta)$ is a differentiable periodic function with period 2π, find the relationship between the Fourier coefficients $a_m(f)$, $b_m(f)$ for $f(\theta)$ and the Fourier coefficients $a_m(f')$, $b_m(f')$ for the derivative $f'(\theta)$. [Hint: use integration by parts]

(c) Show that if $f(\theta)$ is a k times differentiable periodic function with period 2π, and the kth derivative $f^{(k)}(\theta)$ is bounded, then the Fourier coefficients a_m and b_m of $f(\theta)$ are bounded by a constant multiple of $1/m^k$.

Thus, smoothness of $f(\theta)$ is reflected in rapidity of decay of the Fourier coefficients.

6. Find the Fourier series for the function $f(\theta)$ defined by $f(\theta) = \theta^2$ for $-\pi \le \theta \le \pi$ and and then extended to all values of θ by periodicity, $f(\theta + 2\pi) = f(\theta)$. Evaluate your answer at $\theta = 0$ and at $\theta = \pi$, and use your answer to find $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

[5]The sawtooth waveform is approximately what is produced by a violin or other bowed instrument. This is because the bow pulls the string, and then suddenly releases it when the coefficient of static friction is exceeded. The coefficient of dynamic friction is smaller, so once the string is released by the bow, it will tend to continue moving rapidly until the other extreme of its trajectory is reached.



2.5. The Gibbs phenomenon

A function defined on a closed interval is said to be *piecewise continuous* if it is continuous except at a finite set of points, and at those points the left limit and right limit exist although they may not be equal. When we talk of the *size* of a discontinuity of a piecewise continuous function $f(\theta)$ at $\theta = a$, we mean the difference $f(a^+) - f(a^-)$, where

$$f(a^+) = \lim_{\theta \to a^+} f(\theta), \qquad f(a^-) = \lim_{\theta \to a^-} f(\theta)$$

denote the left limit and the right limit at that point. A periodic function is said to be piecewise continuous if it is so on a closed interval forming a period of the function.

Many of the functions encountered in the theory of synthesized sound are piecewise continuous but not continuous. These include waveforms such as the square wave and the sawtooth function.

Denote by $\phi(\theta)$ the piecewise continuous periodic function defined by $\phi(\theta) = (\pi - \theta)/2$ for $0 < \theta < 2\pi$, $\phi(0) = 0$, and $\phi(\theta + 2\pi) = \phi(\theta)$. Then given any piecewise continuous periodic function $f(\theta)$, we may add some finite set of functions of the form $C\phi(\theta + \alpha)$ (with C and α constants) to make the left limits and right limits at the discontinuities agree. We can then just change the values of the function at the discontinuities, which will not affect the Fourier series, to make the function continuous. It follows that in order to understand the Fourier series for piecewise continuous functions in general, it suffices to understand the Fourier series of continuous functions together with the Fourier series of the single function $\phi(\theta)$. The Fourier series of this function (see Exercise 3 of §2.4) is

$$\phi(\theta) = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n}.$$

At the discontinuity ($\theta = 0$), this series converges to zero because all the terms are zero. This is the average of the left limit and the right limit at this point. It follows that for any piecewise continuous periodic function, the Cesàro sums σ_m described in §2.4 converge everywhere, and at the points of discontinuity σ_m converges to the average of the left and right limit at the point:

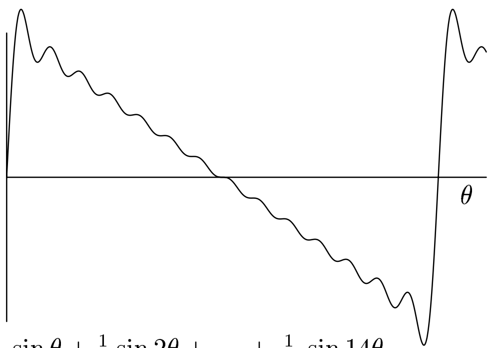
$$\lim_{m \to \infty} \sigma_m(a) = \tfrac{1}{2}(f(a^+) + f(a^-)).$$

A further examination of the function $\phi(\theta)$ shows that the convergence around the point of discontinuity is not as straightforward as one might suppose. Namely, setting

$$\phi_m(\theta) = \sum_{n=1}^{m} \frac{\sin n\theta}{n}, \tag{2.5.1}$$

although it is true that we have *pointwise convergence*, in the sense that for each point a we have $\lim_{m\to\infty} \phi_m(a) = \phi(a)$, this convergence is not uniform.

The definition in analysis of pointwise convergence is that given a value a of θ and given $\varepsilon > 0$, there exists N such that $m \geq N$ implies $|\phi_m(a) - \phi(a)| < \varepsilon$. Uniform convergence means that given $\varepsilon > 0$, there exists N (independent of a) such that for all values a of θ, $m \geq N$ implies $|\phi_m(a) - \phi(a)| < \varepsilon$. What happens with the Fourier series for the above function ϕ is that there is an overshoot, the size of which does not tend to zero as m gets larger. The peak of the overshoot gets closer and closer to the discontinuity though, so that for any particular value a of θ, convergence holds. But choosing ε smaller than the size of the overshoot shows that uniform convergence fails. This overshoot is called the Gibbs phenomenon.[6]



$\sin\theta + \frac{1}{2}\sin 2\theta + \cdots + \frac{1}{14}\sin 14\theta$

To demonstrate the reality of the overshoot, we shall compute its size in the limit. The first step is to differentiate $\phi_m(\theta)$ to find its local maxima and minima. We concentrate on the interval $0 \leq \theta \leq \pi$, since $\phi_m(2\pi - \theta) = -\phi_m(\theta)$. We have

$$\phi_m'(\theta) = \sum_{n=1}^{m} \cos n\theta = \frac{\sin \frac{1}{2}m\theta \, \cos \frac{1}{2}(m+1)\theta}{\sin \frac{1}{2}\theta}$$

(see Exercise 6 of §1.7). So the zeros of $\phi_m'(\theta)$ occur at $\theta = \frac{(2k+1)\pi}{m+1}$ and $\theta = \frac{2k\pi}{m}$, $0 \leq k \leq \lfloor \frac{m-1}{2} \rfloor$.[7]

[6]Josiah Willard Gibbs described this phenomenon in a series of letters to *Nature* in 1898 in correspondence with A. E. H. Love. He seems to have been unaware of the previous treatment of the subject by Henry Wilbraham in his article *On a certain periodic function*, Cambridge & Dublin Math. J. **3** (1848), 198–201.

[7]The notation $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

Now $\sin\frac{1}{2}\theta$ is positive throughout the interval $0 \le \theta \le 2\pi$. At $\theta = \frac{(2k+1)\pi}{m+1}$, $\sin\frac{1}{2}m\theta$ has sign $(-1)^k$ while $\cos\frac{1}{2}(m+1)\theta$ changes sign from $(-1)^k$ to $(-1)^{k+1}$, so that $\phi'_m(\theta)$ changes from positive to negative. It follows that $\theta = \frac{(2k+1)\pi}{m+1}$ is a local maximum of ϕ_m. Similarly, at $\theta = \frac{2k\pi}{m}$, $\cos\frac{1}{2}(m+1)\theta$ has sign $(-1)^k$ while $\sin\frac{1}{2}m\theta$ changes sign from $(-1)^{k-1}$ to $(-1)^k$, so that $\phi'_m(\theta)$ changes from negative to positive. It follows that $\theta = \frac{2k\pi}{m}$ is a local minimum of $\phi_m(\theta)$. These local maxima and minima alternate.

The first local maximum value of $\phi_m(\theta)$ for $0 \le \theta \le 2\pi$ happens at $\frac{\pi}{m+1}$. The value of $\phi_m(\theta)$ at this maximum is

$$\phi_m\left(\tfrac{\pi}{m+1}\right) = \sum_{n=1}^{m} \tfrac{1}{n}\sin\left(\tfrac{n\pi}{m+1}\right) = \frac{\pi}{m+1}\sum_{n=1}^{m}\frac{\sin\left(\frac{n\pi}{m+1}\right)}{\left(\frac{n\pi}{m+1}\right)}.$$

This is the Riemann sum for

$$\int_0^\pi \frac{\sin\theta}{\theta}\,d\theta$$

with $m+1$ equal intervals of size $\frac{\pi}{m+1}$ (note that $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 1$ so that we should define the integrand to be 1 when $\theta = 0$ to make a continuous function on the closed interval $0 \le \theta \le \pi$). Therefore the limit as m tends to infinity of the height of the first maximum point of the sum of the first m terms in the Fourier series for $\phi(\theta)$ is

$$\int_0^\pi \frac{\sin\theta}{\theta}\,d\theta \approx 1.8519370.$$

This overshoots the maximum value $\frac{\pi}{2} \approx 1.5707963$ of the function $\phi(\theta)$ by a factor of 1.1789797. Of course, the size of the discontinuity is not $\frac{\pi}{2}$ but π, so that as a proportion of the size of the discontinuity, the overshoot is about 8.9490%.[8] It follows that for any piecewise continuous function, the overshoot of the Fourier series just after a discontinuity is this proportion of the size of the discontinuity.

After the function overshoots, it then returns to undershoot, then overshoot again, and so on, each time with a smaller value than before. An argument similar to the above shows that the value at the kth critical point of $\phi_m(\theta)$ tends to $\int_0^{k\pi}\frac{\sin\theta}{\theta}\,d\theta$ as m tends to infinity. Thus for example the first undershoot ($k = 2$) has a value with a limit of about 1.4181516, which undershoots $\frac{\pi}{2}$ by a factor of 0.9028233. The undershoot is therefore about 4.8588% of the size of the discontinuity.

The Gibbs phenomenon can be interpreted in terms of the response of an amplifier as follows. No matter how good your amplifier is, if you feed it

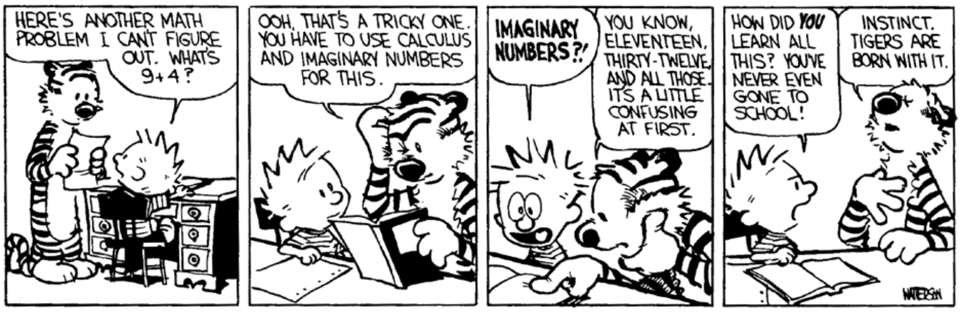
[8]This value was first computed by Maxime Bôcher, *Introduction to the theory of Fourier's series.* Ann. of Math. (2) **7** (1905–6), 81–152. A number of otherwise reputable sources overstate the size of the overshoot by a factor of two for some reason probably associated with uncritical copying.

a square wave, the output will overshoot at the discontinuity by roughly 9%. This is because any amplifier has a frequency beyond which it does not respond. Improving the amplifier can only increase this frequency, but cannot get rid of the limitation altogether.

Manufacturers of cathode ray tubes also have to contend with this problem. The beam is being made to run across the tube from left to right linearly and then switch back suddenly to the left. Much effort goes into preventing the inevitable overshoot from causing problems.

As mentioned above, the Gibbs phenomenon is a good example to illustrate the distinction between pointwise convergence and uniform convergence. For pointwise convergence of a sequence of functions $f_n(\theta)$ to a function $f(\theta)$, it is required that for each value of θ, the values $f_n(\theta)$ must converge to $f(\theta)$. For uniform convergence, it is required that the distance between $f_n(\theta)$ and $f(\theta)$ is bounded by a quantity which depends on n and not on θ, and which tends to zero as n tends to infinity. In the above example, the distance between the nth partial sum of the Fourier series and the original function can at best be bounded by a quantity which depends on n and not on θ, but which tends to roughly 0.28114. So this Fourier series converges pointwise, but not uniformly.

2.6. Complex coefficients



The theory of Fourier series is considerably simplified by the introduction of complex exponentials. See Appendix C for a quick summary of complex numbers and complex exponentials. The relationships (C.1)–(C.3)

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$e^{-i\theta} = \cos\theta - i\sin\theta \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

mean that equation (2.2.1) can be rewritten as[9]

$$f(\theta) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\theta} \tag{2.6.1}$$

[9]Note that we are dealing with complex valued functions of a real periodic variable, and not with functions of a complex variable here.

where $\alpha_0 = \frac{1}{2}a_0$, and for $m > 0$, $\alpha_m = \frac{1}{2}a_m + \frac{1}{2i}b_m$ and $\alpha_{-m} = \frac{1}{2}a_m - \frac{1}{2i}b_m$. Conversely, given a series of the form (2.6.1) we can reconstruct the series (2.2.1) using $a_0 = 2\alpha_0$, $a_m = \alpha_m + \alpha_{-m}$ and $b_m = i(\alpha_m - \alpha_{-m})$ for $m > 0$. Equations (2.2.2)–(2.2.4) are replaced by the single equation[10]

$$\int_0^{2\pi} e^{im\theta}e^{in\theta}\,d\theta = \begin{cases} 2\pi & \text{if } m = -n \\ 0 & \text{if } m \neq -n \end{cases}$$

and equations (2.2.5)–(2.2.7) are replaced by

$$\alpha_m = \frac{1}{2\pi}\int_0^{2\pi} e^{-im\theta} f(\theta)\,d\theta. \tag{2.6.2}$$

Exercises

1. For the square wave example discussed in §2.2, show that

$$\alpha_m = \frac{1}{2\pi}\left(\int_0^{\pi} e^{-im\theta}\,d\theta - \int_{\pi}^{2\pi} e^{-im\theta}\,d\theta\right) = \begin{cases} 2/im\pi & m \text{ odd} \\ 0 & m \text{ even.} \end{cases}$$

so that the Fourier series is

$$\sum_{n=-\infty}^{\infty} \frac{2}{i(2n+1)\pi} e^{i(2n+1)\theta}.$$

2.7. Proof of Fejér's Theorem

We are now in a position to prove Fejér's Theorem 2.4.2. This section may safely be skipped on first reading.

In terms of the complex form of the Fourier series, the partial sums (2.4.1) become

$$s_m = \sum_{n=-m}^{m} \alpha_n e^{in\theta}, \tag{2.7.1}$$

and so the Cesàro sums σ_m are given by

$$\begin{aligned}
\sigma_m(\theta) &= \frac{s_0 + \cdots + s_m}{m+1} \\
&= \frac{1}{m+1}\sum_{j=0}^{m}\sum_{n=-j}^{j} \alpha_n e^{in\theta} \\
&= \frac{1}{m+1}\big(\alpha_{-m}e^{-im\theta} + 2\alpha_{-(m-1)}e^{-i(m-1)\theta} + 3\alpha_{-(m-2)}e^{-i(m-2)\theta} + \dots \\
&\qquad + \cdots + m\alpha_{-1}e^{-i\theta} + (m+1)\alpha_0 e^0 + m\alpha_1 e^{i\theta} + \cdots + \alpha_m e^{im\theta}\big) \\
&= \sum_{n=-m}^{m} \frac{m+1-|n|}{m+1}\alpha_n e^{in\theta}.
\end{aligned}$$

---

[10] Over the complex numbers, to interpret this equation as an orthogonality relation (see the footnote on page 32), the inner product needs to be taken to be $\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\overline{g(\theta)}\,d\theta$.

$$
\begin{aligned}
&= \sum_{n=-m}^{m} \frac{m+1-|n|}{m+1} \left(\frac{1}{2\pi} \int_0^{2\pi} e^{-inx} f(x)\,dx \right) e^{in\theta} \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(x) \left(\sum_{n=-m}^{m} \frac{m+1-|n|}{m+1} e^{in(\theta-x)} \right) dx \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(x) K_m(\theta - x)\,dx
\end{aligned}
$$

where

$$K_m(y) = \sum_{n=-m}^{m} \frac{m+1-|n|}{m+1} e^{iny}.$$

The functions K_m are called the Fejér kernels.

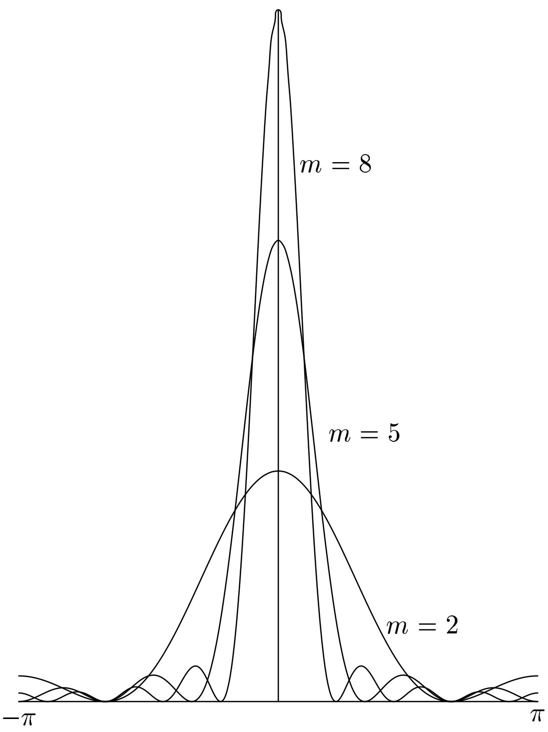
The substitution $y = \theta - x$ shows that

$$\frac{1}{2\pi} \int_0^{2\pi} f(x) K_m(\theta - x)\,dx = \frac{1}{2\pi} \int_0^{2\pi} f(\theta - y) K_m(y)\,dy$$

By examining what happens when a geometric series is squared, for $y \neq 0$ we have

$$
\begin{aligned}
K_m(y) &= \frac{1}{m+1} \big(e^{-imy} + 2e^{-i(m-1)y} + \cdots + (m+1)e^0 + \cdots + e^{imy} \big) \\
&= \frac{1}{m+1} (e^{-i\frac{m}{2}y} + e^{-i(\frac{m}{2}-1)y} + \cdots + e^{i\frac{m}{2}y})^2 \qquad (2.7.2) \\
&= \frac{1}{m+1} \left(\frac{e^{i\frac{m+1}{2}y} - e^{-i\frac{m+1}{2}y}}{e^{i\frac{1}{2}y} - e^{-i\frac{1}{2}y}} \right)^2 \\
&= \frac{1}{m+1} \left(\frac{\sin \frac{m+1}{2}y}{\sin \frac{1}{2}y} \right)^2,
\end{aligned}
$$

and $K_m(0) = m+1$ can also be read off from (2.7.2). Here are the graphs of $K_m(y)$ for some small values of m.



The function $K_m(y)$ satisfies $K_m(y) \geq 0$ for all y; for any $\delta > 0$, $K_m(y) \to 0$ uniformly as $m \to \infty$ on $[\delta, 2\pi - \delta]$; and $\int_0^{2\pi} K_m(y)\,dy = 2\pi$. So

$$\sigma_m(\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta - y)K_m(y)\,dy \approx \frac{1}{2\pi}\int_{-\delta}^{\delta} f(\theta - y)K_m(y)\,dy$$
$$\approx f(\theta)\left(\frac{1}{2\pi}\int_{-\delta}^{\delta} K_m(y)\,dy\right) \approx f(\theta).$$

If $f(\theta)$ is continuous at θ, then by choosing δ small enough, the second approximation may be made as close as desired (independently of m). Then by choosing m large enough, the first and third approximations may be made as close as desired. This completes the proof of Fejér's theorem.

Exercises

1. (i) Substitute equation (2.6.2) in equation (2.7.1) to show that

$$s_m(\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(x)D_m(\theta - x)\,dx$$

where

$$D_m(y) = \sum_{n=-m}^{m} e^{iny}.$$

The functions D_m are called the Dirichlet kernels.

(ii) Use a substitution to show that

$$s_m(\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta - y)D_m(y)\,dy.$$

(iii) By regarding the formula for $D_m(y)$ as a geometric series, show that

$$D_m(y) = \frac{\sin(m+\frac{1}{2})y}{\sin\frac{1}{2}y}.$$

(iv) Show that $|D_m(y)| \le |\operatorname{cosec}\frac{1}{2}y|$

(v) Sketch the graphs of the Dirichlet kernels for small values of m. What happens as m gets large?

2.8. Bessel functions

Bessel functions[11] are the result of applying the theory of Fourier series to the functions $\sin(z\sin\theta)$ and $\cos(z\sin\theta)$ as functions of θ, where z is to be thought of at first as a real (or complex) constant, and later it will be allowed to vary. We shall have two uses for the Bessel functions. One is understanding the vibrations of a drum in §3.5, and the other is understanding the amplitudes of side bands in FM synthesis in §7.13.

Now $\sin(z\sin\theta)$ is an odd periodic function of θ, so its Fourier coefficients a_n (2.2.1) are zero for all n (see §2.3). Since

$$\sin(z\sin(\pi+\theta)) = -\sin(z\sin\theta),$$

the Fourier coefficients b_{2n} are also zero (see §2.3 again). The coefficients b_{2n+1} depend on the parameter z, and so we write $2J_{2n+1}(z)$ for this coefficient. The factor of two simplifies some later calculations. So the Fourier expansion (2.2.1) is

$$\boxed{\sin(z\sin\theta) = 2\sum_{n=0}^{\infty} J_{2n+1}(z)\sin(2n+1)\theta.} \tag{2.8.1}$$

Similarly, $\cos(z\sin\theta)$ is an even periodic function of θ, so the coefficients b_n are zero. Since

$$\cos(z\sin(\pi+\theta)) = \cos(z\sin\theta)$$

we also have $a_{2n+1} = 0$, and we write $2J_{2n}(z)$ for the coefficient a_{2n} to obtain

$$\boxed{\cos(z\sin\theta) = J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z)\cos 2n\theta.} \tag{2.8.2}$$

[11] Friedrich Wilhelm Bessel was a German astronomer and a friend of Gauss. He was born in Minden on July 22, 1784. His working life started as a ship's clerk. But in 1806, he became an assistant at an astronomical observatory in Lilienthal. In 1810 he became director of the then new Prussian Observatory in Königsberg, where he remained until he died on March 17, 1846. The original context (around 1824) of his investigations of the functions that bear his name was the study of planetary motion, see Section 2.11.

The functions $J_n(z)$ giving the Fourier coefficients in these expansions are called the *Bessel functions* of the first kind.

Equations (2.2.5) and (2.2.6) allow us to find the Fourier coefficients $J_n(z)$ in the above expansions as integrals. We obtain

$$2J_{2n+1}(z) = \frac{1}{\pi}\int_0^{2\pi} \sin(2n+1)\theta \sin(z\sin\theta)\,d\theta.$$

The integrand is an even function of θ, so the integral from 0 to 2π is twice the integral from 0 to π,

$$J_{2n+1}(z) = \frac{1}{\pi}\int_0^{\pi} \sin(2n+1)\theta \sin(z\sin\theta)\,d\theta.$$

Now the function $\cos(2n+1)\theta\cos(z\sin\theta)$ is negated when θ is replaced by $\pi-\theta$, so

$$\frac{1}{\pi}\int_0^{\pi} \cos(2n+1)\theta\cos(z\sin\theta)\,d\theta = 0.$$

Adding this into the above expression for $J_{2n+1}(z)$, we obtain

$$\begin{aligned} J_{2n+1}(z) &= \frac{1}{\pi}\int_0^{\pi} [\cos(2n+1)\theta\cos(z\sin\theta) + \sin(2n+1)\theta\sin(z\sin\theta)]\,d\theta \\ &= \frac{1}{\pi}\int_0^{\pi} \cos((2n+1)\theta - z\sin\theta)\,d\theta. \end{aligned}$$

In a similar way, we have

$$2J_{2n}(z) = \frac{1}{\pi}\int_0^{2\pi} \cos 2n\theta\cos(z\sin\theta)\,d\theta$$

which a similar manipulation puts in the form

$$J_{2n}(z) = \frac{1}{\pi}\int_0^{\pi} \cos(2n\theta - z\sin\theta)\,d\theta.$$

This means that we have the single equation for all values of n, even or odd,

$$\boxed{J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - z\sin\theta)\,d\theta} \tag{2.8.3}$$

which can be taken as a definition for the Bessel functions for integers $n \geq 0$. In fact, this definition also makes sense when n is a negative integer,[12] and gives

$$J_{-n}(z) = (-1)^n J_n(z). \tag{2.8.4}$$

This means that (2.8.1) and (2.8.2) can be rewritten as

$$\sin(z\sin\theta) = \sum_{n=-\infty}^{\infty} J_{2n+1}(z)\sin(2n+1)\theta \tag{2.8.5}$$

[12]For non-integer values of n, the above formula is not the correct definition of $J_n(z)$. Rather, one uses the differential equation (2.10.1). See for example Whittaker and Watson, *A course in modern analysis*, Cambridge University Press, 1927, p. 358.

$$\cos(z\sin\theta) = \sum_{n=-\infty}^{\infty} J_{2n}(z)\cos 2n\theta. \tag{2.8.6}$$

We also have

$$\sum_{n=-\infty}^{\infty} J_{2n}(z)\sin 2n\theta = 0$$

$$\sum_{n=-\infty}^{\infty} J_{2n+1}(z)\cos(2n+1)\theta = 0,$$

because the terms with positive subscript cancel with the corresponding terms with negative subscript. So we can rewrite equations (2.8.5) and (2.8.6) as

$$\sin(z\sin\theta) = \sum_{n=-\infty}^{\infty} J_n(z)\sin n\theta \tag{2.8.7}$$

$$\cos(z\sin\theta) = \sum_{n=-\infty}^{\infty} J_n(z)\cos n\theta. \tag{2.8.8}$$

So using equation (1.7.2) we have

$$\begin{aligned}\sin(\phi + z\sin\theta) &= \sin\phi\cos(z\sin\theta) + \cos\phi\sin(z\sin\theta)\\ &= \sin\phi\sum_{n=-\infty}^{\infty} J_n(z)\cos n\theta + \cos\phi\sum_{n=-\infty}^{\infty} J_n(z)\sin n\theta\\ &= \sum_{n=-\infty}^{\infty} J_n(z)(\sin\phi\cos n\theta + \cos\phi\sin n\theta).\end{aligned}$$

Finally, recombining the terms using equation (1.7.2), we obtain

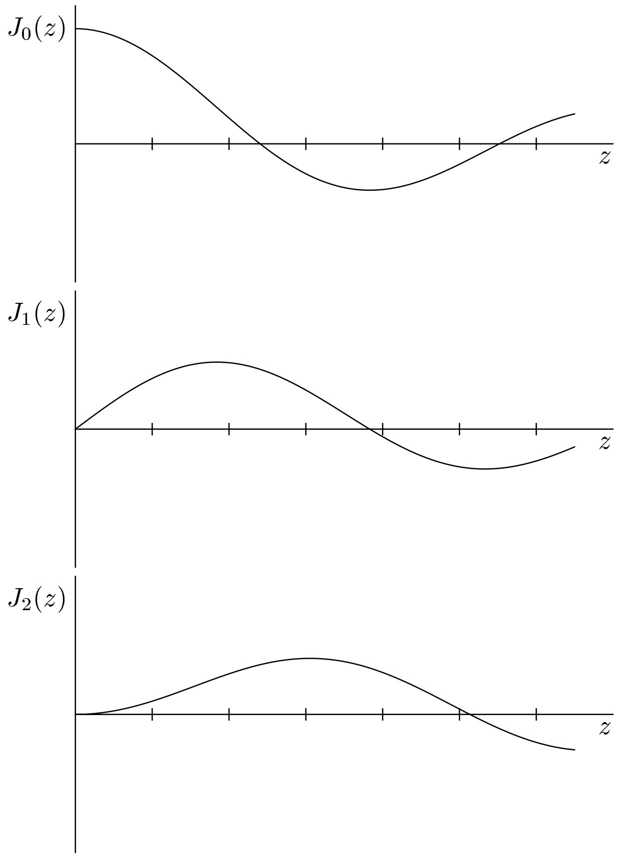
$$\boxed{\sin(\phi + z\sin\theta) = \sum_{n=-\infty}^{\infty} J_n(z)\sin(\phi + n\theta).} \tag{2.8.9}$$

This equation will be of fundamental importance for FM synthesis in §7.13. A similar argument gives

$$\cos(\phi + z\sin\theta) = \sum_{n=-\infty}^{\infty} J_n(z)\cos(\phi + n\theta), \tag{2.8.10}$$

which can also be obtained from equation (2.8.9) by replacing ϕ by $\phi + \frac{\pi}{2}$, or by differentiating with respect to ϕ, keeping z and θ constant.

Here are graphs of the first few Bessel functions:



2.9. Properties of Bessel functions

From equation (2.8.9), we can obtain relationships between the Bessel functions and their derivatives, as follows. Differentiating (2.8.9) with respect to z, keeping θ and ϕ constant, we obtain

$$\sin\theta\,\cos(\phi + z\sin\theta) = \sum_{n=-\infty}^{\infty} J_n'(z)\sin(\phi + n\theta) \tag{2.9.1}$$

On the other hand, multiplying equation (2.8.10) by $\sin\theta$ and using (1.7.7), we have

$$\begin{aligned}\sin\theta\,\cos(\phi + z\sin\theta) &= \sum_{n=-\infty}^{\infty} J_n(z).\tfrac{1}{2}\big(\sin(\phi + (n+1)\theta) - \sin(\phi + (n-1)\theta)\big)\\ &= \sum_{n=-\infty}^{\infty} \tfrac{1}{2}\big(J_{n-1}(z) - J_{n+1}(z)\big)\sin(\phi + n\theta).\end{aligned} \tag{2.9.2}$$

In the last step, we have split the sum into two parts, reindexed by replacing n by $n-1$ and $n+1$ respectively in the two parts, and then recombined the parts.

We would like to compare formulas (2.9.1) and (2.9.2) and deduce that

$$\boxed{J_n'(z) = \tfrac{1}{2}\big(J_{n-1}(z) - J_{n+1}(z)\big)} \tag{2.9.3}$$

In order to do this, we need to know that the functions $\sin(\phi + n\theta)$ are independent. This can be seen using Fourier series as follows.

LEMMA 2.9.1. *If*

$$\sum_{n=-\infty}^{\infty} a_n \sin(\phi + n\theta) = \sum_{n=-\infty}^{\infty} a_n' \sin(\phi + n\theta),$$

as an identity between functions of ϕ and θ, where a_n and a_n' do not depend on θ and ϕ, then each coefficient $a_n = a_n'$.

PROOF. Subtracting one side from the other, we see that we must prove that if $\sum_{n=-\infty}^{\infty} c_n \sin(\phi + n\theta) = 0$ (where $c_n = a_n - a_n'$) then each $c_n = 0$. To prove this, we expand using (1.7.2) to give

$$\sum_{n=-\infty}^{\infty} c_n \sin\phi \cos n\theta + \sum_{n=-\infty}^{\infty} c_n \cos\phi \sin n\theta = 0.$$

Putting $\phi = 0$ and $\phi = \frac{\pi}{2}$ in this equation, we obtain

$$\sum_{n=-\infty}^{\infty} c_n \cos n\theta = 0, \tag{2.9.4}$$

$$\sum_{n=-\infty}^{\infty} c_n \sin n\theta = 0. \tag{2.9.5}$$

Multiply equation (2.9.4) by $\cos m\theta$, integrate from 0 to 2π and divide by π. Using equation (2.2.3), we get $c_m + c_{-m} = 0$. Similarly, from equations (2.9.5) and (2.2.4), we get $c_m - c_{-m} = 0$. Adding and dividing by two, we get $c_m = 0$. □

This completes the proof of equation (2.9.3). As an example, setting $n = 0$ in (2.9.3) and using (2.8.4), we obtain

$$J_1(z) = -J_0'(z). \tag{2.9.6}$$

In a similar way, we can differentiate (2.8.9) with respect to θ, keeping z and ϕ constant to obtain

$$z\cos\theta\, \cos(\phi + z\sin\theta) = \sum_{n=-\infty}^{\infty} nJ_n(z)\cos(\phi + n\theta). \tag{2.9.7}$$

On the other hand, multiplying equation (2.8.10) by $z\cos\theta$ and using (1.7.10), we obtain

$$\begin{aligned} &z\cos\theta\, \cos(\phi + z\sin\theta) \\ &\quad = \sum_{n=-\infty}^{\infty} J_n(z).\tfrac{z}{2}\big(\cos(\phi + (n+1)\theta) + \cos(\phi + (n-1)\theta)\big) \end{aligned}$$

$$= \sum_{n=-\infty}^{\infty} \tfrac{z}{2}\big(J_{n-1}(z) + J_{n+1}(z)\big)\cos(\phi + n\theta). \qquad (2.9.8)$$

Comparing equations (2.9.7) and (2.9.8) and using Lemma 2.9.1, we obtain the recurrence relation

$$\boxed{J_n(z) = \frac{z}{2n}\big(J_{n-1}(z) + J_{n+1}(z)\big).} \qquad (2.9.9)$$

Exercises

1. Show that $\displaystyle\int_0^\infty J_1(z)\,dz = 1.$
[You may use the fact that $\lim\limits_{z\to\infty} J_0(z) = 0$]

2.10. Bessel's equation and power series

Using equations (2.9.3) and (2.9.9), we can now derive the differential equation (2.10.1) for the Bessel functions $J_n(z)$. Using (2.9.3) twice, we obtain

$$\begin{aligned} J_n''(z) &= \tfrac{1}{2}(J_{n-1}'(z) - J_{n+1}'(z)) \\ &= \tfrac{1}{4}J_{n-2}(z) - \tfrac{1}{2}J_n(z) + \tfrac{1}{4}J_{n+2}(z). \end{aligned}$$

On the other hand, substituting (2.9.9) into (2.9.3), we obtain

$$\begin{aligned} J_n'(z) &= \tfrac{1}{2}\left(\tfrac{z}{2(n-1)}(J_{n-2}(z) + J_n(z)) - \tfrac{z}{2(n+1)}(J_n(z) + J_{n+2}(z))\right) \\ &= \tfrac{z}{4(n-1)}J_{n-2}(z) + \tfrac{z}{2(n^2-1)}J_n(z) - \tfrac{z}{4(n-1)}J_{n+2}(z). \end{aligned}$$

In a similar way, using (2.9.9) twice gives

$$\begin{aligned} J_n(z) &= \tfrac{z}{2n}\left(\tfrac{z}{2(n-1)}(J_{n-2}(z) + J_n(z)) + \tfrac{z^2}{2(n+1)}(J_n(z) + J_{n+2}(z))\right) \\ &= \tfrac{z}{4n(n-1)}J_{n-2}(z) + \tfrac{z^2}{n^2-1}J_n(z) + \tfrac{z^2}{4n(n+1)}J_{n+2}(z). \end{aligned}$$

Combining these three formulas, we obtain

$$J_n''(z) + \tfrac{1}{z}J_n'(z) - \tfrac{n^2}{z^2}J_n(z) = -J_n(z),$$

or

$$\boxed{J_n''(z) + \frac{1}{z}J_n'(z) + \left(1 - \frac{n^2}{z^2}\right)J_n(z) = 0.} \qquad (2.10.1)$$

We now discuss the general solution to *Bessel's Equation*, namely the differential equation

$$f''(z) + \frac{1}{z}f'(z) + \left(1 - \frac{n^2}{z^2}\right)f(z) = 0. \qquad (2.10.2)$$

This is an example of a second order linear differential equation, and once one solution is known, there is a general proceedure for obtaining all solutions. In this case, this consists of substituting $f(z) = J_n(z)g(z)$, and finding the differential equation satisfied by the new function $g(z)$. We find that

$$\begin{aligned} f'(z) &= J_n'(z)g(z) + J_n(z)g'(z), \\ f''(z) &= J_n''(z)g(z) + 2J_n'(z)g'(z) + J_n(z)g''(z). \end{aligned}$$

So substituting into Bessel's equation (2.10.2), we obtain

$$\left(J_n''(z) + \frac{1}{z}J_n'(z) + \left(1 - \frac{n^2}{z^2}\right)J_n(z)\right)g(z) + \\ \left(2J_n'(z) + \frac{1}{z}J_n(z)\right)g'(z) + J_n(z)g''(z) = 0.$$

The coefficient of $g(z)$ vanishes by equation (2.10.1), and so we are left with

$$\left(2J_n'(z) + \frac{1}{z}J_n(z)\right)g'(z) + J_n(z)g''(z) = 0, \tag{2.10.3}$$

This is a separable first order equation for $g'(z)$, so we separate the variables

$$\frac{g''(z)}{g'(z)} = -2\frac{J_n'(z)}{J_n(z)} - \frac{1}{z}$$

and integrate to obtain

$$\ln|g'(z)| = -2\ln|J_n(z)| - \ln|z| + C$$

where C is the constant of integration. Exponentiating, we obtain

$$g'(z) = \frac{B}{zJ_n(z)^2}$$

where $B = \pm e^C$. Alternatively, we could have obtained this directly from equation (2.10.3) by multiplying by $zJ_n(z)$ to see that the derivative of $zJ_n(z)^2g'(z)$ is zero.

Integrating again, we obtain

$$g(z) = A + B\int\frac{dz}{zJ_n(z)^2}$$

where the integral sign denotes a chosen antiderivative. Finally, it follows that the general solution to Bessel's equation is given by

$$f(z) = AJ_n(z) + BJ_n(z)\int\frac{dz}{zJ_n(z)^2}. \tag{2.10.4}$$

The function

$$Y_n(z) = \frac{2}{\pi}J_n(z)\int\frac{dz}{zJ_n(z)^2},$$

for a suitable choice of constant of integration, is called Neumann's Bessel function of the second kind, or Weber's function. The factor of $2/\pi$ is introduced (by most, but not all authors) so that formulas involving $J_n(z)$ and $Y_n(z)$ look similar; we shall not go into the details. From the above integral, it is not hard to see that $Y_n(z)$ tends to $-\infty$ as z tends to zero from above; we shall be more explicit about this towards the end of this section.

Next, we develop the power series for $J_n(z)$. We begin with $J_0(z)$. Putting $z = \theta = 0$ in equation (2.8.2), we see that $J_0(0) = 1$. By (2.8.4), $J_0(z)$ is an even function of z, so we look for a power series of the form

$$J_0(z) = 1 + a_2 z^2 + a_4 z^4 + \cdots = \sum_{k=0}^{\infty} a_{2k} z^{2k}$$

where $a_0 = 1$. Then

$$J_0'(z) = 2a_2 z + 4a_4 z^3 + \cdots = \sum_{k=1}^{\infty} 2k a_{2k} z^{2k-1},$$

$$J_0''(z) = 2 \cdot 1\, a_2 + 4 \cdot 3\, a_4 z^2 + \cdots = \sum_{k=1}^{\infty} 2k(2k-1) a_{2k} z^{2k-2}.$$

Putting $n = 0$ in equation (2.10.1) and comparing coefficients of a_{2k-2}, we obtain

$$2k(2k-1)a_{2k} + 2k a_{2k} + a_{2k-2} = 0,$$

or

$$(2k)^2 a_{2k} = -a_{2k-2}.$$

So starting with $a_0 = 1$, we obtain $a_2 = -1/2^2$, $a_4 = 1/(2^2 \cdot 4^2)$, ..., and by induction on k, we have

$$a_{2k} = \frac{(-1)^k}{2^2 \cdot 4^2 \ldots (2k)^2} = \frac{(-1)^k}{2^k (k!)^2}.$$

So we have

$$J_0(z) = 1 - \frac{z^2}{2^2} + \frac{z^4}{2^2 \cdot 4^2} - \frac{z^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k}}{(k!)^2}. \tag{2.10.5}$$

Since the coefficients in this power series are tending to zero very rapidly, it has an infinite radius of convergence.[13] So it is uniformly convergent, and can be differentiated term by term. It follows that the sum of the power series satisfies Bessel's equation, because that's how we chose the coefficients. We have already seen that there is only one solution of Bessel's equation with value 1 at $z = 0$, which completes the proof that the sum of the power series is indeed $J_0(z)$.

Differentiating equation (2.10.5) term by term and using (2.9.6), we see that

$$J_1(z) = \frac{z}{2} - \frac{z^3}{2^2 \cdot 4} + \frac{z^5}{2^2 \cdot 4^2 \cdot 6} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{1+2k}}{k!(1+k)!}.$$

[13]For any value of z, the ratio of successive terms tends to zero, so by the ratio test the series converges.

Now using equation (2.9.9) and induction on n, we find that

$$\boxed{J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k(\frac{z}{2})^{n+2k}}{k!(n+k)!},} \tag{2.10.6}$$

with infinite radius of convergence.

From the power series, we see that for small positive values of z, $J_n(z)$ is equal to $z^n/2^n n!$ plus much smaller terms. So $1/zJ_n(z)^2$ is equal to $2^{2n}(n!)^2 z^{-2n-1}$ plus much smaller terms, and $\int 1/zJ_n(z)^2\,dz$ is equal to $-2^{2n-1}n!(n-1)!z^{-2n}$ plus much smaller terms. Finally, $Y_n(z)$ is equal to $-2^n(n-1)!z^{-n}/\pi$ plus much smaller terms. In particular, this shows that $Y_n(z) \to -\infty$ as $z \to 0^+$.

Exercises

1. Replace θ by $\frac{\pi}{2} - \theta$ in equations (2.8.1) and (2.8.2) to obtain the Fourier series for $\sin(z\cos\theta)$ and $\cos(z\cos\theta)$.

2. Show that $y = J_n(\alpha x)$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(\alpha^2 - \frac{n^2}{x^2}\right)y = 0.$$

Show that the general solution to this equation is given by $y = AJ_n(\alpha x) + BY_n(\alpha x)$.

3. Show that $y = \sqrt{x}J_n(x)$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} + \left(1 + \frac{\frac{1}{4} - n^2}{x^2}\right)y = 0.$$

Find the general solution of this equation.

4. Show that $y = J_n(e^x)$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} + (e^{2x} - n^2)y = 0.$$

Find the general solution of this equation.

5. The following exercise is another route to Bessel's differential equation (2.10.1).

(a) Differentiate equation (2.8.9) twice with respect to z, keeping ϕ and θ constant.

(b) Differentiate equation (2.8.9) twice with respect to θ, keeping z and ϕ constant.

(c) Divide the result of (b) by z^2 and add to the result of (a), and use the relation $\sin^2\theta + \cos^2\theta = 1$. Deduce that

$$\sum_{n=-\infty}^{\infty} \left(J_n''(z) + \frac{1}{z}J_n'(z) + \left(1 - \frac{n^2}{z^2}\right)J_n(z)\right)\sin(\phi + z\theta) = 0.$$

(d) Finally, use Lemma 2.9.1 to show that Bessel's equation (2.8.9) holds.

(The following exercises suppose some knowledge of complex analysis in order to give an alternative development of the power series and recurrence relations for the Bessel functions)

6. Show that

$$J_n(z) = \frac{1}{2\pi}\int_0^{\pi} e^{i(n\theta - z\sin\theta)}\,d\theta + \frac{1}{2\pi}\int_0^{\pi} e^{-i(n\theta - z\sin\theta)}\,d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(n\theta - z\sin\theta)}\,d\theta.$$

Substitute $t = e^{i\theta}$ (so that $\frac{1}{2i}(t - \frac{1}{t}) = \sin\theta$) to obtain

$$J_n(z) = \frac{1}{2\pi i}\oint t^{-n-1}e^{\frac{1}{2}z(t-\frac{1}{t})}\,dt \tag{2.10.7}$$

where the contour of integration goes counterclockwise once around the unit circle. Use Cauchy's integral formula to deduce that $J_n(z)$ is the coefficient of t^n in the Laurent expansion of $e^{\frac{1}{2}z(t-\frac{1}{t})}$:

$$e^{\frac{1}{2}z(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(z)t^n.$$

7. Substitute $t = 2s/z$ in (2.10.7) to obtain

$$J_n(z) = \frac{1}{2\pi i}\left(\frac{z}{2}\right)^n \oint s^{-n-1}e^{s-\frac{z^2}{4s}}\,ds.$$

Discuss the contour of integration. Expand the integrand in powers of z to give

$$J_n(z) = \frac{1}{2\pi i}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{z}{2}\right)^{n+2k}\oint s^{-n-k-1}e^s\,ds$$

and justify the term by term integration. Show that the residue of the integrand at $s = 0$ is $1/(n+k)!$ when $n + k \geq 0$ and is zero when $n + k < 0$. Deduce the power series (2.10.6).

8. (a) Use the power series (2.10.6) to show that

$$J_n(z) = \tfrac{z}{2n}(J_{n-1}(z) + J_{n+1}(z)).$$

(b) Differentiate the power series (2.10.6) term by term to show that

$$J_n'(z) = \tfrac{1}{2}(J_{n-1}(z) - J_{n+1}(z)).$$

Further reading on Bessel functions:

Milton Abramowitz and Irene A. Stegun, *Handbook of mathematical functions*, National Bureau of Standards, 1964, reprinted by Dover in 1965 and still in print. This contains extensive tables of many mathematical functions including $J_n(z)$ and $Y_n(z)$.

Frank Bowman, *Introduction to Bessel functions*, reprinted by Dover in 1958 and still in print.

G. N. Watson, *A treatise on the theory of Bessel functions* [**111**] is an 800 page tome on the theory of Bessel functions. This work contains essentially everything that was known in 1922 about these functions, and is still pretty much the standard reference.

E. T. Whittaker and G. N. Watson, *Modern Analysis*, Cambridge University Press, 1927, chapter XVII.

2.11. Fourier series for FM feedback and planetary motion

We shall see in §7.13 that in the theory of FM synthesis, feedback is represented by an equation of the form

$$\phi = \sin(\omega t + z\phi), \tag{2.11.1}$$

where ω and z are constants with $|z| \leq 1$.

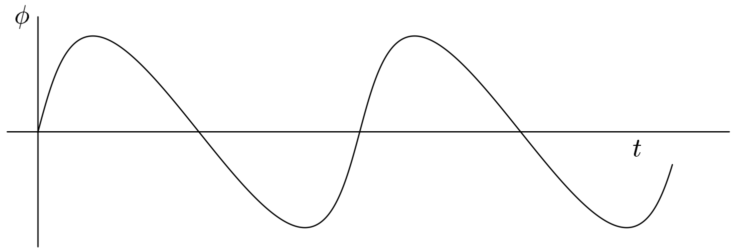
In the theory of planetary motion, Kepler's laws imply that the angle θ subtended at the center (not the focus) of the elliptic orbit by the planet, measured from the major axis of the ellipse, satisfies

$$\omega t = \theta - z \sin\theta \tag{2.11.2}$$

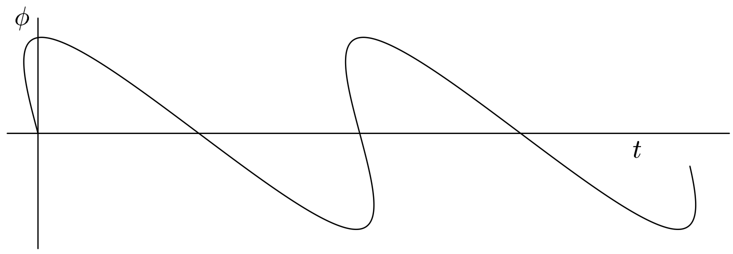
where z is the eccentricity[14] of the ellipse, a number in the range $0 \leq z \leq 1$, and $\omega = 2\pi\nu$ is angular velocity.

Both of these equations define periodic functions of t, namely ϕ in the first case and $\sin\theta = (\theta - \omega t)/z$ in the second. In fact, they are really just different ways of writing the same equation. To get from equation (2.11.2) to (2.11.1), we use the substitution $\theta = \omega t + z\phi$. To go the other way, we use the inverse substitution $\phi = (\theta - \omega t)/z$.

To graph ϕ as a function of t, it is best to use θ as a parameter and set $t = (\theta - z\sin\theta)/\omega$, $\phi = \sin\theta$. Here is the result when $z = \frac{1}{2}$:



When $|z| > 1$, the parametrized form of the equation still makes sense, but it is easy to see that the resulting graph does not define ϕ uniquely as a function of t. Here is the result when $z = \frac{3}{2}$:



[14]The eccentricity of an ellipse is defined to be the distance from the center to the focus, as a proportion of the major radius.

In this section, we examine equation (2.11.2), and find the Fourier coefficients of $\phi = \sin\theta$ as a function of t, regarding z as a constant. The answer is given in terms of Bessel functions. In fact, the solution of this equation in the context of planetary motion was the original motivation for Bessel to introduce his functions $J_n(z)$.[15]

First, for convenience we write $T = \omega t$. Next, we observe that provided $|z| \le 1$, $\theta - z\sin\theta$ is a strictly increasing fuction of θ whose domain and range are the whole real line. It follows that solving equation (2.11.2) gives a unique value of θ for each T, so that θ may be regarded as a continuous function of T. Furthermore, adding 2π to both θ and T, or negating both θ and T does not affect equation (2.11.2), so $z\phi = z\sin\theta = \theta - T$ is an odd periodic function of T with period 2π. So it has a Fourier expansion

$$z\phi = \sum_{n=1}^{\infty} b_n \sin nT. \tag{2.11.3}$$

The coefficients b_n can be calculated directly using equation (2.2.6):

$$b_n = \frac{1}{\pi}\int_0^{2\pi} z\phi \sin nT\, dT = \frac{2}{\pi}\int_0^{\pi} z\phi \sin nT\, dT.$$

Integrating by parts gives

$$b_n = \frac{2}{\pi}\left[-z\phi\frac{\cos nT}{n}\right]_0^{\pi} + \frac{2}{\pi}\int_0^{\pi} z\,\frac{d\phi}{dT}\,\frac{\cos nT}{n}\,dT.$$

We have $\phi = 0$ when $T = 0$ or $T = \pi$, so the first term vanishes. Rewriting the second term, we obtain

$$b_n = \frac{2}{n\pi}\int_0^{\pi} \cos nT\,\frac{d(z\phi)}{dT}\,dT.$$

Now $\displaystyle\int_0^{\pi} \cos nT\, dT = 0$, so we can rewrite this as

$$\begin{aligned} b_n &= \frac{2}{n\pi}\int_0^{\pi} \cos nT\,\frac{d(z\phi + T)}{dT}\,dT = \frac{2}{n\pi}\int_0^{\pi} \cos nT\,\frac{d\theta}{dT}\,dT \\ &= \frac{2}{n\pi}\int_0^{\pi} \cos nT\, d\theta. \end{aligned}$$

In the last step, we have used the fact that as T increases from 0 to π, so does θ. Substituting $T = \theta - z\sin\theta$ now gives

$$b_n = \frac{2}{n\pi}\int_0^{\pi} \cos(n\theta - nz\sin\theta)\, d\theta.$$

Comparing with equation (2.8.3) finally gives

$$b_n = \frac{2}{n}J_n(nz).$$

[15] Bessel, *Untersuchung der Theils der planetarischen Störungen, welcher aus der Bewegung der Sonne entsteht*, Berliner Abh. (1826), 1–52.

Substituting back into equation (2.11.3) gives

$$\boxed{\phi = \sin\theta = \sum_{n=1}^{\infty} \frac{2J_n(nz)}{nz} \sin n\omega t.} \tag{2.11.4}$$

So this equation gives the Fourier series relevant to both feedback in FM synthesis (2.11.1) and planetary motion (2.11.2).

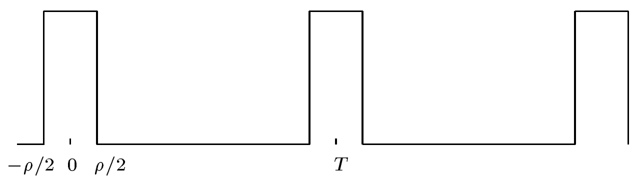
2.12. Pulse streams

In this section, we examine streams of square pulses. The purpose of this is twofold. First, in analog synthesizers[16] one method for obtaining a time varying frequency spectrum is to use pulse width modulation (PWM). A low frequency oscillator (LFO, §7.7) is used to control the pulse width of a square wave, while keeping the fundamental frequency constant. The second point is that by keeping the pulse width constant and decreasing the frequency, we motivate the definition of Fourier transform, to be introduced in §2.13.

Let us investigate the frequency spectrum of the square wave given by

$$f(t) = \begin{cases} 1 & 0 \le t < \rho/2 \\ 0 & \rho/2 \le t < T - \rho/2 \\ 1 & T - \rho/2 \le t < T \end{cases}$$

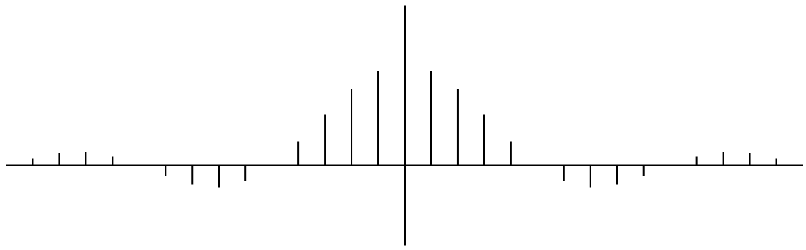
where ρ is some number between 0 and T, and $f(t+T) = f(t)$.



The Fourier coefficients are given by

$$\alpha_m = \frac{1}{T}\int_{-\rho/2}^{\rho/2} e^{-2\pi i m t/T}\, dt = \frac{1}{\pi m}\sin(\pi m \rho/T).$$

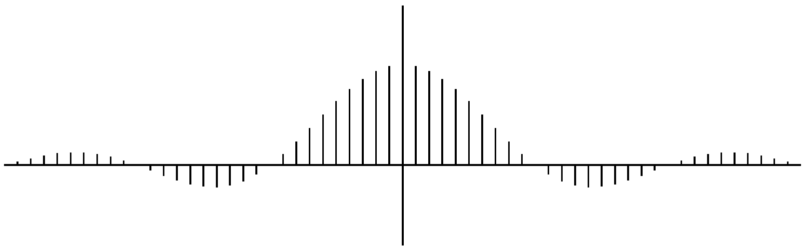
For example, if $T = 5\rho$, the frequency spectrum is as follows



[16]This is also used in some of the more modern analog modeling synthesizers such as the Roland JP-8000/JP-8080.

If we keep ρ constant and increase T, the shape of the spectrum stays the same, but vertically scaled down in proportion, so as to keep the energy density along the horizontal axis constant. It makes sense to rescale to take account of this, and to plot $T\alpha_m$ instead of α_m. If we do this, and increase T while keeping ρ constant, all that happens is that the graph fills in. So for example, removing every second peak from the original square wave

then the spectrum fills in as follows.



Letting T tend to infinity while keeping ρ constant, we obtain the *Fourier transform* of a single square pulse, which (after suitable scaling) is the function $\sin(\nu)/\nu$. Here, ν is a continuously variable quantity representing frequency.

2.13. The Fourier transform

The theory of Fourier series, as described in §§2.2–2.4, decomposes periodic waveforms into infinite sums of sines and cosines, or equivalently (§2.6) complex exponential functions of the form e^{int}. It is often desirable to analyse nonperiodic functions in a similar way. This leads to the theory of Fourier transforms. The theory is more beset with conditions than the theory of Fourier series. In particular, the theory only describes functions which tend to zero for large positive or negative values of the time variable t. To deal with this from a musical perspective, we introduce the theory of windowing. The point is that any actual sound is not really periodic, since periodic functions have no starting point and no end point. Moreover, we don't really want a frequency analysis of, for example, the whole of a symphony, because the answer would be dominated by extremely phase sensitive low frequency information. We'd really like to know at each instant what the frequency spectrum of the sound is, and to plot this frequency spectrum against time. Now, it turns out that it doesn't really make sense to ask for the instantaneous frequency spectrum of a sound, because there's not enough information. We really need to know the waveform for a time window around each point, and analyse that. Small window sizes give information which is more localized in time, but the frequency components are smeared out along the spectrum. Large window sizes give information in which the frequency components are more accurately described, but more smeared out along the time axis. This limitation is inherent to the process, and has nothing to do with how accurately the waveform is measured. In this respect, it resembles the Heisenberg uncertainty principle.[17]

If $f(t)$ is a real or complex valued function of a real variable t, then its *Fourier transform* $\hat{f}(\nu)$ is the function of a real variable ν defined by[18]

$$\boxed{\hat{f}(\nu) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i \nu t}\, dt.} \tag{2.13.1}$$

Existence of a Fourier transform for a function assumes convergence of the above integral, and this already puts restrictions on the function $f(t)$. A reasonable condition which ensures convergence is the following. A function $f(t)$ is said to be L^1, or *absolutely integrable* on $(-\infty, \infty)$ if the integral $\int_{-\infty}^{\infty} |f(t)|\, dt$ converges.

Calculating the Fourier transform of a function is usually a difficult process. As an example, we now calculate the Fourier transform of $e^{-\pi t^2}$. This function is unusual, in that it turns out to be its own Fourier transform.

THEOREM 2.13.1. *The Fourier transform of* $e^{-\pi t^2}$ *is* $e^{-\pi \nu^2}$.

PROOF. Let $f(t) = e^{-\pi t^2}$. Then

$$\hat{f}(\nu) = \int_{-\infty}^{\infty} e^{-\pi t^2} e^{-2\pi i \nu t}\, dt$$

---

[17]In fact, this is more than just an analogy. In quantum mechanics, the probability distributions for position and velocity of a particle are related by the Fourier transform, with an extra factor of $\hbar/m$, where $\hbar$ is Planck's constant and m is the mass. The Heisenberg uncertainty principle applies to the expected deviation from the average value of any two quantities related by the Fourier transform, and says that the product of these expected deviations is at least $\frac{1}{2}$. So in the quantum mechanical context the product is at least $\hbar/2m$, because of the extra factor.

[18]There are a number of variations on this definition to be found in the literature, depending mostly on the placement of the factor of 2π. The way we have set it up means that the variable ν directly represents frequency. Most authors delete the 2π from the exponential in this definition, which amounts to using the angular velocity ω instead. This means that they either have a factor of $1/2\pi$ appearing in formula (2.13.3), causing an annoying asymmetry, or a factor of $1/\sqrt{2\pi}$ in both (2.13.1) and (2.13.3).

Strictly speaking, the meaning of equation (2.13.1) should be

$$\lim_{a \to -\infty} \lim_{b \to \infty} \int_a^b f(t) e^{-2\pi i \nu t}\, dt.$$

However, under some conditions this double limit may not exist, while

$$\lim_{R \to \infty} \int_{-R}^{R} f(t) e^{-2\pi i \nu t}\, dt$$

may exist. This weaker symmetric limit is called the *Cauchy principal value* of the integral. Principal values are often used in the theory of Fourier transforms.

$$= \int_{-\infty}^{\infty} e^{-\pi(t^2+2i\nu t)}\,dt$$
$$= \int_{-\infty}^{\infty} e^{-\pi((t+i\nu)^2+\nu^2)}\,dt.$$

Substituting $x = t + i\nu$, $dx = dt$, we obtain

$$\hat{f}(\nu) = \int_{-\infty}^{\infty} e^{-\pi(x^2+\nu^2)}\,dx. \tag{2.13.2}$$

This form of the integral makes it obvious that $\hat{f}(\nu)$ is positive and real, but it is not obvious how to evaluate the integral. It turns out that it can be evaluated using a trick. The trick is to square both sides, and then regard the right hand side as a double integral.

$$\hat{f}(\nu)^2 = \int_{-\infty}^{\infty} e^{-\pi(x^2+\nu^2)}\,dx \int_{-\infty}^{\infty} e^{-\pi(y^2+\nu^2)}\,dy$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\pi(x^2+y^2+2\nu^2)}\,dx\,dy.$$

We now convert this double integral over the (x, y) plane into polar coordinates (r, θ). Remembering that the element of area in polar coordinates is $r\,dr\,d\theta$, we get

$$\hat{f}(\nu)^2 = \int_0^{2\pi}\int_0^{\infty} e^{-\pi(r^2+2\nu^2)}\,r\,dr\,d\theta.$$

We can easily perform the integration with respect to θ, since the integrand is constant with respect to θ. And then the other integral can be carried out explicitly.

$$\hat{f}(\nu)^2 = \int_0^{\infty} 2\pi r e^{-\pi(r^2+2\nu^2)}\,dr$$
$$= \left[-e^{-\pi(r^2+2\nu^2)}\right]_0^{\infty}$$
$$= e^{-2\pi\nu^2}.$$

Finally, since equation (2.13.2) shows that $\hat{f}(\nu)$ is positive, taking square roots gives $\hat{f}(\nu) = e^{-\pi\nu^2}$ as desired. □

The inversion formula is the following, which should be compared with Theorem 2.4.1.

THEOREM 2.13.2. *Let $f(t)$ be a piecewise C^1 function (i.e., on any finite interval, $f(t)$ is C^1 except at a finite set of points) which is also L^1. Then at points where $f(t)$ is continuous, its value is given by the* **inverse Fourier transform**

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\nu)e^{2\pi i\nu t}\,d\nu. \tag{2.13.3}$$

(Note the change of sign in the exponent from equation (2.13.1)*)*
At discontinuities, the expression on the right of this formula gives the average of the left limit and the right limit, $\frac{1}{2}(f(t^+) + f(t^-))$*, just as in §2.5.*

Just as in the case of Fourier series, it is not true that a piecewise continuous L^1 function satisfies the conclusions of the above theorem. But a device analogous to Cesàro summation works equally well here. The analogue of averaging the first n sums is to introduce a factor of $1 - |\nu|/R$ into the integral defining the inverse Fourier transform, before taking principal values.

THEOREM 2.13.3. *Let $f(t)$ be a piecewise continuous L^1 function. Then at points where $f(t)$ is continuous, its value is given by*

$$f(t) = \lim_{R\to\infty} \int_{-R}^{R} \Big(1 - \frac{|\nu|}{R}\Big) \hat{f}(\nu) e^{2\pi i\nu t}\, d\nu.$$

At discontinuities, this formula gives $\frac{1}{2}(f(t^+) + f(t^-))$*.*

How does the Fourier transform tell us about the frequency distribution in the original function? Well, just as in §2.6, the relations (C.1)–(C.3) tell us how to rewrite complex exponentials in terms of sines and cosines, and vice-versa. So the values of $\hat{f}$ at ν and at $-\nu$ tell us not only about the magnitude of the frequency component with frequency ν, but also the phase. If the original function $f(t)$ is real valued, then $\hat{f}(-\nu)$ is the complex conjugate $\hat{f}(\nu)$. The *energy density* at a particular value of ν is defined to be the square of the amplitude $|\hat{f}(\nu)|$,

$$\text{Energy Density} = |\hat{f}(\nu)|^2.$$

Integrating this quantity over an interval will measure the total energy corresponding to frequencies in this interval. But note that both ν and $-\nu$ contribute to energy, so if only positive values of ν are used, we must remember to double the answer.

The usual way to represent the *frequency spectrum* of a real valued signal is to represent the amplitude and the phase of $f(\nu)$ separately for positive values of ν. Recall from Appendix C that in polar coordinates, we can write $f(\nu)$ as $re^{i\theta}$, where $r = |f(\nu)|$ is the amplitude of the correpsonding frequency component and θ is the phase. So r is always nonnegative, and we take θ to lie between $-\pi$ and π. Then $f(-\nu) = \overline{f(\nu)} = re^{-i\theta}$, so we have already represented this information.

Parseval's formula states that the total energy of a signal is equal to the total energy in its spectrum:

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \int_{-\infty}^{\infty} |\hat{f}(\nu)|^2\, d\nu.$$

More generally, if $f(t)$ and $g(t)$ are two functions, it states that

$$\int_{-\infty}^{\infty} f(t)\overline{g(t)}\, dt = \int_{-\infty}^{\infty} \hat{f}(\nu)\overline{\hat{g}(\nu)}\, d\nu. \tag{2.13.4}$$

The term *white noise* refers to a waveform whose spectrum is flat; for pink noise, the spectrum level decreases by 3dB per octave, while for brown noise (named after Brownian motion), the spectrum level decreases by 6dB per octave.

The windowed Fourier transform was introduced by Gabor,[19] and is described as follows. Given a windowing function $\psi(t)$ and a waveform $f(t)$, the windowed Fourier transform is the function of two variables

$$\mathcal{F}_\psi(f)(p,q) = \int_{-\infty}^{\infty} f(t)e^{-2\pi iqt}\psi(t-p)\,dt,$$

for p and q real numbers. This may be thought of as using all possible time translations of the windowing function, and pulling out the frequency components of the result.

Exercises

1. Download a copy of *Spectrogram* from

http://www.monumental.com/rshorne/gram.html

This is a freeware realtime audio frequency analysing program for a PC running Windows 95/98/ME. Plug a microphone into the audio card on your PC and use this program to watch the frequency spectrum of sounds such as your voice, any musical instruments you may have around, and so on.

2. Find $\int_{-\infty}^{\infty} e^{-x^2}\,dx$.
[Hint: Square the integral and convert to polar coordinates, as in the proof of Theorem 2.13.1]

3. Show that if a is a constant then the Fourier transform of $f(at)$ is $\frac{1}{a}\hat{f}(\frac{\nu}{a})$.

4. Show that if a is a constant then the Fourier transform of $f(t-a)$ is $e^{-2\pi ia\nu}\hat{f}(\nu)$.

2.14. The Poisson summation formula

When we come to study digital music in Chapter 7, we shall need to use the Poisson summation formula.

THEOREM 2.14.1 (Poisson's summation formula).

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n). \tag{2.14.1}$$

PROOF. Define

$$g(\theta) = \sum_{n=-\infty}^{\infty} f\left(\frac{\theta}{2\pi} + n\right).$$

[19] D. Gabor, *Theory of communication*, J. Inst. Electr. Eng. 93 (1946), 429–457.

Then the left hand side of the desired formula is $g(0)$. Furthermore, $g(\theta)$ is periodic with period 2π, $g(\theta+2\pi)=g(\theta)$. So we may apply the theory of Fourier series to $g(\theta)$. By equation (2.6.1), we have

$$g(\theta)=\sum_{n=-\infty}^{\infty}\alpha_n e^{in\theta}$$

and by equation (2.6.2), we have

$$\begin{aligned}\alpha_m &= \frac{1}{2\pi}\int_0^{2\pi} g(\theta)e^{-im\theta}\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\sum_{n=-\infty}^{\infty} f\left(\frac{\theta}{2\pi}+n\right)e^{-im\theta}\,d\theta \\ &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_0^{2\pi} f\left(\frac{\theta}{2\pi}+n\right)e^{-im\theta}\,d\theta \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f\left(\frac{\theta}{2\pi}\right)e^{-im\theta}\,d\theta \\ &= \int_{-\infty}^{\infty} f(t)e^{-2\pi imt}\,dt \\ &= \hat{f}(m).\end{aligned}$$

The third step above consists of piecing together the real line from segments of length 2π. The fourth step is given by the substitution $\theta=2\pi t$. Finally, we have

$$\sum_{n=-\infty}^{\infty} f(n)=g(0)=\sum_{n=-\infty}^{\infty}\alpha_n=\sum_{n=-\infty}^{\infty}\hat{f}(n). \qquad \square$$

2.15. The Dirac delta function

Dirac's delta function $\delta(t)$ is defined by the following properties:

(i) $\delta(t)=0$ for $t\neq 0$, and

(ii) $\displaystyle\int_{-\infty}^{\infty}\delta(t)\,dt=1$.

Think of $\delta(t)$ as being zero except for a spike at $t=0$, so large that the area under it is equal to one. The awake reader will immediately notice that these properties are contradictory. This is because changing the value of a function at a single point does not change the value of an integral, and the function is zero except at one point, so the integral should be zero. Later in this section, we'll explain the resolution of this problem, but for the moment, let's continue as though there were no problem, and as though equations (2.13.1) and (2.13.3) work for functions involving $\delta(t)$.

It is often useful to shift the spike in the definition of the delta function to another value of t, say $t = t_0$, by using $\delta(t - t_0)$ instead of $\delta(t)$. The fundamental property of the delta function is that it can be used to pick out the value of another function at a desired point by integrating. Namely, if we want to find the value of $f(t)$ at $t = t_0$, we notice that $f(t)\delta(t-t_0) = f(t_0)\delta(t-t_0)$, because $\delta(t - t_0)$ is only nonzero at $t = t_0$. So

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)\,dt = \int_{-\infty}^{\infty} f(t_0)\delta(t - t_0)\,dt = f(t_0)\int_{-\infty}^{\infty} \delta(t - t_0)\,dt = f(t_0).$$

Next, notice what happens if we take the Fourier transform of a delta function. If $f(t) = \delta(t - t_0)$ then by equation (2.13.1)

$$\hat{f}(\nu) = \int_{-\infty}^{\infty} \delta(t - t_0)e^{-2\pi i\nu t}\,dt = e^{-2\pi i\nu t_0}.$$

So the Fourier transform of $\frac{1}{2}(\delta(t - t_0) + \delta(t + t_0))$ is

$$\tfrac{1}{2}(e^{-2\pi i\nu t_0} + e^{2\pi i\nu t_0}) = \cos(2\pi\nu t_0)$$

(see equation (C.2)).

Conversely, if we apply the inverse Fourier transform (2.13.3) to the function $\hat{f}(\nu) = \delta(\nu - \nu_0)$, we obtain $f(t) = e^{2\pi i\nu_0 t}$. So we can think of the Dirac delta function concentrated at a frequency ν_0 as the Fourier transform of a complex exponential. Similarly, $\frac{1}{2}(\delta(\nu - \nu_0) + \delta(\nu + \nu_0))$ is the Fourier transform of a cosine wave $\cos(2\pi\nu_0 t)$ with frequency ν_0.

The relationship between Fourier series and the Fourier transform can be made more explicit in terms of the delta function. Suppose that $f(t)$ is a periodic function of t of the form $\sum_{n=-\infty}^{\infty} \alpha_n e^{in\theta}$ (see equation (2.6.1)) where $\theta = 2\pi\nu_0 t$. Then we have

$$\hat{f}(\nu) = \sum_{n=-\infty}^{\infty} \alpha_n\delta(\nu - n\nu_0).$$

So the Fourier transform has a spike at plus and minus each frequency component, consisting of a delta function multiplied by the amplitude of that frequency component.

So what kind of a function is $\delta(t)$? The answer is that it really isn't a function at all, it's a *distribution*, sometimes also called a *generalized function.* A distribution is only defined in terms of what happens when we multiply by a function and integrate. Whenever a delta function appears, there is an implicit integration lurking in the background.

More formally, one starts with a suitable space of *test functions*,[20] and a distribution is defined as a continuous linear map from the space of test functions to the complex numbers (or the real numbers, according to context).

A function $f(t)$ can be regarded as a distribution, namely we identify it with the linear map taking $g(t)$ to $\int_{-\infty}^{\infty} f(t)g(t)\,dt$, as long as this makes sense. The delta function is the distribution which corresponds to the linear map taking a test function $g(t)$ to $g(0)$. It is easy to see that this distribution does not come from an ordinary function in the above way. The argument is given at the beginning of this section.

There is one warning that must be stressed at this stage. Namely, it does not make sense to multiply distributions. So for example, the square of the delta function does not make sense as a distribution. After all, what would $\int_{-\infty}^{\infty} \delta(t)^2 g(t)\,dt$ be? It would have to be $\delta(0)g(0)$, which isn't a number!

However, distributions can be multiplied by functions. The value of a distribution times $f(t)$ on $g(t)$ is equal to the value of the original distribution on $f(t)g(t)$.

To illustrate how to manipulate distributions, let us find $t\frac{d}{dt}(\delta(t))$. Integration by parts shows that if $g(t)$ is a test function, then

$$\int_{-\infty}^{\infty} t\frac{d}{dt}(\delta(t))g(t)\,dt = -\int_{-\infty}^{\infty} \delta(t)\frac{d}{dt}(tg(t))\,dt = -\int_{-\infty}^{\infty} \delta(t)(tg'(t)+g(t))\,dt.$$

Now $t\delta(t)=0$, so this gives $-f(0)$. If two distributions take the same value on all test functions, they are by definition the same distribution. So we have

$$\boxed{t\frac{d}{dt}(\delta(t)) = -\delta(t).}$$

The reader should be warned, however, that extreme caution is necessary when playing with equations of this kind.

It is also useful at this stage to go back to the proof of Fejér's theorem give in §2.7. Basically, the reason why this proof works is that the functions $K_m(y)$ are finite approximations to the distribution $2\pi\delta(y)$. Approximations to delta functions, used in this way, are called *kernel functions*, and they play a very important role in the theory of partial differential equations, analogous to the role they play in the proof of Fejér's theorem.

[20] In the context of the theory of Fourier transforms, it is usual to start with the *Schwartz space* $\mathcal{S}$ consisting of infinitely differentiable functions $f(t)$ with the property that there is a bound not depending on m and n for the value of any derivative $f^{(m)}(t)$ times any power t^n of t $(m,n\geq 0)$. So these functions are very smooth and all their derivatives tend to zero very rapidly as $|t|\to\infty$. An example of a function in $\mathcal{S}$ is the function e^{-t^2}. The sum, product and Fourier transform of functions in $\mathcal{S}$ are again in $\mathcal{S}$. For the purpose of saying what it means for a linear map on $\mathcal{S}$ to be continuous, the distance between two functions $f(t)$ and $g(t)$ in $\mathcal{S}$ is defined to be the largest distance between the values of $t^n f^{(m)}(t)$ and $t^n g^{(m)}(t)$ as m and n run through the nonnegative integers. The space of distributions defined on $\mathcal{S}$ is written $\mathcal{S}'$. Distributions in $\mathcal{S}'$ are called *tempered* distributions.

Exercises

1. Find the Fourier transform of the sine wave $f(t) = \sin(2\pi\nu_0 t)$ in terms of the Dirac delta function.

2. Show that if C is a constant then

$$\delta(Ct) = \frac{1}{|C|}\delta(t).$$

3. The Heaviside function $H(t)$ is defined by

$$H(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0. \end{cases}$$

Prove that the derivative of $H(t)$ is equal to the Dirac delta function $\delta(t)$. [Hint: Use integration by parts]

4. Show that $t\delta(t) = 0$.

Further reading:

F. G. Friedlander and M. Joshi, *Introduction to the theory of distributions*, second edition, CUP, 1998.

2.16. Convolution

The Fourier transform does not preserve multiplication. Instead, it turns it into *convolution*. If $f(t)$ and $g(t)$ are two functions, their convolution $f * g$ is defined by

$$(f * g)(t) = \int_{-\infty}^{\infty} f(s)g(t-s)\,ds.$$

The corresponding verb is to *convolve* the function f with the function g. It is easy to check the following properties of convolution.

(i) (commutativity) $f * g = g * f$.

(ii) (associativity) $(f * g) * h = f * (g * h)$.

(iii) (distributivity) $f * (g + h) = f * g + f * h$.

(iv) (identity element) $\delta * f = f * \delta = f$.

Here, δ denotes the Dirac delta function.

THEOREM 2.16.1. (i) $\widehat{f * g}(\nu) = \hat{f}(\nu)\hat{g}(\nu)$,
(ii) $\widehat{fg}(\nu) = (\hat{f} * \hat{g})(\nu)$.

PROOF. To prove part (i), from the definition of convolution we have

$$\begin{aligned} \widehat{f * g}(\nu) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s)g(t-s)e^{-2\pi i\nu t}\,ds\,dt \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s)g(u)e^{-2\pi i\nu(s+u)}\,ds\,du \end{aligned}$$

$$= \left(\int_{-\infty}^{\infty} f(s)e^{-2\pi i\nu s}\,ds\right)\left(\int_{-\infty}^{\infty} g(u)e^{-2\pi i\nu u}\,du\right)$$
$$= \hat{f}(\nu)\hat{g}(\nu).$$

Here, we have made the substitution $u = t - s$. Part (ii) follows from part (i) by the Fourier inversion formula (2.13.3); in other words, by reversing the roles of t and ν. □

Part (i) of this theorem can be interpreted in terms of frequency filters. Applying a frequency filter to an audio signal is supposed to have the effect of multiplying the frequency distribution by a filter function. So in the time domain, this corresponds to convolving the signal with the inverse Fourier transform of the filter function.

The output of a filter is usually taken to depend only on the input at the current and previous times. Looking at the formula for convolution, this corresponds to the statement that the inverse Fourier transform of the filter function should be zero for negative values of its argument.

Further reading:

Curtis Roads, *Sound transformation by convolution*, appears as article 12 of Roads et al [**94**], pages 411–438.

2.17. Wavelets

The wavelet transform is a relative of the windowed Fourier transform, in which all possible time translations and dilations are applied to a given window, to give a function of two variables as the transform. The exponential functions used in the windowed Fourier transforms are no longer present, but in some sense they are replaced by the use of dilations on the windowing function.

To be more precise, a *wavelet* is a function $\psi(t)$ of a real variable t which satisfies the *admissibility condition*

$$0 < c_\psi < \infty$$

where c_ψ is the constant defined by

$$c_\psi = 2\pi \int_{-\infty}^{\infty} \frac{|\widehat{\psi}(\nu)|^2}{|\nu|}\,d\nu.$$

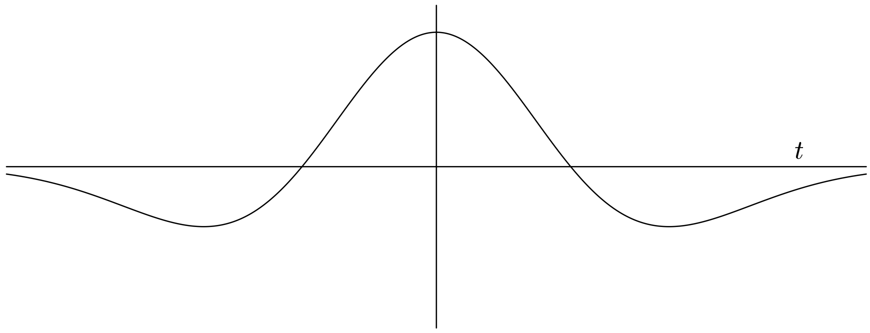
The wavelet ψ is chosen once and for all, and is interpreted as the shape of the window. The *wavelet transform* $L_\psi(f)$ of a waveform f is defined as the function of two variables

$$L_\psi(f)(a,b) = \frac{1}{\sqrt{|a|c_\psi}} \int_{-\infty}^{\infty} f(t)\psi\big(\frac{t-b}{a}\big)\,dt$$

for real $a \neq 0$ and b.

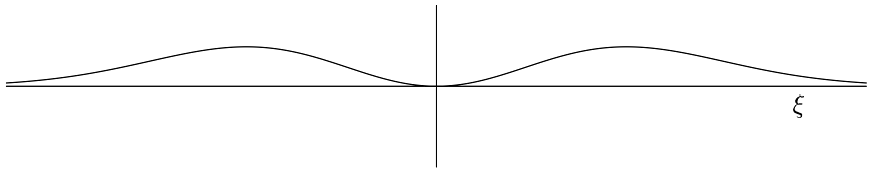
An example of a wavelet often used in practise is the *Mexican hat*, defined by

$$\psi(t) = (1 - t^2)e^{-t^2/2}.$$



The Fourier transform of the Mexican hat is

$$\widehat{\psi}(\nu) = \nu^2 e^{-\nu^2/2}$$



and we have $c_\psi = 1$.

The inverse wavelet transform L^*_ψ with respect to ψ is defined as follows. If $g(a,b)$ is a function of two real variables, then $L^*_\psi(g)$ is the function of the single real variable t defined by

$$L^*_\psi(g)(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{|a|c_\psi}} g(a,b)\psi\big(\frac{t-b}{a}\big)\,\frac{da\,db}{a^2}.$$

Note that at $a = 0$ the integrand is not defined, so the integral with respect to a simply misses out this value.

THEOREM 2.17.1. *If $f(t)$ is a square integrable function of a real variable t then $L^*_\psi L_\psi f$ agrees with f at almost all values of t, and in particular, at all points where $f(t)$ is continuous.*

Further reading:

G. Evangelista, *Wavelet representations of musical signals*, appears as article 4 in Roads et al [**94**], pages 127–154.

R. Kronland–Martinet, *The wavelet transform for the analysis, synthesis, and processing of speech and music sounds*, Computer Music Journal 12 (4) (1988), 11–20.

A. K. Louis, P. Maaß and A. Rieder, *Wavelets, theory and applications*, Wiley, 1997. ISBN 0471967920.

Stéphane Mallat, *A wavelet tour of signal processing*, Academic Press, 1998. ISBN 0124666051.

P. Polotti and G. Evangelista, *Fractal additive synthesis via harmonic-band wavelets*, Computer Music Journal 25 (3) (2001), 22–37.

Curtis Roads, *The computer music tutorial* [**93**], pages 581–589.