Random matrix ensembles for quantum decoherence

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0 . Purpose of this talk:

- Present a very simple toy model
- Based on very standard ideas:
  - spin and coherent states *Takahashi & Shibata (1975)*
  - random matrix hamiltonians *Mello, Pereyra & Kumar (1988), Lutz & Weidenmuller (1999), etc.*
  - which have been much applied for the spin 1/2 (Q-bit, 2 level system)
    MPK (1988), Esposito & Gaspard (2003), Lebowitz, Pastur & Lytova (2004 & 2007), Struntz, Haake & Braun (2002), etc.
- But some (relatively) novel aspects
  - general spin j (from quantum to classical spin)
  - generic interaction (novel random matrix ensembles)
- It allows to study analytically several aspects decoherence
- In particular the crossover between unitary quantum dynamics and stochastic diffusion in classical phase space for the spin

1 . The model

A quantum SU(2) spin $\mathcal{S}$ + an external system $\mathcal{E}$ $\quad \mathcal{H} = \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{E}$

spin $= j \qquad \dim(\mathcal{H}_\mathcal{S}) = 2j+1 \qquad \dim(\mathcal{H}_\mathcal{E}) = N \gg j$

Single spin:

For large spin $j \to \infty$ the spin becomes a classical object
Classical phase space is the 2-sphere
The coherent states behave as quasi classical states

$$|\vec{n}\rangle \quad , \quad (\vec{n}\cdot\vec{\mathbf{S}})|\vec{n}\rangle = j|\vec{n}\rangle$$

Dynamics of the coupled spin:

$$H = H_\mathcal{S} \otimes \mathbf{1}_\mathcal{E} + H_{\mathcal{S}\mathcal{E}} + \mathbf{1}_\mathcal{S} \otimes H_\mathcal{E}$$

The Hamiltonians:

- Slow spin dynamics $\quad H_\mathcal{S} = 0$
  (no dissipative & thermalisation effects)
- Dynamic of the external system generic $\quad H_\mathcal{E} \to H_{\mathcal{S}\mathcal{E}}$

The interaction Hamiltonian

The interaction hamiltonian is given by a Gaussian random matrix ensemble, with the only constraint that the ensemble in invariant under

$$\underbrace{SU(2)}_{\text{spin}} \times \underbrace{U(N)}_{\text{external system}}$$

For this, go to Wigner representation of spin operators

$$\langle r\alpha|H|s\beta\rangle = H^{rs}_{\alpha\beta} \to W^{(lm)}_{\alpha\beta} \qquad \mathbf{j}\otimes\mathbf{j} = \mathbf{0}\oplus\mathbf{1}\oplus\cdots\oplus\mathbf{2j}$$

$$A_{rs} = \langle r|A|s\rangle \qquad W^{(l,m)}_{A} = \sum_{r,s=-j}^{j} \sqrt{\frac{2l+1}{2j+1}} \left\langle \begin{matrix} j & l \\ r & m \end{matrix} \middle| \begin{matrix} j \\ s \end{matrix} \right\rangle A_{rs}$$

It is enough to take for the $W^{(lm)}_{\alpha\beta}$ independent gaussian random variables with zero mean and variance depending only on l and with the Hermiticity constraint.

$$\text{Var}\left(\mathrm{Re}|\mathrm{Im}(W^{(lm)}_{\alpha\beta})\right) = \Delta(l) \qquad W^{(l,m)}_{\alpha\beta} = (-1)^m \overline{W}^{(l,-m)}_{\beta\alpha}$$

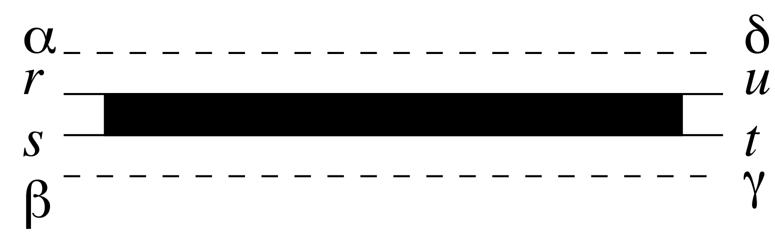
We thus get a matrix ensemble characterized by the variances

$$\Delta = \{\Delta(l)\,,\ l = 0, 1, \cdots 2j\}$$

NB: The *l=m=0* term represents the $H_{\mathcal{E}}$ Hamiltonian

The 2-points correlator is the average over this «G*U(2)xU(N)*E» matrix ensemble, and is

$$\overline{H^{rs}_{\alpha\beta} H^{tu}_{\gamma\delta}} = \delta_{\alpha\delta}\delta_{\beta\gamma}\mathcal{D}_{rs,tu}$$



$$\mathcal{D}_{rs,tu} = \quad \delta_{s-r,t-u} \sum_{l=0}^{2j} \Delta(l) \frac{2l+1}{2j+1} \left\langle \begin{matrix} j & l \\ s & r-s \end{matrix} \middle| \begin{matrix} j \\ r \end{matrix} \right\rangle \left\langle \begin{matrix} j & l \\ t & u-t \end{matrix} \middle| \begin{matrix} j \\ u \end{matrix} \right\rangle$$

It can be represented by a standard ribbon propagator for the N indices, with a more complicated structure for the spin indices, but still planar.

2 . The evolution functional

separable state $\to$ entangled state $\to$ mixed state for $\mathcal{S}$

$$|\psi_0\rangle \otimes |\phi_0\rangle \to |\Phi(t)\rangle \ , \quad \rho_{\mathcal{S}}(t) = \mathrm{tr}_{\mathcal{E}}(|\Phi(t)\rangle\langle\Phi(t)|$$

Evolution functional *(POVM, completely positive map, ...)*

$$\rho_{\mathcal{S}}(t) = \mathcal{M}(t)\cdot\rho_{\mathcal{S}}(0) \ , \quad \mathcal{M}(t)\cdot\star = \mathrm{tr}_{\mathcal{E}}\left(e^{-itH}\left(\star\otimes\rho_{\mathcal{E}}(0)\right)e^{itH}\right)$$

For simplicity, start from a random state $|\psi_{\mathrm{E}}\rangle$

Then the evolution functional is

$$\mathcal{M}(t) = \oint\frac{dx}{2i\pi}\oint\frac{dy}{2i\pi}\ e^{it(x-y)}\,\mathcal{G}(x,y)$$

$$\mathcal{G}(x,y) = \frac{1}{N}\mathrm{tr}_{\mathcal{E}}\left[\frac{1}{x-H}\otimes_{\mathcal{S}}\cdot_{\mathcal{E}}\frac{1}{y-H}\right]$$

tensor product on $\mathcal{H}_{\mathcal{S}}$

ordinary product on $\mathcal{H}_{\mathcal{E}}$

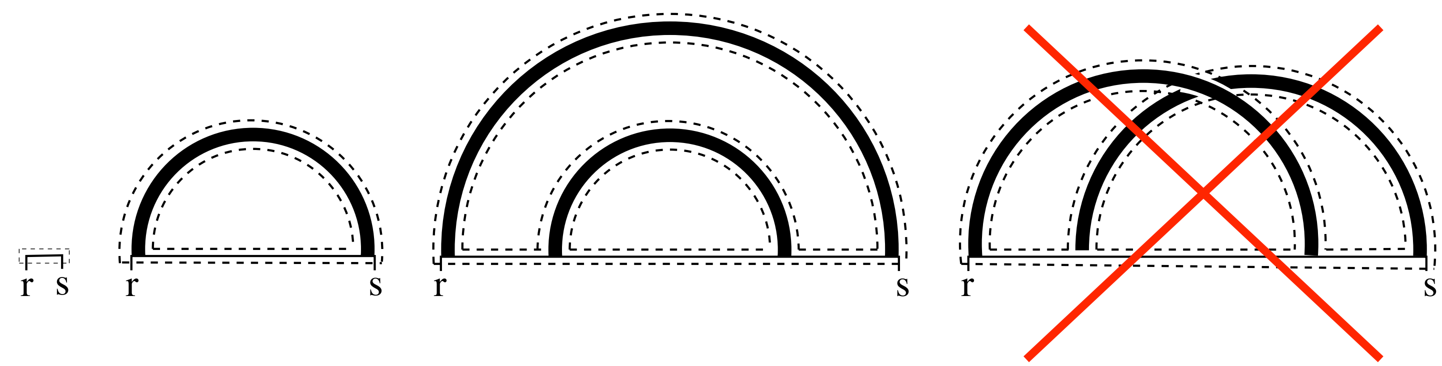
!!!

We take the large N limit (large external system) and make the average over H, assuming self averaging as usual.

It is useful to start from the single resolvent

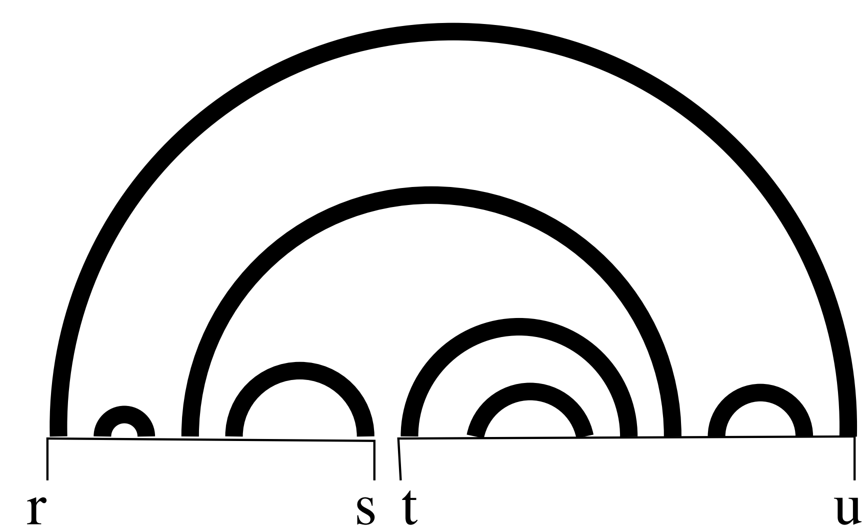
$$\mathcal{H}(x) = \frac{1}{N}\mathrm{tr}_{\mathcal{E}}\left[\frac{1}{x-H}\right]$$

$\overline{\mathcal{H}(x)}$ is given by a sum of planar rainbow diagrams



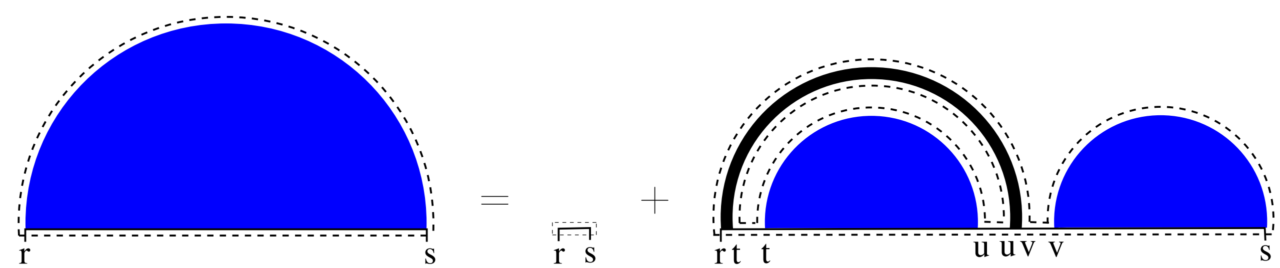
+ ....

$\overline{\mathcal{G}(x,y)}$ is also given by a sum of planar diagrams of the standard form



These resolvents obey recursion relations

Thanks to the SU(2) invariance, the solution of these equations takes a simple diagonal form in the Wigner representation



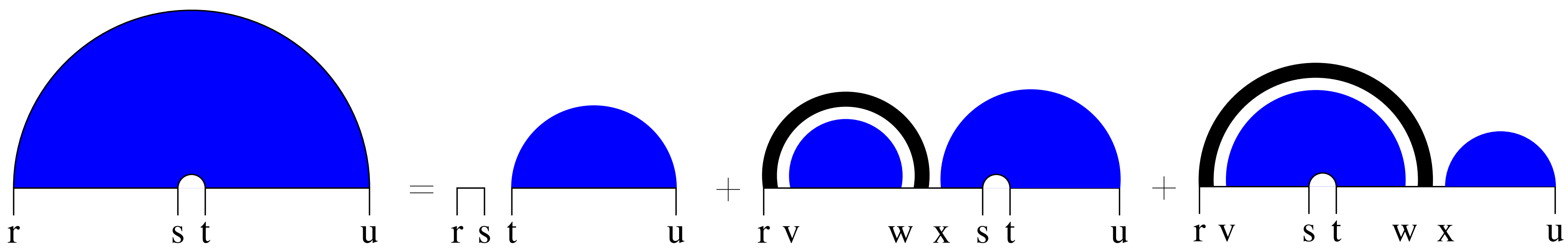
$$\overline{\mathcal{H}}_{rs}(x) = \delta_{rs}\widehat{\mathcal{H}}(x)$$

with

$$\widehat{\mathcal{H}}(x) = \frac{1}{2\widehat{\Delta}(0)}\left(x - \sqrt{x^2 - 4\widehat{\Delta}(0)}\right)$$

$$\widehat{\Delta}(0) = N\sum_{l=0}^{2j}\frac{2l+1}{2j+1}\,\Delta(l)$$

Resolvent for a single Wigner matrix (semi circle law)

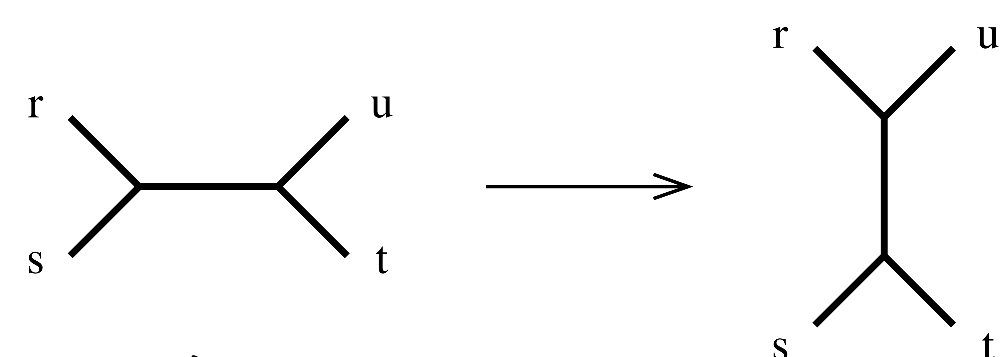


$$\overline{\mathcal{G}}_{rs,tu}(x,y) \rightarrow \overline{\mathcal{W}}_{\mathcal{G}}^{(l_1,m_1),(l_2,m_2)}(x,y) = \delta_{l_1 l_2}\delta_{m_1+m_2,0}\,(-1)^{m_1}\widehat{\mathcal{G}}^{(l)}(x,y)$$

and

$$\widehat{\mathcal{G}}^{(l)}(x,y) = \frac{\widehat{\mathcal{H}}(x)\widehat{\mathcal{H}}(y)}{1-\widehat{\Delta}(l)\widehat{\mathcal{H}}(x)\widehat{\mathcal{H}}(y)}$$

with a bit of $\mathrm{SU}(2)$ algebra



$$\widehat{\Delta}(l) = N\sum_{l'=0}^{2j}\Delta(l')(2l'+1)(-1)^{2j+l'+l}\begin{Bmatrix} j & j & l' \\ j & j & l \end{Bmatrix}$$

6-j symbol

The evolution functional for the density matrix of the spin $\rho_{\mathcal{S}}(t)$ takes a simple diagonal form in the Wigner representation basis

$$\rho_{\mathcal{S}\,rs}(t) \;\to\; W_{\mathcal{S}}^{(l,m)}(t) = \widehat{\mathcal{M}}^{(l)}(t)\cdot W_{\mathcal{S}}^{(l,m)}(0)$$

with the kernel given by a universal decoherence function

$$\widehat{\mathcal{M}}^{(l)}(t) = M(t/\tau_0, Z(l))$$

depending on a rescaled time $t' = t/\tau_0$ and a factor $Z(l)$

$$\tau_0 = 1/\sqrt{\widehat{\Delta}(0)} \qquad\qquad Z(l) = \frac{\widehat{\Delta}(l)}{\widehat{\Delta}(0)}$$

τ_0 is the dynamical time scale of the system (more later)

The parameter $Z(l)$ depends on the spin sector l considered.

2.2 . The $Z(l)$ function

The l dependence of the factor $Z(l)$ depends on the initial variances of the $\mathrm{GU}(2)$ ensemble for the Hamiltonian.

$$\widehat{\Delta}(l) = N \sum_{l'=0}^{2j} \Delta(l')(2l'+1)(-1)^{2j+l'+l} \begin{Bmatrix} j & j & l' \\ j & j & l \end{Bmatrix} \longleftarrow \text{6-j symbol}$$

$$Z(l) = \widehat{\Delta}(l)/\widehat{\Delta}(0) \qquad\qquad Z(l) \in [-1, 1]$$

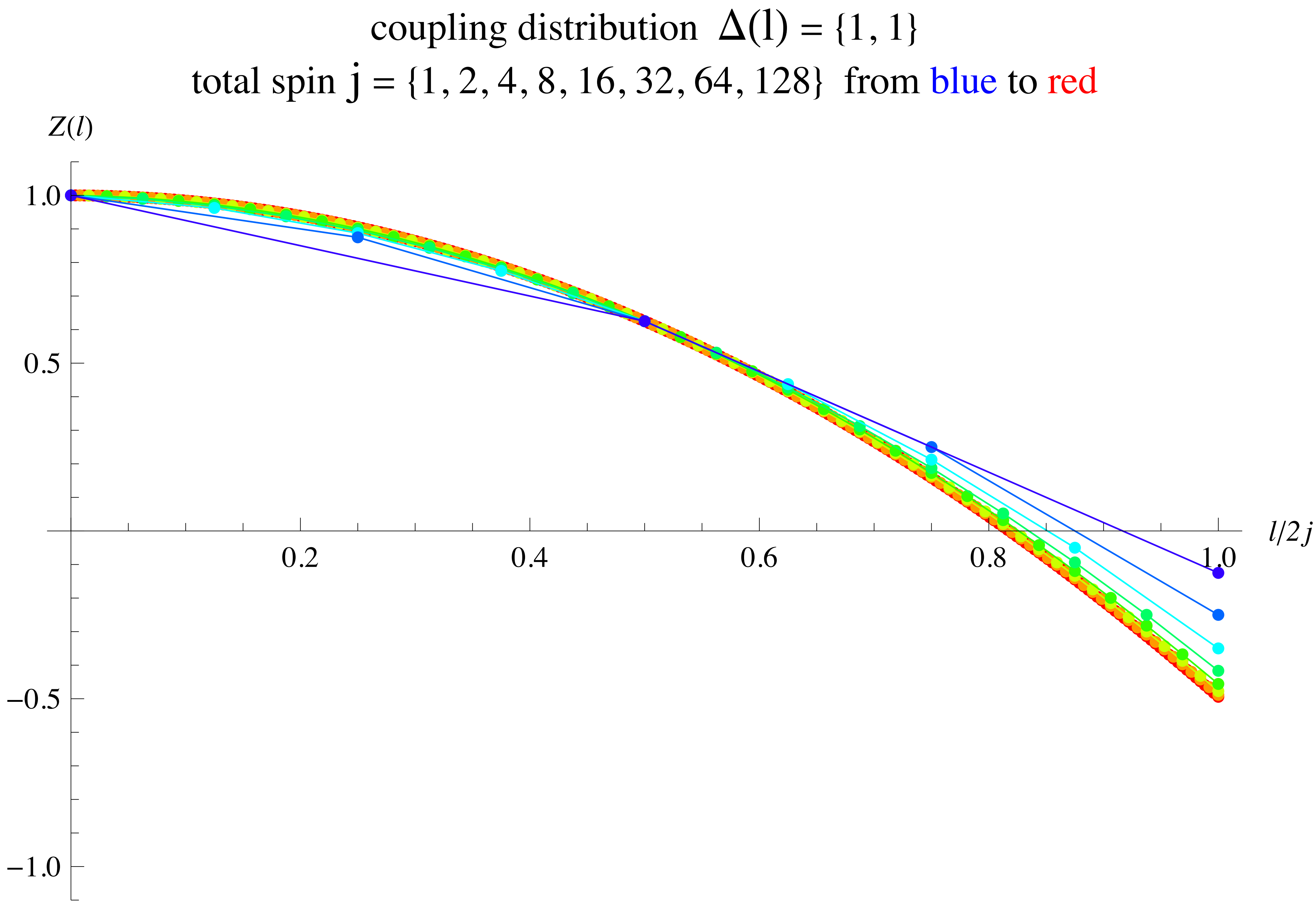
$Z(l)$ is maximal for $l=0$
$Z(l)$ takes a scaling form in the large spin limit

$$Z(l) = \widehat{\Delta}(l)/\widehat{\Delta}(0) \to Y(x) \text{ with } x = l/2j$$

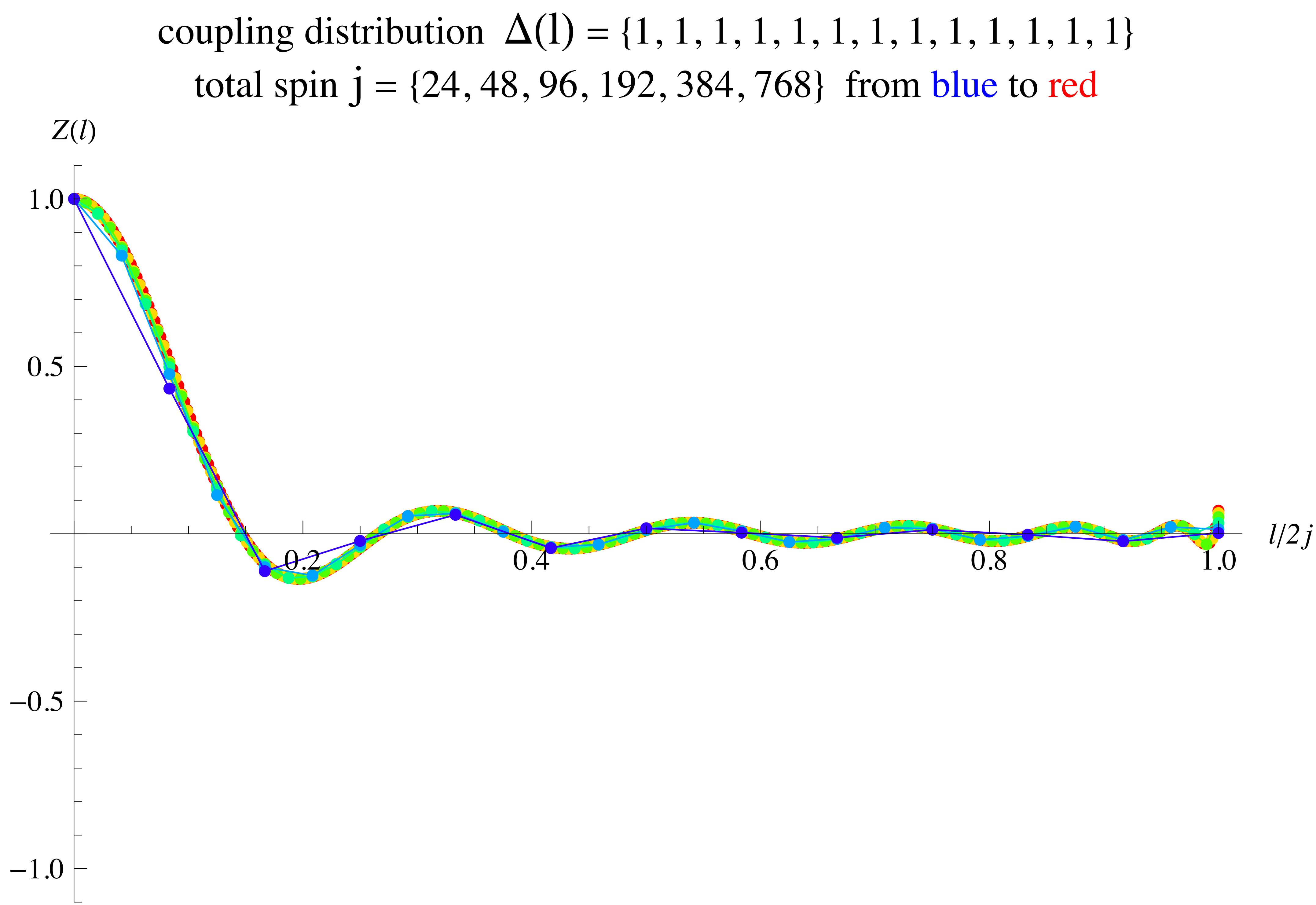
Its small l behavior is quadratic in l

$$Z(l) = 1 - l(l+1)\,\frac{1}{4}\frac{D_0}{j(j+1)} + \cdots \quad , \quad D_0 = \frac{\sum_{l'=1}^{l_0} \bar{\Delta}(l')\,(2l'+1)\,l'(l'+1)}{\sum_{l'=0}^{l_0} \bar{\Delta}(l')(2l'+1)}$$

Example 1: l=0 and 1 channels only



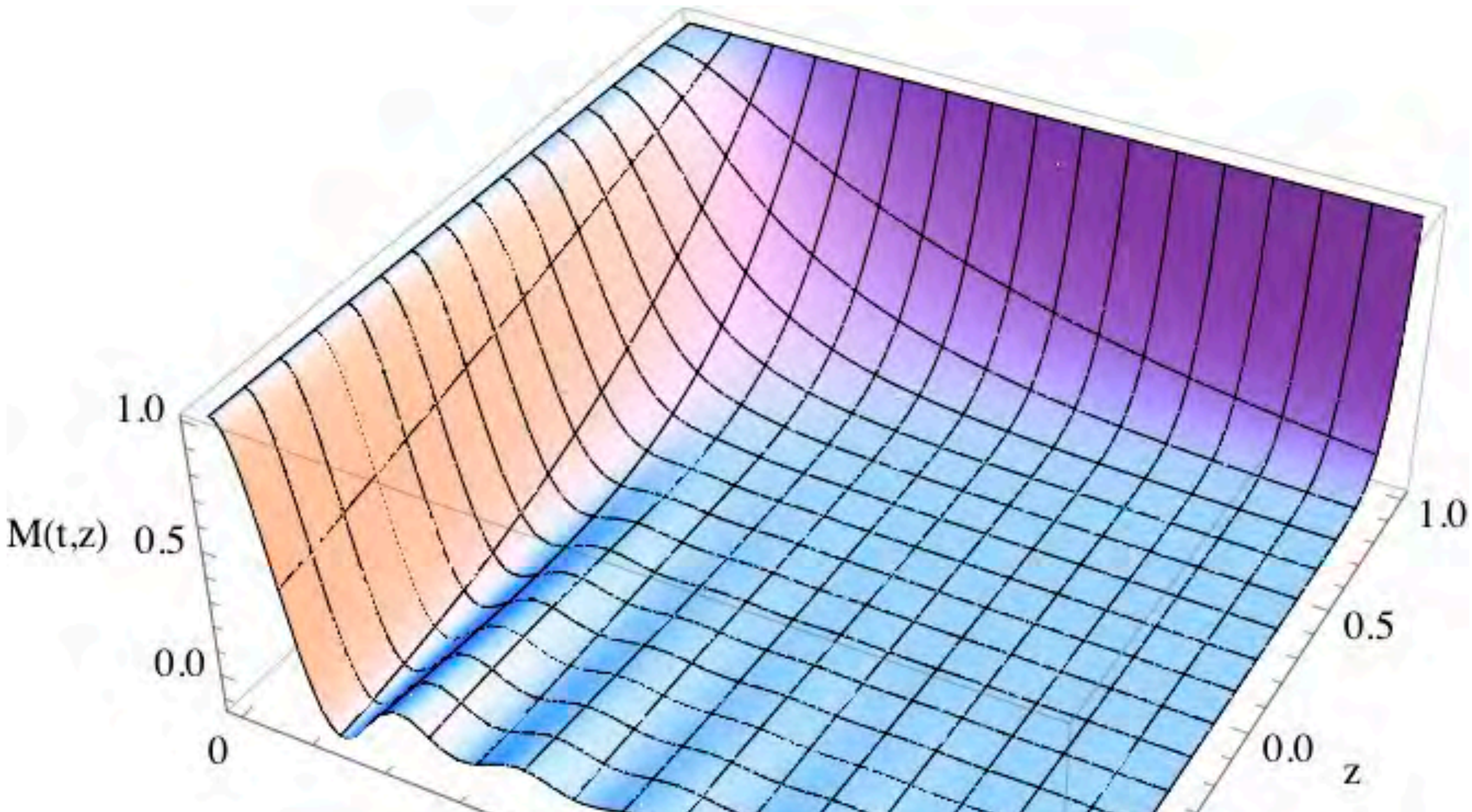
Example 2: l=0 to 12 channels



2.3 . The decoherence function (a generalized hypergeometric function)

$$M(t,Z)=\oint\frac{dx}{2\mathrm{i}\pi}\oint\frac{dy}{2\mathrm{i}\pi}\,\mathrm{e}^{-\mathrm{i}t(x-y)}\,\frac{H(x)H(y)}{1-Z\,H(x)H(y)}\quad,\qquad H(x)=\frac{1}{2}(x-\sqrt{x^2-4})$$

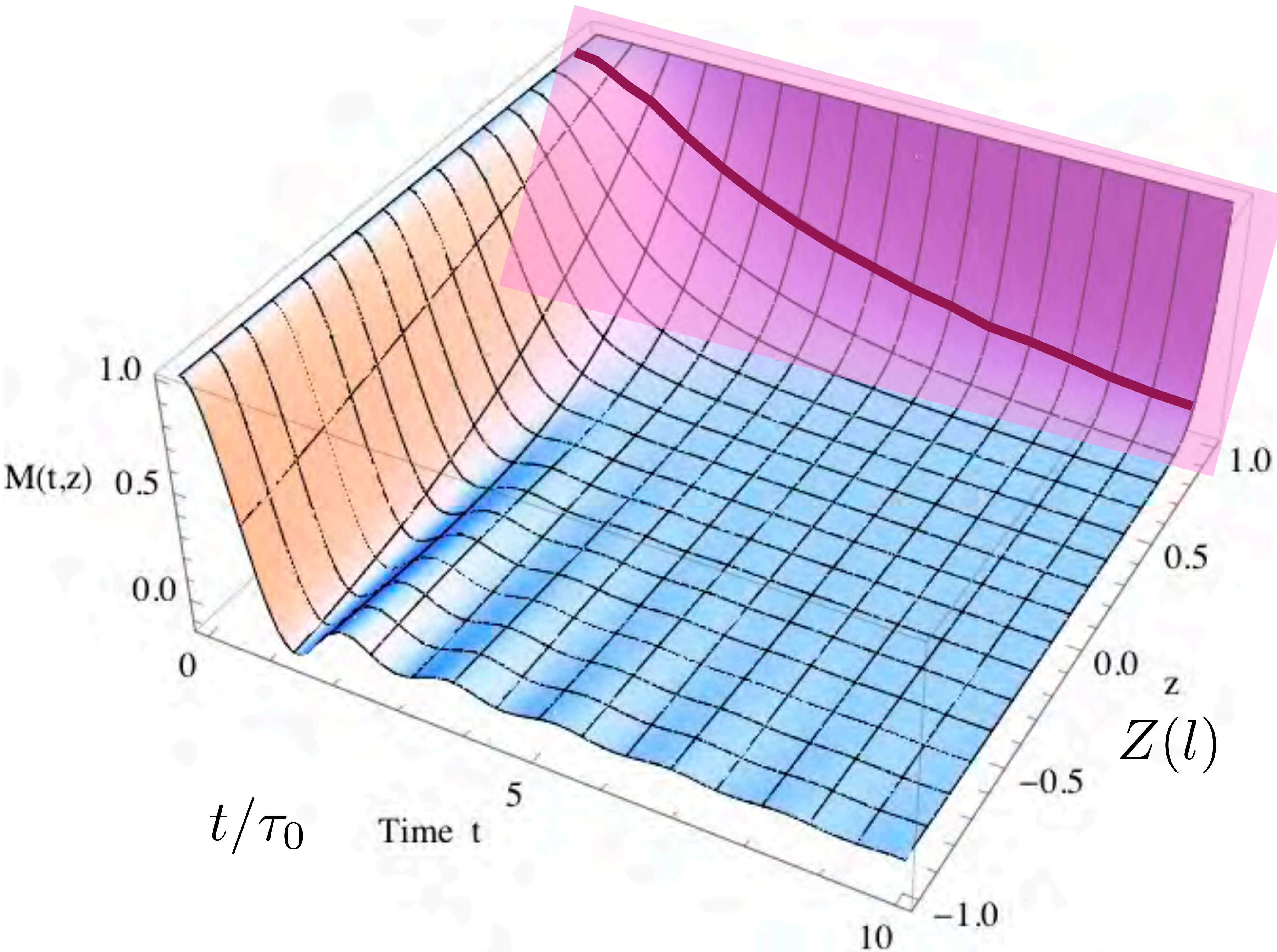
$$=\sum_{m=0}^{\infty}\sum_{n=0}^{m}t^{2m}\,z^{n}\,(-1)^{m+n}\,\frac{2(2m+1)(n+1)^2(2m)!}{m!(m+1)!(m-n)!(m+n+2)!}$$



large time limit: fast algebraic decay with t except for Z close to unity

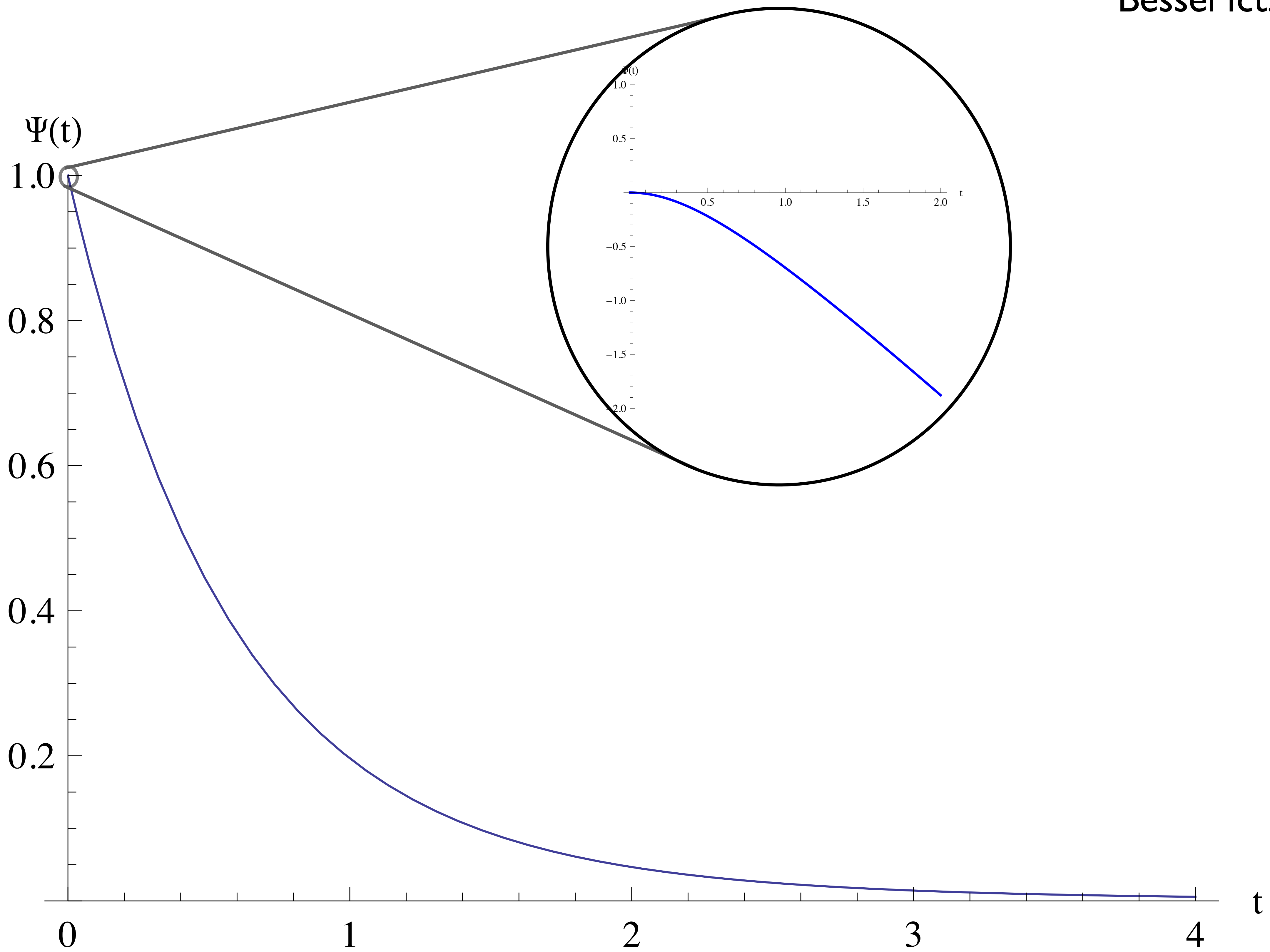
$$\boxed{M(t,z)=\frac{1}{2\pi}\,t^{-3}\left(\frac{1+z}{(1-z)^3}-\frac{1-z}{(1+z)^3}\sin(4t)\right)\left(1+\mathcal{O}(t^{-1})\right)}$$

$Z \to 1$ scaling $\quad M(t', z) = \Psi(t'')$ with $t'' = t'(1-z)$



$Z \to 1$ scaling function $\qquad \Psi(t'') \;=\; \dfrac{1}{2\pi}\displaystyle\int_{-2}^{2} dx\,\sqrt{4-x^2}\; e^{-t''\,\sqrt{4-x^2}}$

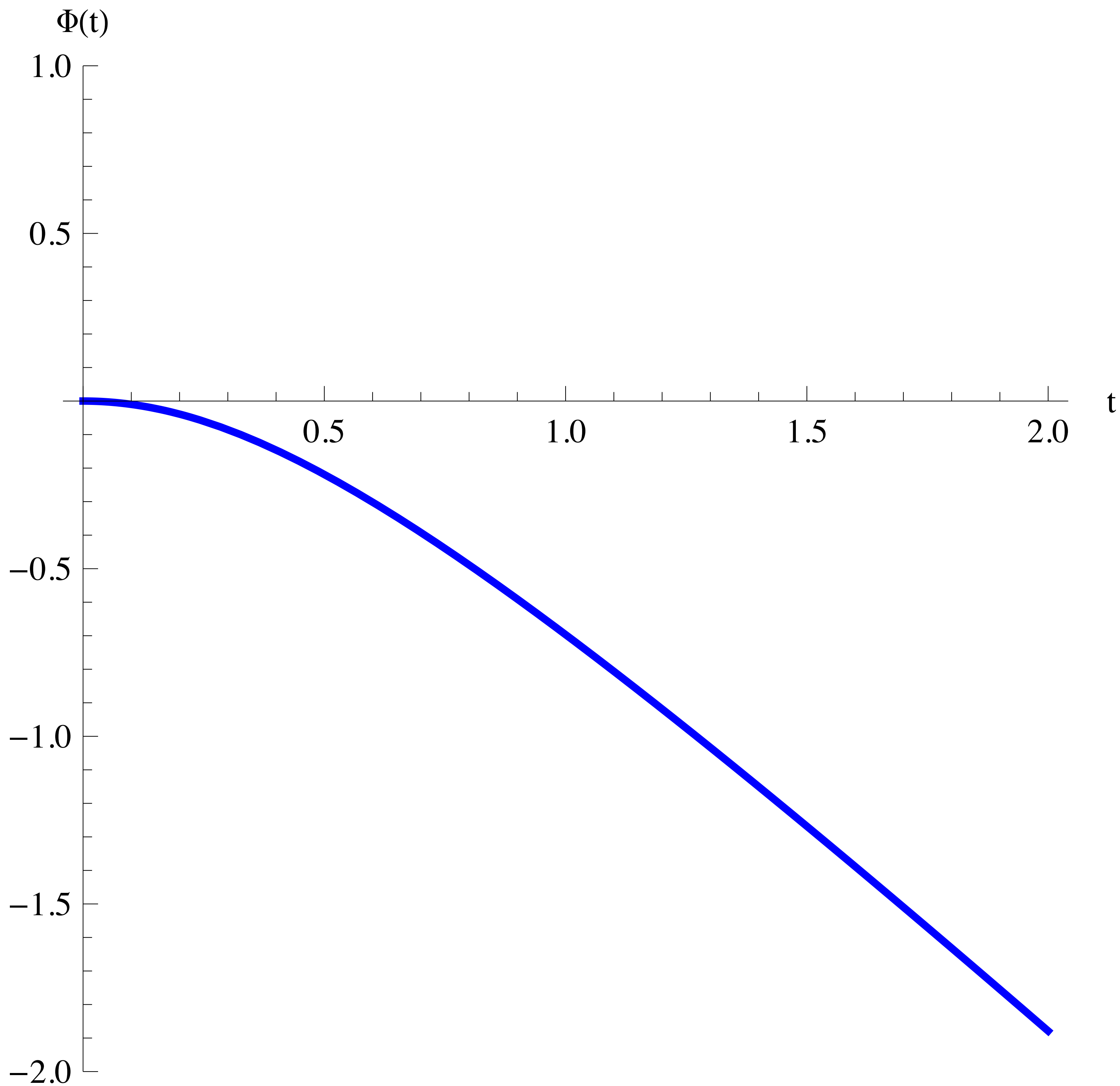
Bessel fct.



small t and $Z \simeq 1$ behavior

$$M(t,z) = 1 + (1-z)\,\Phi(t) + \cdots$$

$$\Phi(t) = 1 - {}_1F_2\left(-\tfrac{1}{2}; 1, 2; -4t^2\right)$$



3 . Evolution of coherent and incoherent states

We can easily study analytically and illustrate the evolution on the matrix density of the spin, starting from a pure spin state $|\psi\rangle$

$$|\psi\rangle \to \rho = |\psi\rangle\langle\psi| \to W^{(l,m)} \to W(\vec{n}) = \sum_{l,m} W^{(l,m)}\, Y_l^m(\vec{n})$$

Wigner distribution = function on the sphere

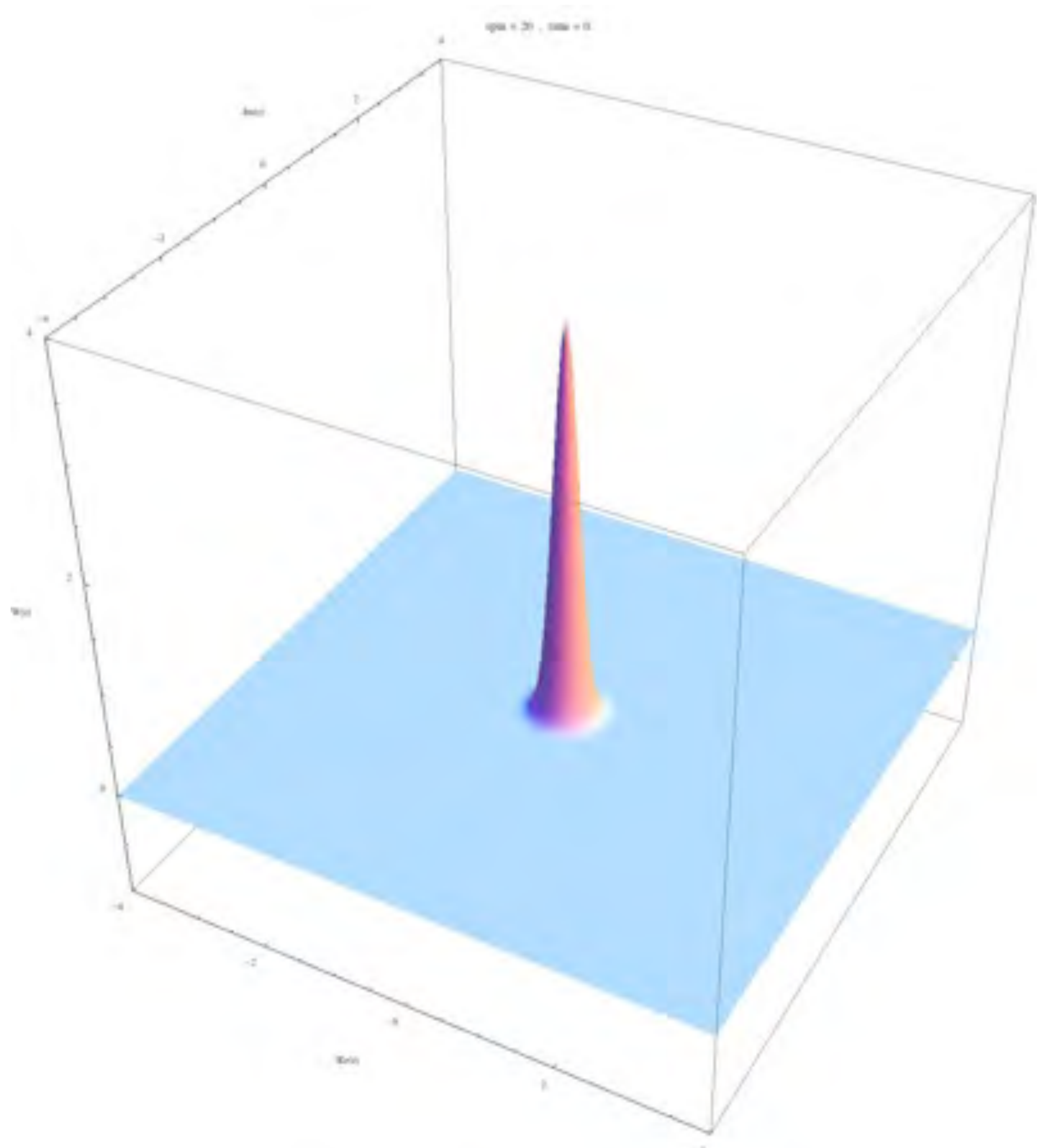
Coherent state

$$|\vec{n}\rangle = \sum_{m=-j}^{j} \sqrt{\frac{(2j)!}{(j+m)!\,(j-m)!}}\, \cos(\theta/2)^{j+m}\, \sin(\theta/2)^{j-m} \mathrm{e}^{-\mathrm{i}m\phi} |m\rangle$$

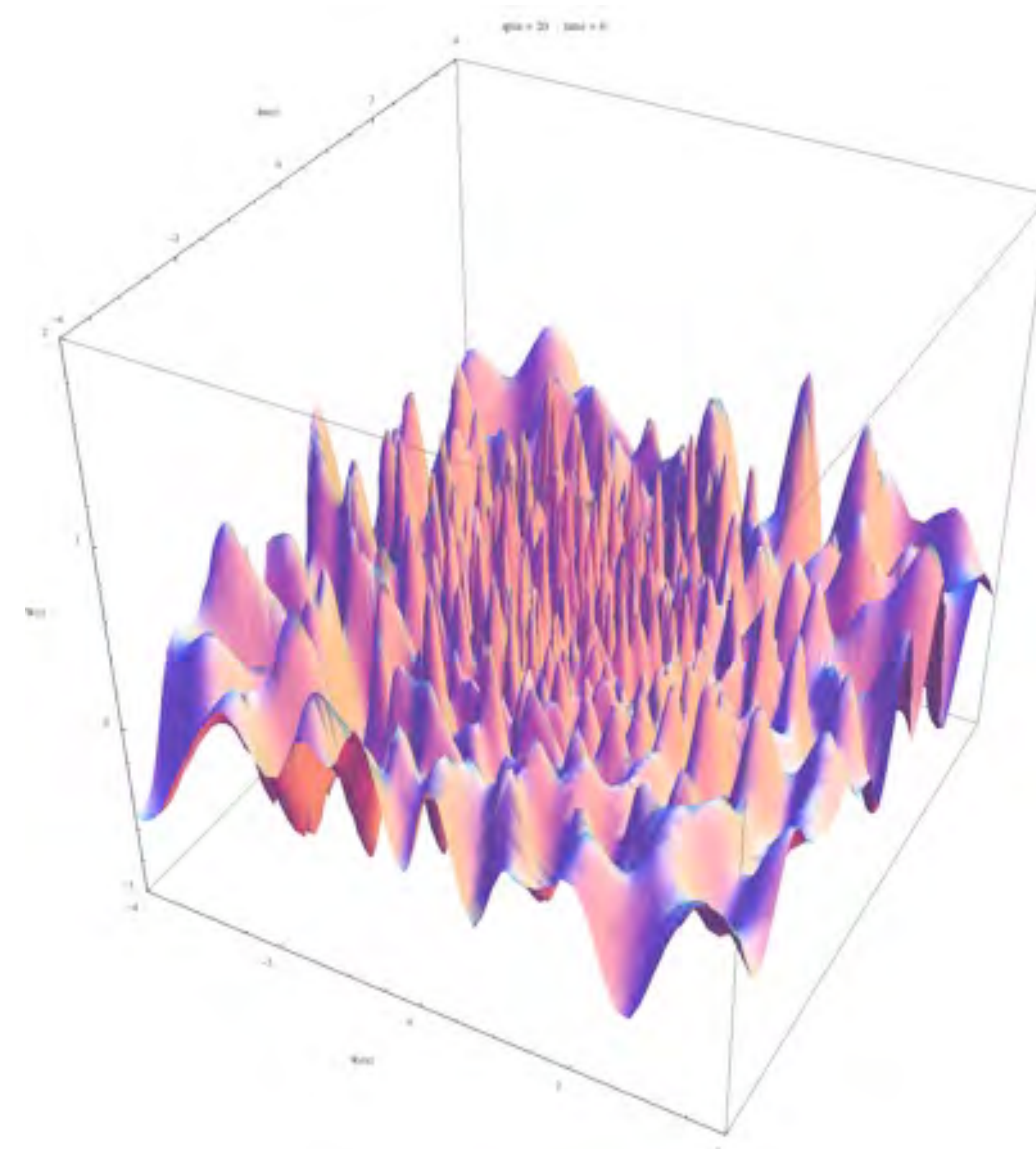
$$W^{(l)}_{\mathrm{c.s.}} = \frac{2l+1}{\sqrt{2j+1}} \exp\left(-\frac{l^2}{2j}\right) \qquad\qquad l \sim \sqrt{j}$$

Coherent states are the most localised states on the sphere

- Coherent states look like a Gaussian on the unit sphere with width $\Delta_\theta = 1/\sqrt{j}$
- Random states look like random functions on the unit sphere



coherent state



random state

stereographic projection and *j=20*

4. The time scales of decoherence

There are 4 time scales $\tau_0 \leq \tau_1 \ll \tau_2 \ll \tau_3$

τ_0 dynamical time scale for the whole system

τ_1 decoherence time scale for generic states $l \gg \sqrt{j}$

τ_2 evolution time scale for coherent states (onset of quantum diffusion)

τ_3 equilibration time for quantum diffusion

For our simple model with Gaussian Hamiltonian ensembles

$$\tau_0 = 1/\parallel H_{\mathcal{S}\mathcal{E}} + H_{\mathcal{E}} \parallel \qquad \frac{\tau_0}{\tau_1} = \left(\frac{\parallel H_{\mathcal{S}\mathcal{E}} \parallel}{\parallel H_{\mathcal{S}\mathcal{E}} + H_{\mathcal{E}} \parallel} \right)^2$$

$$\boxed{\frac{\tau_1}{\tau_2} = \left(\frac{\parallel [\vec{\mathbf{S}}, H_{\mathcal{S}\mathcal{E}}] \parallel}{\parallel \vec{\mathbf{S}} \parallel \, \parallel H_{\mathcal{S}\mathcal{E}} \parallel} \right)^2} \qquad \frac{\tau_2}{\tau_3} = \frac{1}{j}$$

$H_{\mathcal{E}} \leftarrow l = 0$ term

$H_{\mathcal{S}\mathcal{E}} \leftarrow l \neq 0$ terms

with the «L₂ norm» for operators $\parallel A \parallel^2 = \dfrac{\mathrm{tr}(A^\dagger A)}{\mathrm{tr}(1)}$

The ratio $\tau_2 \gg \tau_1$ is large iff the commutator $\left[\vec{S}, H_{\mathcal{S}\mathcal{E}}\right]$ is «small»

$$\left[\vec{S}, H_{\mathcal{S}\mathcal{E}}\right] \ll \vec{S} \times H_{\mathcal{S}\mathcal{E}}$$

Coherent states are robust against decoherence and play the role of pointer states if

$$\Delta(l) \neq 0 \text{ for } l \leq l_0 \text{ and } j \gg l_0^2$$

The dynamics of decoherence depends on the details of the Hamiltonian ensemble

$$\Delta = \{\Delta(l),\ l = 0, \cdots l_0\}$$

Beyond the decoherence time scale τ_1 , the dynamics of coherent states is much simpler and exhibit some universal features.

5 . Quantum diffusion

For $\tau_1 \ll t \ll \tau_2$ only semiclassical coherent states survive

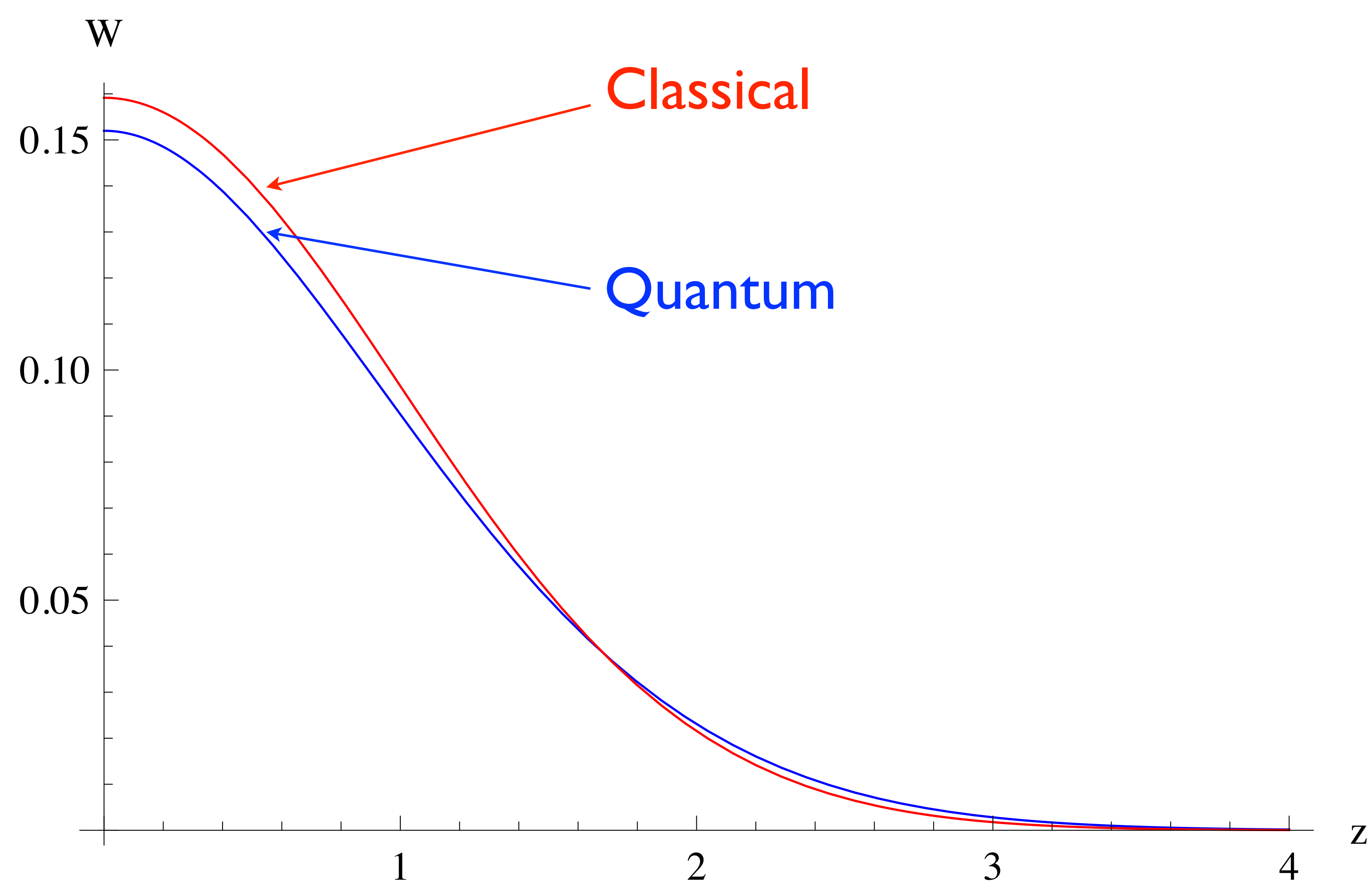
For $\tau_2 < t$ coherent states start to become mixed states $\quad j \gg 1$

This is an effect of quantum diffusion, i.e. the remaining weak effect of the external system on the coherent states.

The width of the distribution function in phase space is found to grow like $\Delta_\theta(t) \propto \sqrt{t}$

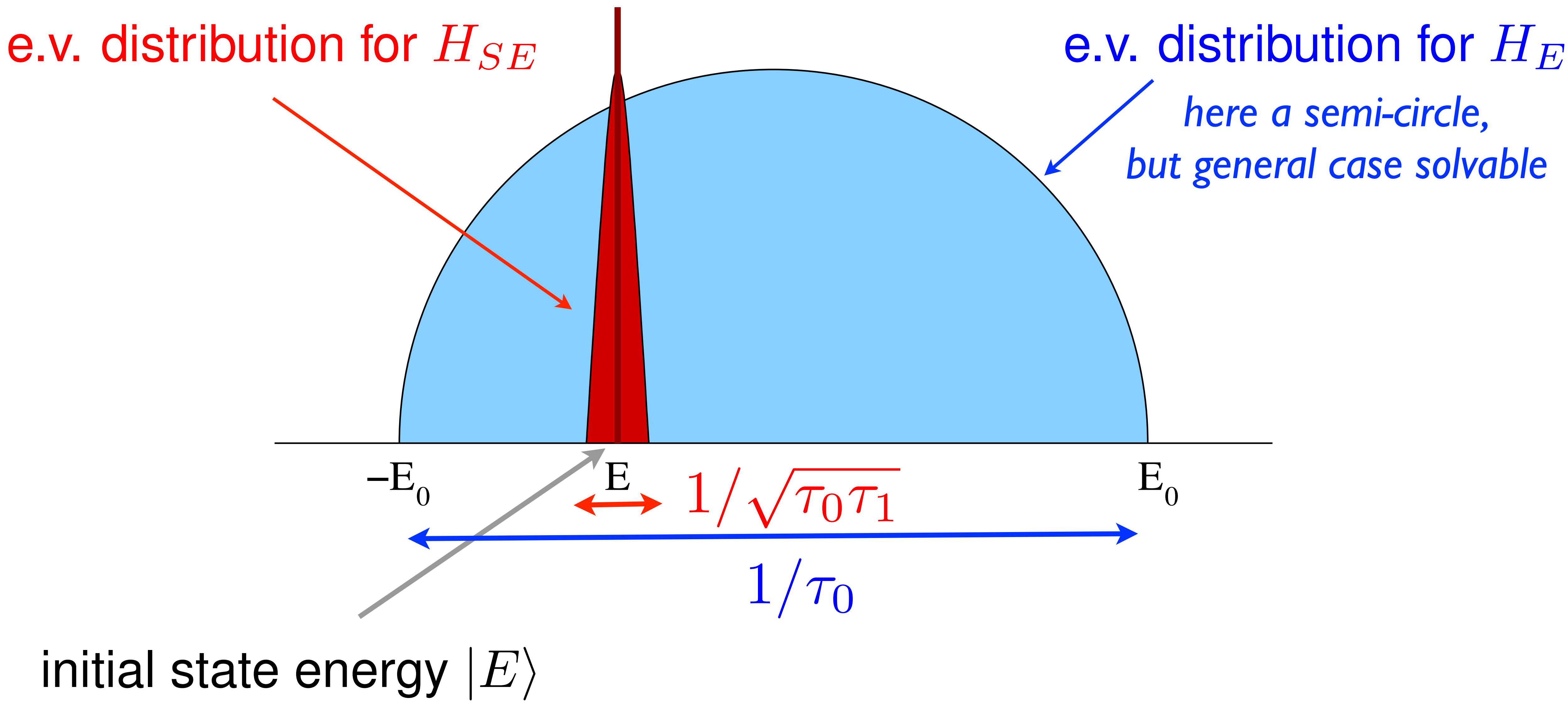
This suggests a random walk in phase space

But the probability profile can be computed and is not a Gaussian ! This is a signal that the evolution is not a Markovian short range process, even at large times!



V - Dynamics and initial conditions for $\mathcal{E}$

The calculation can be extended to a general Hamiltonian for the external system with a general eigenvalue distribution, and to a given initial state $|\phi_\mathcal{E}\rangle$ such as an energy eigenstate



The calculations and the explicit solutions are less simple

Spin part: now sum over the $l>0$ sectors

$$Z'(l)) = \frac{\hat{\Delta}'(l)}{\hat{\Delta}'(0)} \qquad \hat{\Delta}'(l) = N\hat{D}(l) = \sum_{l'=1}^{2j} \tilde{\Delta}(l')(2l'+1)(-1)^{2j+l'+l_1} \begin{Bmatrix} j & j & l' \\ j & j & l_1 \end{Bmatrix}$$

Random matrix part: now involves the Hilbert transform of the E d.o.s.

$$\tilde{C}(x) = \int \mathrm{d}E\, \frac{\nu(E)}{w-E} \qquad w = W(x) = x - \hat{\Delta}'\tilde{C}(x) \qquad \hat{\Delta}'(0) = \hat{\Delta}'$$

Decoherence function: depends on E (energy at $t=0$) and l (spin channel)

$$\hat{\mathcal{M}}^{(l)}(t,E) = \oint \frac{\mathrm{d}x_1}{2\mathrm{i}\pi} \oint \frac{\mathrm{d}x_2}{2\mathrm{i}\pi} \frac{\mathrm{e}^{-\mathrm{i}t(x_1-x_2)}}{(W(x_1)-E)(W(x_2)-E)} \times \frac{1}{(1-Z'(l)) + Z'(l)((x_1-x_2)/(W(x_1)-W(x_2)))}$$

We are led to the following conjecture

External system: fast dynamics + initial energy eigenstate

If $\tau_1 \gg \tau_0$ and if one starts from an energy eigenstate $|E\rangle$ then the diffusion is Markovian and the diffusion coefficient is

$$D_{\rm diff} = 2\pi\, \rho_{\mathcal{E}}(E)\, \overline{\left|\langle\Phi|[\vec{\mathbf{S}}, H_{\mathcal{S}\mathcal{E}}]|\Phi'\rangle\right|^2}$$

d.o.s. of the external system

typical size of a matrix element of the commutator

This is a Golden Rule formula

Not too surprising, one must be able to write a master equation for the evolution of the density matrix

If the initial state is a quantum superposition of energy eigenstates

$$|\phi_{\mathcal{E}}\rangle = \sum_E \phi(E)|E\rangle$$

we expect that the diffusive regime will be a randomisation of the collection of Markovian diffusion processes $\mathbb{P}(E)$

Each diffusion process $\mathbb{P}(E)$ is a RW with diffusion constant $D(E)$

$$|\vec{n}\rangle \otimes |E\rangle \xrightarrow{t} \sum_{\vec{n}'} \Psi(E,\vec{n}';t)\,|\vec{n}'\rangle \otimes |E,\vec{n}';\vec{n},t\rangle$$

The processes are taken with probability weight $W(E) = |\phi(E)|^2$

The $|E,\vec{n}';\vec{n},t\rangle$ are all ~ orthogonals

This reflects the decoherence between energy eigenstates (of the the external system) induced by the coupling with the large spin

Conclusion

A simple but rich model

A starting point to study more realistic physical models with interesting dynamics (works in progress/project)

- Dissipation and thermalisation (add a specific dynamics for the spin, e.g. external field).
- Study backreaction of the spin on the environment (decoherence in the environment by the spin)
- More physical models for the environment (bath of oscillators, of spins) and the couplings
- Relation with standard approximations used in open quantum systems: Master equations and Lindbladians, RWA (*Rotating Wave Approximation)*, TCL *(Time Convolutionless Limit*)
- Finite N effects, large N versus large j limits
- Multi-times functions (relation with quantum stochastic processes)

Is these interesting/new mathematics?

- Random representations matrix models: Consider a group G and a representation R of G onto some V. Classify the (Gaussian) Random matrix ensembles (in VxV) invariants under G. What are their properties?