Kasteleyn Operators from Mirror Symmetry

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joint with D. Treumann and E. Zaslow

- Fix an embedded bipartite graph $\Gamma \subset T^2$, edge weighting $\mathcal{E} : \Gamma_1 \to \mathbb{C}^\times$, and Kasteleyn orientation $\kappa : \Gamma_1 \to \{\pm 1\}$.

Theorem (Treumann-W.-Zaslow)

The following coherent sheaves on $(\mathbb{C}^\times)^2$ are isomorphic:

1. the spectral transform of the Kasteleyn operator $K(x, y)$ of $(\Gamma, \mathcal{E}, \kappa)$, and
2. the homological mirror of the conjugate Lagrangian $L_\Gamma \subset T^*T^2$, promoted to an object of the wrapped Fukaya category $\mathcal{W}(T^*T^2)$ using $(\mathcal{E}, \kappa)$.

$$\begin{tikzcd}
 & \text{combinatorics} \arrow[dl, "\substack{\text{spectral}\\ \text{transform of}\\ K(x,y)}"'] \arrow[dr, "\substack{T^2 \quad T^*T^2 \\ \cup \qquad \cup \\ \Gamma \mapsto L_\Gamma}"] & \\
\text{algebraic} \arrow[rr, leftrightarrow] & & \text{symplectic}
\end{tikzcd}$$

Dimer Models and the Kasteleyn Operator

- $\kappa : \Gamma_1 \to \{\pm 1\}$ is a Kasteleyn orientation if for each face F of Γ,

$$\prod_{E \subset \partial F} \kappa(E) = \begin{cases} +1 & \text{if } |\partial F| \equiv 2 \mod 4 \\ -1 & \text{if } |\partial F| \equiv 0 \mod 4 \end{cases}$$

- Choose generators $x, y \in H_1(T^2, \mathbb{Z})$ and closed curves γ_x, γ_y in T^2 representing their Poincaré duals.

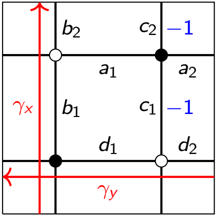
Definition

$K(x, y)$ is the $(\Gamma_0^b \times \Gamma_0^w)$-matrix-valued Laurent polynomial with entries

$$K(x, y)_{\langle v_b, v_w \rangle} = \sum_{v_b \overset{E}{\text{---}} v_w} \mathcal{E}(E)\kappa(E)x^{\langle \gamma_x, E\rangle} y^{\langle \gamma_y, E\rangle}.$$

Dimer Models and the Kasteleyn Operator

- An example:



$$K(x,y) = \begin{bmatrix} a_1 + a_2 x & b_1 + b_2 y^{-1} \\ -c_1 - c_2 y & d_1 + d_2 x^{-1} \end{bmatrix}$$

$$\det K(x,y) = a_1 d_1 + a_2 d_2 + b_1 c_1 + b_2 c_2 + a_1 d_2 x^{-1} + a_2 d_1 x + b_1 c_2 y + b_2 c_1 y^{-1}$$

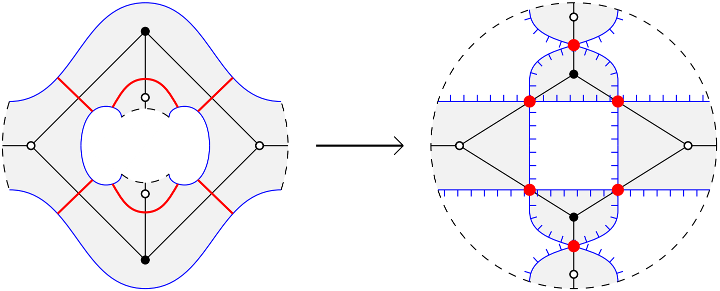
- Kenyon-Okounkov-Sheffield: dimer model on lift on Γ to $\mathbb{R}^2$ controlled by **spectral transform** of $K(x,y)$, the cokernel of

$$\mathbb{C}[x^{\pm 1}, y^{\pm 1}]^{\Gamma_0^w} \xrightarrow{K(x,y)} \mathbb{C}[x^{\pm 1}, y^{\pm 1}]^{\Gamma_0^b}.$$

- This is a rank one coherent sheaf on $(\mathbb{C}^\times)^2$ supported on the **spectral curve** $\{\det K(x,y) = 0\} \subset (\mathbb{C}^\times)^2$.
- Up to isomorphism, only depends on $\mathcal{E}$ up to gauge $\implies$ defines family of objects parametrized by $H^1(\Gamma, \mathbb{C}^\times) \cong (\mathbb{C}^\times)^{b_1(\Gamma)}$ in $\mathrm{Coh}(\mathbb{C}^\times)^2$.

Symplectic Geometry of Bipartite Graphs

- Goncharov-Kenyon: Γ embedded bipartite graph in a surface S $\implies$ Poisson structure on $(\mathbb{C}^\times)^{b_1(\Gamma)}$ via **conjugate surface** L_Γ.
- Start with **zig-zag paths** of Γ, immersed (co-)oriented curves with exactly one crossing on each edge of Γ. These divides S into "white", "black", and "alternating" regions.
- Define L_Γ by blowing up S at crossings, then taking the closure of the white and black regions:



Symplectic Geometry of Bipartite Graphs

- Recall T^*S has exact symplectic form $\omega = d\lambda$, $\lambda = \sum p_i dq_i$.
- A surface $L \subset T^*S$ is exact Lagrangian if $\lambda|_L$ is exact.
- Write ∂T^*S for the fiberwise boundary of T^*S, and let $\Lambda_\Gamma \subset \partial T^*S$ denote the conormal lift of the zig-zag paths of Γ.

Theorem (Shende-Treumann-W.-Zaslow)

The surface L_Γ embeds into T^*S as an exact Lagrangian asymptotic to Λ_Γ, canonical up to Hamiltonian isotopy.

- Corollary: equipping L_Γ with local system, spin structure defines a family of objects parametrized by $H^1(\Gamma, \mathbb{C}^\times) \cong (\mathbb{C}^\times)^{b_1(\Gamma)}$ in the partially wrapped Fukaya category $\mathcal{W}(T^*S, \Lambda_\Gamma)$.
- Useful model: can identify $\mathcal{W}(T^*S, \Lambda_\Gamma)$ with $Sh_{\Lambda_\Gamma}(T^*S)$, the category of constructible sheaves with singular support asymptotic to Λ_Γ (Ganatra-Pardon-Shende).

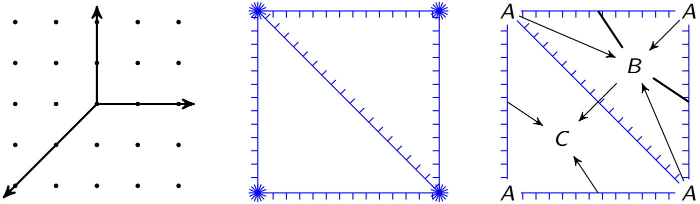
- Given a fan $\Sigma \subset \mathbb{R}^n$, write X_Σ for the associated toric compactification of $(\mathbb{C}^\times)^n$.

Theorem (Fang-Liu-Treumann-Zaslow, Kuwagaki)

There exists a Legendrian subset $\Lambda_\Sigma \subset \partial T^* T^n$ and an equivalence

$$Sh_{\Lambda_\Sigma}(T^n) \cong \mathrm{Coh}(X_\Sigma).$$

- Example (after Beilinson): $X_\Sigma = \mathbb{P}^2$



$\Sigma \subset \mathbb{R}^2$ $\quad$ $\Lambda_\Sigma \subset \partial T^* T^2$ $\quad$ $\mathcal{F} \in Sh_{\Lambda_\Sigma}(T^2)$

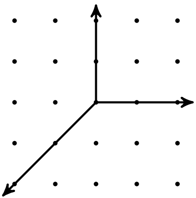
Toric Mirror Symmetry

- Given Γ, let $\Sigma(\Gamma)$ denote the complete fan in $H^1(T^2, \mathbb{R}) \cong \mathbb{R}^2$ with rays generated by the classes of the zig-zag paths of Γ.

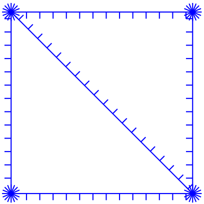
Lemma

If Γ is consistent there is a Legendrian isotopy from Λ_Γ into Λ_Σ.

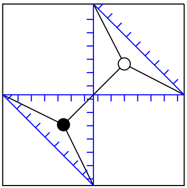
- Note: consistency does not restrict curves which appear.
- Caveat: should really consider stacky fans in general.
- Guillermou-Kashiwara-Schapira: an isotopy as above induces a fully faithful functor $Sh_{\Lambda_\Gamma}(T^2) \hookrightarrow Sh_{\Lambda_\Sigma}(T^2)$.
- Example: a graph for $\mathbb{P}^2$



$\Sigma \subset \mathbb{R}^2$



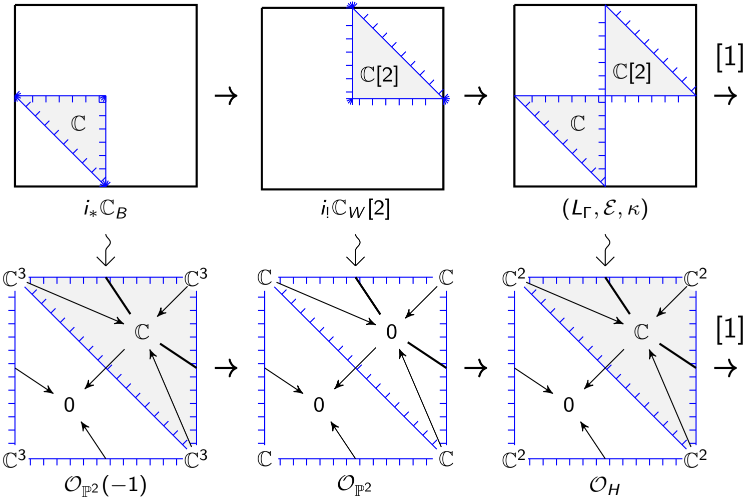
$\Lambda_\Sigma \subset \partial T^* T^2$



Γ and Λ_Γ

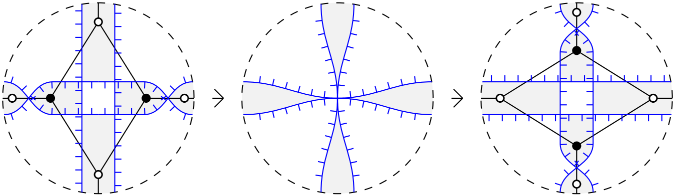
Example of Main Result: A Hyperplane $H \subset \mathbb{P}^2$

- The triple $(L_\Gamma, \mathcal{E}, \kappa)$ defines a sheaf in $Sh_{\Lambda_\Gamma}(T^2)$ (top right), which we can isotope to a sheaf in $Sh_{\Lambda_\Sigma}(T^2)$ (bottom right).



Bonus: Discrete Integrability via Mirror Symmetry

- The **square move** is a basic operation on bipartite graphs. It acts on Λ_Γ by a Legendrian isotopy.



Theorem (Shende-Treumann-W.-Zaslow)

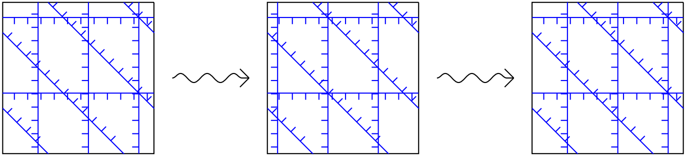
$\Gamma \mapsto \Gamma'$ square move $\implies$ functor $Sh_{\Lambda_\Gamma}(T^2) \to Sh_{\Lambda_{\Gamma'}}(T^2)$ relates families $(\mathbb{C}^\times)^{b_1(\Gamma)}$ and $(\mathbb{C}^\times)^{b_1(\Gamma')}$ by cluster $\mathcal{X}$-transformation.

Corollary (Goncharov-Kenyon)

Γ, Γ' differ by square move, $\mathcal{E}$, $\mathcal{E}'$ by cluster $\mathcal{X}$-transformation, and κ, κ' in the obvious way $\implies$ Kasteleyn operators associated to $(\Gamma, \mathcal{E}, \kappa)$ and $(\Gamma', \mathcal{E}', \kappa')$ have the same spectral transform.

Bonus: Discrete Integrability via Mirror Symmetry

- Corollary: a periodic sequence of square moves of Γ acts by a Legendrian autoisotopy of Λ_Γ. Conjugating this by an isotopy $\Lambda_\Gamma \to \Lambda_{\Sigma(\Gamma)}$ gives an autoisotopy of its image, for example:



- The group of such autoisotopies is $\mathbb{Z}^{|\Sigma_1|}$, hence is equivalent to the group of line bundles $\mathcal{O}(\sum_i n_i D_i)$ on $X_{\Sigma(\Gamma)}$.

Theorem (Treumann-W.-Zaslow)

Mirror symmetry intertwines the autoequivalence of $Sh_{\Sigma(\Gamma)}(T^2)$ defined by the isotopy $(n_i) \in \mathbb{Z}^{|\Sigma_1|}$ with tensoring by $\mathcal{O}(\sum_i n_i D_i)$.

- Corollary: the action of a periodic square move sequence on spectral data is given by tensoring with the corresponding $\mathcal{O}(\sum_i n_i D_i)$ (c.f. Fock-Marshakov, Goncharov).