Integrable systems in higher Teichmuller theory

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Integrability, Combinatorics, and Representations
Giens, September 3, 2019

Quantum Teichmüller theory

Fock–Goncharov: a *quantum higher Teichmüller theory* is an assignment

$$(S, G) \rightsquigarrow \left(\mathcal{X}^q_{G,S}, V^\lambda_{G,S}\right)$$

where

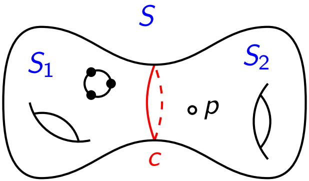
- S is a 2-dimensional topological surface with boundary ∂S and possibly marked points on ∂S;
- G is a simple Lie group;

and

- Γ_S is the mapping class group of the surface S;
- $\chi^q_{G,S}$ is an algebra = a quantization of the moduli space of (decorated) G-local systems on S;
- $V^\lambda_{G,S}$ is a Γ_S-equivariant Hilbert space representation of $\chi^q_{G,S}$ with the central character λ.

By construction, for any closed simple curve $c \in S$ there exists a commutative subalgebra $A_c \subset \chi^q_{G,S}$ generated by quantized traces of the monodromy along c. If c contracts to a *puncture*, i.e. a component of ∂S without marked points, then A_c is central and contributes to the central character λ in $V^\lambda_{G,S}$.

Consider a surface S which we cut along a closed simple curve c into $S = S_1 \sqcup_c S_2$:



By construction, there exists a map

$$\chi^q_{G,S_1} \otimes \chi^q_{G,S_2} \longrightarrow \chi^q_{G,S}.$$

Modular functor conjecture

Conjecture (Fock–Goncharov '09)

1. *As a representation of* $\chi^q_{G,S_1} \otimes \chi^q_{G,S_2}$ *one has*

$$V^{\lambda}_{G,S} \simeq \int_{\mathcal{C}_+}^{\oplus} V^{\lambda,\nu}_{G,S_1} \otimes V^{\lambda,-\nu}_{G,S_2}\, m(\nu) d\nu,$$

   where e^{ν} *are the eigenvalues of the monodromy along* c, $m(\nu)$ *is the Sklyanin measure, and* $\mathcal{C}_+ \subset \mathbb{R}^{\mathrm{rk}(\mathfrak{g})}$ *is the positive Weyl chamber.*

2. *This decomposition is* Γ_S*-equivariant.*

Theorem (Teschner '07)

A version of this conjecture holds for $G = SL_2(\mathbb{C})$.

Idea of the proof

Theorem (Schrader–S '17)

For $G = SL_n$, there exists a family of desired unitary equivalences

$$V^{\lambda}_{G,S} \simeq \int^{\oplus}_{\mathcal{C}_+} V^{\lambda,\nu}_{G,S_1} \otimes V^{\lambda,-\nu}_{G,S_2}\, m(\nu) d\nu,$$

Idea: diagonalize the subalgebra A_c. *Quantum cluster structure* gives an infinite family of unitary transformations on $V_{G,S}$, which allows you to bring the generators of A_c to the Hamiltonians of the (quantum relativistic) Coxeter-Toda system. For general G can still bring them to the Hamiltonians of the full Toda.

Questions:

- How do the decompositions relate to each other?
- Are they Γ_S-equivariant?
- How about closing punctures?

$\mathcal{X}_{G,S}$ as a cluster variety

Fock–Goncharov: The semi-classical moduli space $\mathcal{X}_{G,S}$ is a *cluster Poisson variety*:

1. it has an atlas of toric charts
$$\mathcal{T}_Q \colon (\mathbb{C}^*)^d \longrightarrow \mathcal{X}_{G,S},$$
labelled by quivers Q.
2. The Poisson brackets between toric coordinates are log-canonical:
$$\{Y_j, Y_k\} = \epsilon_{kj} Y_j Y_k$$
and are determined by the adjacency matrix ϵ_{jk} of Q.
3. Each chart has exactly d "adjacent" charts. The gluing data is given by certain subtraction-free rational transformations: for each $1 \leqslant k \leqslant d$ there is a *cluster mutation* μ_k *in direction* k.

Quantization of cluster charts

Promote each cluster chart to a quantum torus algebra

$$\mathcal{T}_Q^q = \left\langle \hat{Y}_1, \ldots, \hat{Y}_d \,\middle|\, \hat{Y}_j \hat{Y}_k = q^{2\epsilon_{kj}} \hat{Y}_k \hat{Y}_j \right\rangle .$$

The quantum "gluing data" is realized via *quantum cluster mutations* μ_k^q, which are algebra automorphisms of conjugation by the *quantum dilogarithm* $\Gamma_q(\hat{Y}_k)$, where

$$\Gamma_q(X) = \prod_{n=1}^{\infty} \frac{1}{1 + q^{2n+1} X}.$$

In other words

$$\mu_k^q = \mathrm{Ad}_{\Gamma_q(\hat{Y}_k)} .$$

Quantization of cluster varieties

Remark

The quantum dilogarithm is a q-analogue of a Γ*-function:*

$$\Gamma_q(q^2X) = (1+qX)\Gamma_q(X).$$

This remark guarantees that quantum mutations provide isomorphisms

$$\mu_k^q\colon \operatorname{Frac}(\mathcal{T}_Q^q) \simeq \operatorname{Frac}(\mathcal{T}_{\mu_k(Q)}^q)$$

Definition

The algebra $\mathcal{X}_{G,S}^q = \mathcal{O}_q(\mathcal{X}_{G,S})$ is the subalgebra of any quantum chart $\mathcal{T}_Q^q$, consisting of those elements that stay Laurent under any finite sequence of cluster mutations.

Positive representations

Set

$$q = e^{\pi i b^2} \qquad \text{where} \qquad b^2 \in \mathbb{R}_{>0} \setminus \mathbb{Q}.$$

Embed each quantum cluster chart $\mathcal{T}_Q^q$ into a Heisenberg algebra

$$\mathcal{H} = \left\langle \hat{y}_1, \dots, \hat{y}_d \,\middle|\, [\hat{y}_j, \hat{y}_k] = \frac{1}{2\pi i} \epsilon_{kj} \right\rangle,$$

by the homomorphism

$$\hat{Y}_j \mapsto e^{2\pi b \hat{y}_j}.$$

$\mathcal{H}$ has irreducible Hilbert space representations in which the generators $\hat{Y}_j$ act by positive self-adjoint operators.

Non-compact quantum dilogarithm

Problem: the series for Γ_q does not converge when $|q| = 1$.

Luckily, there is a *non-compact quantum dilogarithm* function $\varphi(z)$ is the unique solution of the pair of difference equations

$$\varphi(z - ib^{\pm 1}/2) = (1 + e^{2\pi b^{\pm 1} z})\varphi(z + ib^{\pm 1}/2).$$

Now, we get

$$\mu_k^q = \mathrm{Ad}_{\varphi(-\hat{y}_k)}\,.$$

Since

$$z \in \mathbb{R} \implies |\varphi(z)| = 1,$$

and each $\hat{y}_k$ is self-adjoint, quantum cluster mutations become **unitary** operators.

Positive representations

Embedding $\chi^q_{G,S}$ into quantum cluster charts, and pulling back their natural representations, we obtain a (family of unitary equivalent) representations $V^\lambda_{G,S}$.

For each triangulation of S with vertices at punctures or marked points, there is a quiver Q and the cluster chart $\mathcal{T}_Q$. Flips of diagonals are realized by a specific sequence of $\approx n^3/6$ mutations.

The quantum dilogarithm satisfies the *pentagon identity*:

$$[\hat{p}, \hat{x}] = \frac{1}{2\pi i} \quad \Longrightarrow \quad \varphi(\hat{p})\varphi(\hat{x}) = \varphi(\hat{x})\varphi(\hat{p}+\hat{x})\varphi(\hat{p}).$$

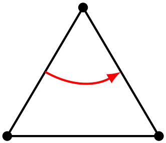
So we get a unitary representation of the cluster modular group(oid) (the one generated by flips). It contains Γ_S.

Quantum monodromies

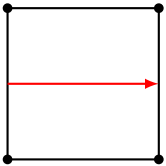
The moduli space of *decorated* G-local systems implies, in particular, that there is a trivialization of the system along each open component of $\partial S \setminus \{\text{marked points}\}$. So, each path γ that starts and ends on such a component defines a quantum monodromy

$$M_\gamma \in \mathrm{Mat}_n(\mathbb{C}) \otimes \mathcal{X}^q_{G,S}.$$

These monodromies satisfy RLL-relations. Therefore, you get homomorphisms from RLL-algebras to $\mathcal{X}^q_{G,S}$. The following two pictures, in fact, represent injective homomorphisms.



$U_q(\mathfrak{b})$



$\mathcal{O}_q(G^{w_0,w_0})$

Relation to quantum groups

- If S is disk with 4 marked points, there is an embedding $O_q(G^{w_0,w_0}) \hookrightarrow \chi^q_{G,S}$;
- If S is a cylinder with 1 marked points on each boundary, there is an embedding $O_q(G^{w_0,w_0} / \operatorname{Ad} H) \hookrightarrow \chi^q_{G,S}$;

Theorem (Schrader–S, Ip '16)

Let S be a **punctured** *disk with 2 marked points, then there is an embedding:* $U_q(\mathfrak{g}_n) \hookrightarrow \chi^q_{G,S}$.

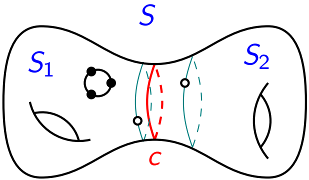
Theorem in progress: Goncharov–Shen showed that each puncture gives rise to a natural Weyl group action on $\chi^q_{G,S}$. For S a punctured disk with 2 marked points:

$$U_q(\mathfrak{g}) \simeq (\chi^q_{G,S})^W$$

Task: find a unitary equivalence:

$$V_{G,S}^{\lambda} \simeq \int_{\mathcal{C}_+}^{\oplus} V_{G,S_1}^{\lambda,\nu} \otimes V_{G,S_2}^{\lambda,-\nu} \, m(\nu) d\nu,$$

Idea: find a triangulation in which c is contained in a cylinder. Then A_c is generated by the Hamiltonians of the full Toda system.



There exists a sequence of quantum cluster mutations sending Hamiltonians of the full Toda system, to the Hamiltonians of the Coxeter-Toda system.

New task: diagonalize quantum Coxeter-Toda Hamiltonians.

Quantization of Coxeter–Toda system

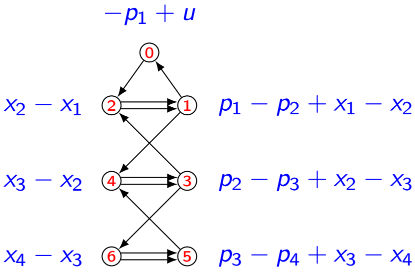
Consider the Heisenberg algebra $\mathcal{H}_n$ generated by $\{x_j, p_j\}_{j=1}^n$

$$[p_j, x_k] = \frac{\delta_{jk}}{2\pi i}$$

acting on $L^2(\mathbb{R}^n)$, via

$$p_j \mapsto \frac{1}{2\pi i}\frac{\partial}{\partial x_j}$$

The representation of the quantum torus algebra for the Coxeter–Toda quiver:



e.g. $\hat{Y}_2$ acts by multiplication by $e^{2\pi b(x_2 - x_1)}$.

Construction of quantum Hamiltonians

Theorem (Schrader–S)

Consider the **Baxter operator** $Q_n(u)$ *obtained by mutating consecutively at* $0, 1, 2, \ldots, 2n-2$*. Then*

1. *The unitary operators* $Q_n(u)$ *satisfy*

$$[Q_n(u), Q_n(v)] = 0,$$

2. *If* $A_n(u) = Q_n(u - ib/2) Q_n(u + ib/2)^{-1}$*, then one can expand*

$$A_n(u) = \sum_{k=0}^{n} H_k U^k, \quad U := e^{2\pi b u}$$

and the commuting operators $H_1, \ldots, H_n$ *quantize the* GL_n *Coxeter–Toda Hamiltonians.*

The Dehn twist

Additionally, there is a **Dehn twist operator** realized as mutations at all even nodes postcomposed by $e^{\pi i(p_1^2+\cdots+p_n^2)}$:

$$\tau_n = e^{\pi i(p_1^2+\cdots+p_n^2)}\varphi(x_2-x_1)\ldots\varphi(x_n-x_{n-1})$$

which commutes with the Baxter operator

$$[\tau_n, Q_n(u)] = 0$$

Problem: Construct complete set of joint eigenfunctions, the *b-Whittaker functions*, for operators $Q_n(u), \tau_n$.

Example

For example, for $n = 1$ we have

$$Q_1(u) = \varphi(p_1 + u), \qquad \tau_1 = e^{\pi i p_1^2}$$

Then the function

$$\Psi_\lambda(x_1) = e^{2\pi i \lambda x_1}$$

satisfies

$$Q_1(u)\Psi_\lambda(x_1) = \varphi(\lambda + u)\Psi_\lambda(x_1),$$
$$\tau_1 \Psi_\lambda(x_1) = e^{\pi i \lambda^2}\Psi_\lambda(x_1).$$

Here we make sense of the operator $\varphi(p + u)$ via the Fourier transform formula for the quantum dilogarithm:

$$\mathrm{const} \cdot \varphi(w) = \int \frac{e^{2\pi i t(w - c_b)}}{\varphi(t - c_b)} dt, \qquad c_b = i\frac{b + b^{-1}}{2}.$$

Construction of b-Whittaker functions

Set $\mathcal{R}_n(u)$ to be the same as the Baxter operator $Q_n(u)$ but without the last mutation. We then define

$$\Psi_{\boldsymbol{\lambda}}(\boldsymbol{x}) := \mathcal{R}_n(c_b - \lambda_n) \ldots \mathcal{R}_2(c_b - \lambda_2) \cdot e^{2\pi b(\boldsymbol{\lambda} \cdot \boldsymbol{x})},$$

where

$$\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n), \qquad \boldsymbol{x} = (x_1, \ldots, x_n).$$

Unitarity of the b-Whittaker transform

Theorem (Schrader–S)

The b-Whittaker transform

$$\mathcal{W}\colon L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n, m(\boldsymbol{\lambda})d\boldsymbol{\lambda}),$$
$$(\mathcal{W}[f])(\boldsymbol{\lambda}) = \int_{\mathbb{R}^n} \overline{\Psi^{(n)}_{\boldsymbol{\lambda}}(\boldsymbol{x})} f(\boldsymbol{x}) d\boldsymbol{x}$$

is a unitary equivalence. Moreover

$$\mathcal{W} \circ \tau = e^{\pi i(\lambda_1^2+\cdots+\lambda_n^2)} \circ \mathcal{W},$$
$$\mathcal{W} \circ Q_n(u) = \prod_{j=1}^{n} \varphi(u-\lambda_j) \circ \mathcal{W},$$
$$\mathcal{W} \circ H_k^{(n)} = e_k(\boldsymbol{\Lambda}^{-1}) \circ \mathcal{W},$$

where e_k is the elementary symmetric function and $\Lambda = e^{2\pi b\lambda}$.

Unitarity of the b-Whittaker transform

Writing all the $R_n(\lambda)$ as integral operators, we get an explicit Givental-type integral formula for the b-Whittaker functions.

Moreover, using the cluster construction of the b-Whittaker functions, we prove the following:

Theorem (Schrader–S)

The b-Whittaker transform

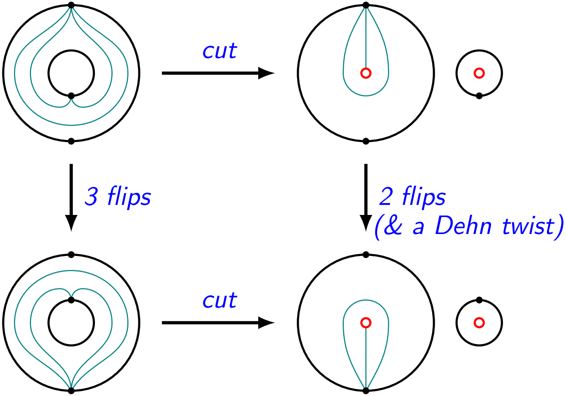
$$(\mathcal{W}[f])(\boldsymbol{\lambda}) = \int_{\mathbb{R}^n} \overline{\Psi^{(n)}_{\boldsymbol{\lambda}}(\boldsymbol{x})} f(\boldsymbol{x}) d\boldsymbol{x}$$

is a unitary equivalence.

Γ_S-equivariance

Question: is our recipe for cutting surfaces Γ_S-equivariant?

Turns out that it is enough to check the following case:



For $G = SL_2$ this is equivalent to the eigenproblem for the Baxter operator: $\mathcal{W} \circ Q_2(u) = \varphi(u-\lambda)\varphi(u+\lambda) \circ \mathcal{W}$.

Γ_S-equivariance

In general, the equality

$$\mathcal{W} \circ \{3 \text{ flips}\} = \{2 \text{ flips}\} \circ \mathcal{W},$$

can be shown by applying the pentagon relation

$$\varphi(p)\varphi(x) = \varphi(x)\varphi(p+x)\varphi(p)$$

together with relation

$$p_n f(\boldsymbol{x}) = 0 \implies \varphi(x_n + \alpha)\varphi(p_n + x_n + \alpha + c_b) f(\boldsymbol{x}) = f(\boldsymbol{x}).$$

Closing punctures

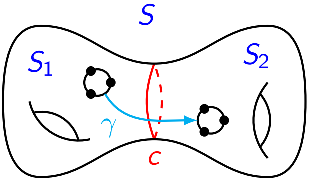
There remains one more problem: what if we cannot include the cutting cycle into a cylinder? In that case, we need to make an additional cut, and show in a similar fashion that the result does not depend on the cut. Alternatively, we can drill a puncture, and show that nothing depends on that puncture.

How do we drill/close punctures? Let $S^\times$ be a surface S with additional puncture.

Then we have $\mathcal{X}^q_{G,S^\times} \subset \mathcal{X}^q_{G,S} \otimes U_q(\mathfrak{g})$, and we can find a subset $V \simeq V^\lambda_{G,S}$ of (tempered) distributions in $V^\lambda_{G,S^\times}$, on which $U_q(\mathfrak{g})$ acts by the counit. Equivalently, we're setting monodromy around the puncture to be trivial. This gives us an embedding $\mathcal{X}^q_{G,S}\big|_V \hookrightarrow \mathcal{X}^q_{G,S^\times}\big|_V$.

Monodromies across the cutting cycle

The following question appears to be very instructive: what happens with monodromies M_γ, γ is transversal to c, when we cut along c?



In fact, $M_\gamma = M_2 C M_1$, and the *Coxeter transport matrix* C is the only one that is affected by cutting.

For $G = SL_n$ we have

$$C = \tau_n^{j-1}\left(e^{-2\pi b x_n} H_{k-1}^{(n-1)}\right).$$

Coxeter transport matrix

Let $\mu, \nu \subset B_{k,n-k}$ be a pair of Young diagrams fitting in a box with $n-k$ rows and k columns. That is

$$\mu = (\mu_1, \dots, \mu_{n-k}), \qquad \nu = (\nu_1, \dots, \nu_{n-k})$$

$$\mu_i \leqslant \mu_{i+1}, \quad \nu_i \leqslant \nu_{i+1}, \qquad \text{and} \qquad \mu_{n-k}, \nu_{n-k} \leqslant k.$$

Set

$$\rho = (1, 2, \dots, n-k),$$

and define C_μ^ν to be the submatrix of C at the intersection of rows $\nu + \rho$ with columns $\mu + \rho$.

In the next slide, let us for simplicity work classically, i.e. set $q = 1$, and consider Poisson algebras instead of algebras of differential operators.

Whittaker transformed Coxeter transport matrix

Consider a Poisson algebra:

$$\mathcal{A}(n) = \mathbb{C}\left[D_j, \Lambda_j\right]_{j=1}^n, \quad \{D_j, \Lambda_k\} = \delta_{jk} D_j \Lambda_k.$$

Let f, g be a pair of symmetric functions. Set

$$R_k^{(n)}[f, g] = \sum_{\substack{J \subset [1,\dots,n] \\ |J|=k}} f(\mathbf{\Lambda}_J) g(\mathbf{\Lambda} \setminus \mathbf{\Lambda}_J) \cdot \mathfrak{D}_J$$

with

$$\mathfrak{D}_J = \prod_{r \in J} \prod_{s \notin J} (1 - \Lambda_{i,s} \Lambda_{i,r}^{-1})^{-1} D_r.$$

Theorem (Schrader–S)

$$\mathcal{W} \circ \det(C_\mu^\nu) = \tau^{1-k} \circ R_k^{(n)}[s_\mu, s_{\nu^*}],$$

where s_μ, s_{ν^*} are Schur functions and $\mu^* = (B_{k,n-k} \setminus \mu)^t$.

Macdonald operators

Example: Consider Macdonald operator

$$M_k = \sum_{\substack{J\subset[1,\dots,n]\\ |J|=k}} \prod_{\substack{j\in J\\ \ell\notin J}} \frac{t\Lambda_\ell - \Lambda_j}{\Lambda_\ell - \Lambda_j} D_J.$$

Rewriting

$$M_k = \tau^{n-k} \sum_{\mu\subset B_{k,n-k}} (-t)^{|\mu|} R_k[s_\mu, s_{\mu^*}]$$

we get

$$\mathcal{W} \circ \sum_{\mu\subset B_{k,n-k}} (-t)^{|\mu|} \det\left(C^{\mu}_{\mu^*}\right) = M_k \circ \mathcal{W}.$$

Thank you!