Cluster varieties, supersymmetric gauge theories and spin chains

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Integrability Combinatorics and Representations

Presqu'île de Giens, Hyères, September 2019

based on

joint work with Kolya Semenyakin, arXiv:1905.09921, to appear in JHEP

extension of:

Cluster integrable systems and q-Painlevé equations, JHEP 02 (2018) 077, arXiv:1711.02063

Cluster Toda chains and Nekrasov functions, Faddeev volume in Theor. Math. Phys. **198** (2019) 157-187, arXiv:1804.10145

with M. Bershtein & P. Gavrylenko;

Loop groups, Clusters, Dimers and Integrable systems, arXiv:1401.1606
with V. Fock, ...

Plan

Cluster integrable systems:

- Newton polygons Δ;
- Quivers $\mathcal{Q}$ and (X- and A-) cluster varieties;
- Integrable flows: $\mathcal{G}_\Delta \subset \mathcal{G}_\mathcal{Q}$ (MCG of cluster variety)

Deautonomization:

- Discrete flows from quiver mutations;
- $q = \prod x \neq 1$: integrability lost!
- q-difference 'RG-equations', bilinear in A-cluster representation.

Plan

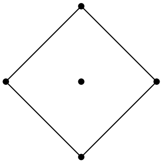
Examples:

- $g = 1$ curves $\equiv SU(2)$ 5d gauge theories $\equiv$ q-Painlevé family;
- Relativistic Toda chains $\equiv$ 5d pure gauge theories $SU(N)_k$;
- $\mathfrak{gl}_N$ spin chains on M-sites $\equiv SU(N)^{\otimes(M-1)}$ linear gauge quivers;
- Generic Δ: topological strings on local CY ...

Solutions:

- $q = 1$ finite gap case;
- Deautonomization: Θ-functions $\Rightarrow$ (dual) Nekrasov functions;
- Quantization of cluster variety $\equiv$ refinement $q \to (q, p = e^{\hbar})$;
- Topological string amplitudes and tropical limit ...

Newton polygon



Newton Polygon (up to $SA(2,\mathbb{Z})$-tranform):

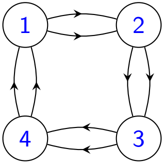
$$f_\Delta(\lambda,\mu) = \sum_{(a,b)\in\Delta} \lambda^a \mu^b f_{a,b} = \lambda + \frac{1}{\lambda} + \mu + \frac{z}{\mu} + H = 0 \qquad (1)$$

5d SW curve, VEV H and coupling z: hamiltonian of relativistic Toda chain.

Remark: renormalizations of λ, μ and f_Δ fix 3 of coefficients $\{f_{a,b}\}$ in the equation.

Poisson quiver

X-cluster Poisson variety with (mutation class of) quiver $\mathcal{Q}$:



encoding logarithmically constant Poisson bracket

$$\{x_i, x_j\} = \epsilon_{ij} x_i x_j, \qquad i, j = 1, \ldots, |\mathcal{Q}| \tag{2}$$

with the *skew*-symmetric matrix

$$\epsilon_{ij} = -\epsilon_{ji} = \#\text{arrows } (i \to j) = \pm 2 \tag{3}$$

Obviously $q = x_1 x_2 x_3 x_4$ and $z = x_1 x_3$ are in the center of Poisson algebra.

Cluster integrable system

a la Goncharov+Kenyon and/or Fock+AM:

- Defined by *any* convex NP $\Delta \subset \mathbb{Z}^2 \subset \mathbb{R}^2$ for a curve $\Sigma \subset \mathbb{C}^\times \times \mathbb{C}^\times$

$$f_\Delta(\lambda,\mu) = \sum_{(a,b)\in\Delta} \lambda^a \mu^b f_{a,b} = 0. \tag{4}$$

- Realized on a Poisson X-cluster variety $\mathcal{X}$, $\dim \mathcal{X} = 2\mathrm{Area}(\Delta)$. Poisson structure

$$\{x_i, x_j\} = \epsilon_{ij} x_i x_j, \qquad \{x_i\} \in \left(\mathbb{C}^\times\right)^{2\mathrm{Area}(\Delta)}. \tag{5}$$

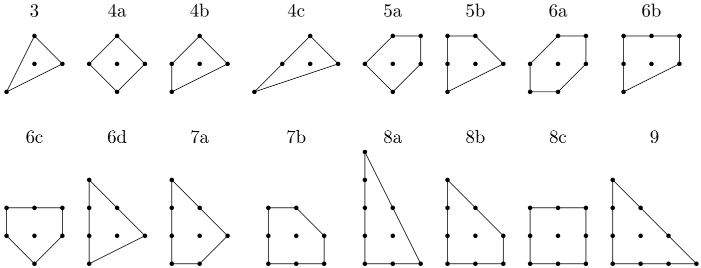
  is encoded in a quiver $\mathcal{Q}$, with $\epsilon_{ij} = \#\mathrm{arrows}(i \to j)$.
  Straightforward quantization (refinement! - spin chains?).

- Integrability: Pick's formula

$$2\mathrm{Area}(\Delta) - 1 = (B-3) + 2g \tag{6}$$

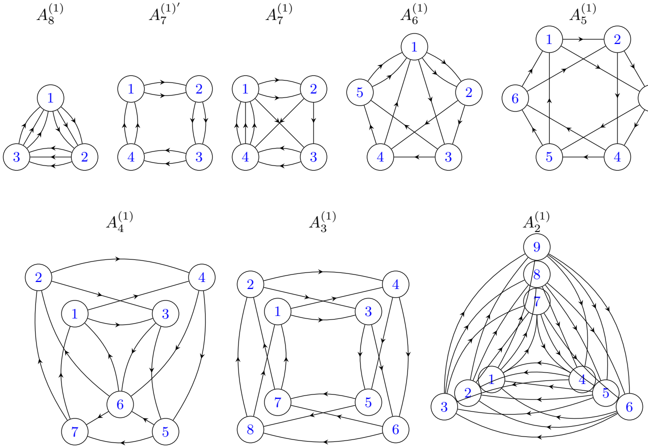
Examples: Painlevé Newton polygons Δ

with a single internal point and $3 \leq B \leq 9$ boundary points:



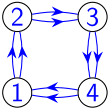
Here Σ: $f_\Delta(\lambda,\mu) = \sum_{(a,b)\in\Delta} \lambda^a \mu^b f_{a,b} = 0$ is always a torus $g = 1$.

Examples: Painlevé quivers $\mathcal{Q}$



Mutations: Poisson maps

Symmetries are generated by *mutations* on X-cluster variety:



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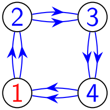

Mutation μ_1

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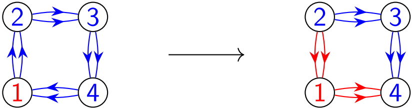

Mutation μ_1

Reverse all incoming
and outgoing arrows
$x_1' = 1/x_1$

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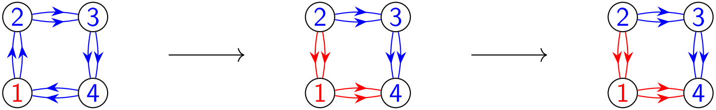

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Complete cycles through mutation vertex
$x_4' = x_4(1+x_1)^2$
$x_2' = x_2(1+1/x_1)^{-2}$

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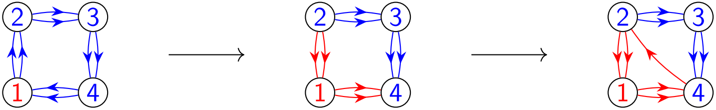

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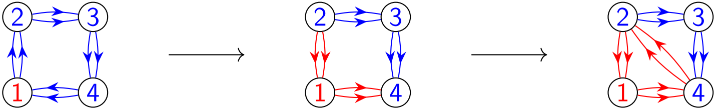

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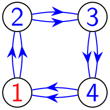

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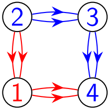

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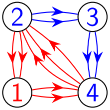

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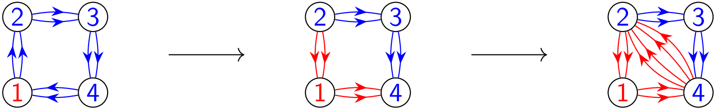

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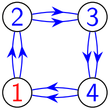
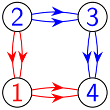


Mutation μ_1

$\longrightarrow$



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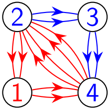

Complete cycles through mutation vertex
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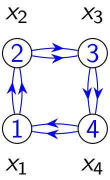
$\mu_j : \epsilon_{ik} \mapsto -\epsilon_{ik},$ if $i = j$ or $k = j$, $\quad \epsilon_{ik} \mapsto \epsilon_{ik} + \frac{\epsilon_{ij}|\epsilon_{jk}| + \epsilon_{jk}|\epsilon_{ij}|}{2}$ otherwise.

Algebraic (bi-rational!) transformations of the variables:

$$\mu_j : x_j \mapsto x_j^{-1}, \qquad x_i \mapsto x_i \left(1 + x_j^{\mathrm{sgn}\,\epsilon_{ij}}\right)^{\epsilon_{ij}}, \quad i \neq j. \qquad \{x_i', x_k'\} = \epsilon_{ik}' x_i' x_k'$$

Cluster automorphisms

All combinations of mutations and permutations of vertices, preserving quiver $\mathcal{G}_{\mathcal{Q}} \supset \mathcal{G}_{\Delta}$ (discrete flows of IS).
Example – the flow $T \in \mathcal{G}_{\mathcal{Q}}$ of two-particle Toda chain:



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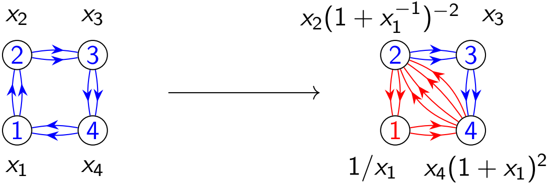

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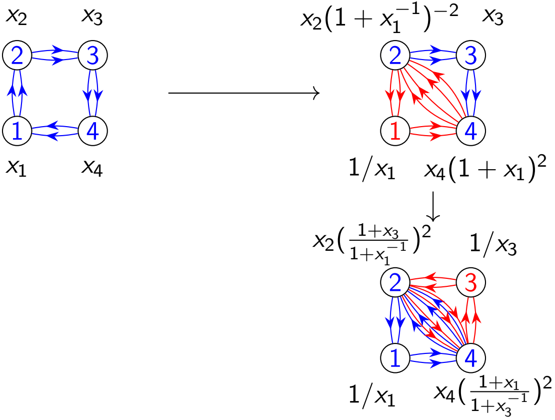

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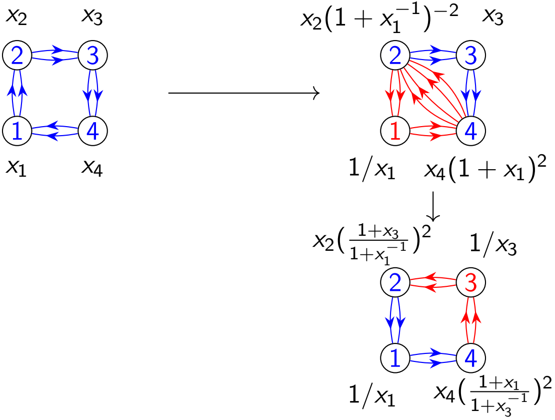

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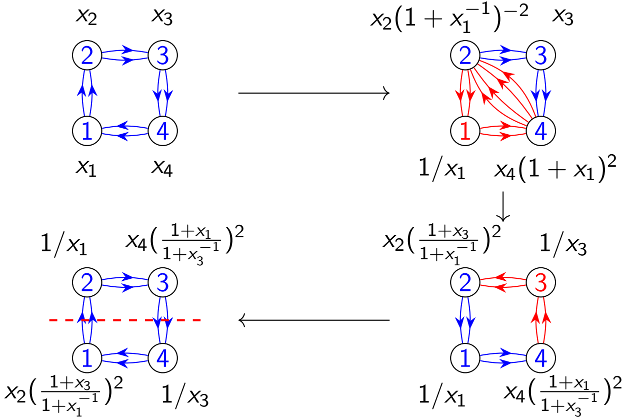


Deautonomization

For $q = 1$ the flow T

$$T : (x_1, x_2, x_3, x_4) \mapsto \left(x_2 \left(\frac{1 + x_3}{1 + x_1^{-1}} \right)^2, x_1^{-1}, x_4 \left(\frac{1 + x_1}{1 + x_3^{-1}} \right)^2, x_3^{-1} \right)$$

preserves the Hamiltonian $H = \sqrt{x_1 x_2} + \frac{1}{\sqrt{x_1 x_2}} + \sqrt{\frac{x_1}{x_2}} + z\sqrt{\frac{x_2}{x_1}}$.

Let $x_1 x_2 x_3 x_4 = q \neq 1$ (*no* integrability!)

$$T : (x_1, x_2, z, q) \mapsto \left(x_2 \left(\frac{x_1 + z}{x_1 + 1} \right)^2, x_1^{-1}, qz, q \right)$$

Casimir z as "time" $x_1 = x(z)$, $x_2 = x^{-1}(q^{-1}z)$, $T : x(z) \mapsto x(qz)$, satisfying

$$x(qz)x(q^{-1}z) = \left(\frac{x(z) + z}{x(z) + 1} \right)^2$$

or q-Painlevé III_3 equation $P(A_7^{(1)'})$.

Tau-functions

For the tau-functions $x(z) = z^{1/2}\frac{\tau_1(z)^2}{\tau_0(z)^2}$ one gets bilinear (non-autonomous!) *Hirota equations*

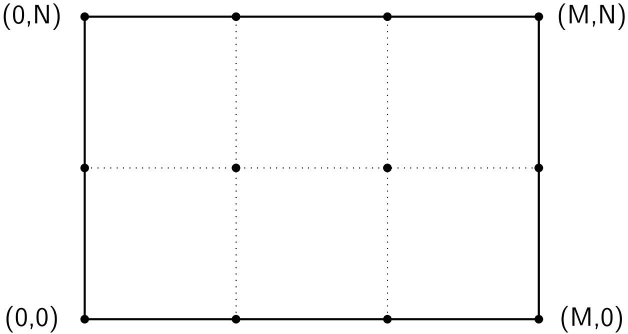
$$\tau_0(qz)\tau_0(q^{-1}z) = \tau_0(z)^2 + z^{1/2}\tau_1(z)^2$$
$$\tau_1(qz)\tau_1(q^{-1}z) = \tau_1(z)^2 + z^{1/2}\tau_0(z)^2$$

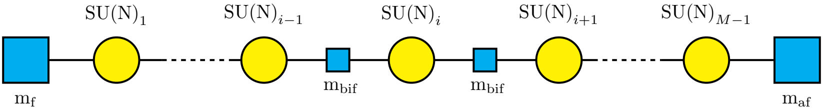
"Generic phenomenon": for the $SU(N)_k$-Toda family ($Y^{N,k}$-geometry)

$$\tau_j\left(qz\right)\tau_j\left(q^{-1}z\right) = \tau_j(z)^2 + z^{1/N}\tau_{j+1}\left(q^{k/N}z\right)\tau_{j-1}\left(q^{-k/N}z\right)$$
$$j \in \mathbb{Z}/N\mathbb{Z}$$

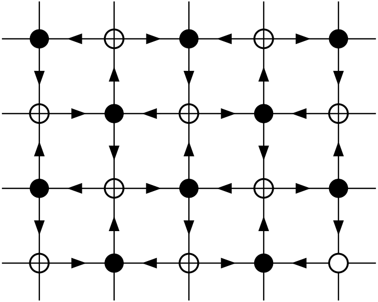
Spin chains: known only for 'exchange zig-zag' transformations ...

Quiver gauge theories and spin chains





Poisson quivers: examples



- Toda: $2 \times N$ fundamental domain of square lattice;
- XXZ-type spin chain: $N \times M$ 'fence-net' domain of the same square lattice.

'Dual' to the GK bipartite graph ...

Lie-group construction: Poisson submanifolds in (co-extended) affine groups.

- Toda: 'Coxeter' words in double Weyl groups $s_N\bar{s}_N \dots s_1\bar{s}_1$ of $\widehat{sl}_N$ or $(s_0\bar{s}_0 s_1\bar{s}_1)^N$ in double $W(\widehat{sl}_2)$;
- Spin chain:

$$(s_M\bar{s}_M \dots s_1\bar{s}_1\Lambda)^N$$

in coextended $W(\widehat{sl}_N)$ (or $W(\widehat{sl}_M)$).

Dirac-Kasteleyn operator of the structure

$$\mathfrak{D}(\lambda,\mu) = \sum E \otimes A + E \otimes C$$

gives Lax matrices

$$L \sim C^{-1}A$$

Toda chains: cluster versus Darboux variables

- Poisson quiver $\mathcal{Q}$ (fundamental domain with $2N$ vertices of square lattice):

$$\{y_i, x_j\} = \widehat{C}_{ij} y_i x_j, \qquad i, j \in \mathbb{Z}/N\mathbb{Z}$$

gives the bracket with affine Cartan matrix;

- Cluster versus canonical co-ordinates

$$x_i = \exp(-(\alpha_i \cdot q)), \quad y_i = \exp(\alpha_i \cdot (P + q)), \qquad i \in \mathbb{Z}/N\mathbb{Z}$$

and

$$P = p + \frac{\partial}{\partial q}\left(\frac{1}{2}\sum_{k=1}^{N} \mathrm{Li}_2\left(-\exp(\alpha_k \cdot q)\right)\right)$$

Cluster versus spin variables

- Poisson quiver $\mathcal{Q}$ ($N \times M$ 'fence-net' domain of square lattice):

$$\{x^{\times}_{ia}, x^{+}_{jb}\} = (\delta_{i,j+1}\delta_{ab} + \delta_{ij}\delta_{a+1,b} - \delta_{ij}\delta_{ab} - \delta_{i,j+1}\delta_{a+1,b})x^{\times}_{ia}x^{+}_{jb},$$

$$\{x^{\times}_{ia}, x^{\times}_{jb}\} = \{x^{+}_{ia}, x^{+}_{jb}\} = 0, \qquad i,j \in \mathbb{Z}/N\mathbb{Z}, \qquad a,b \in \mathbb{Z}/M\mathbb{Z}$$

- Relation to spin variables:

$$x^{\times}_{i,a} = e^{-2(S^0_a)_i}$$

but

$$x^{+}_{i,a} = -(S^{+}_{a-1})_{i+1}(S^{-}_{a})_i \frac{e^{(S^0_a)_{i+1}+(S^0_{a-1})_i}}{\cosh{(S^0_{a-1})_{i+1}}\cosh{(S^0_a)_i}}$$

- Spin chains: inhomogeneous (casimir functions), yet classical ...

Cluster MCG

- local $\mathbb{P}^2$:

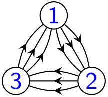
$$\mathcal{G}_{\mathcal{Q}} \simeq \mathcal{G}_{\Delta} \simeq \mathbb{Z}/3\mathbb{Z}$$

- Toda or pure $SU(2)$ or local $\mathbb{P}^1 \times \mathbb{P}^1$:

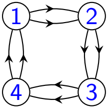
$$\mathcal{G}_{\mathcal{Q}} \simeq Dih_4 \ltimes W(A_1^{(1)}) \supset \mathcal{G}_{\Delta} \simeq \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

Painlevé quivers and their MCG

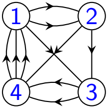
$\mathbb{Z}/3\mathbb{Z}$



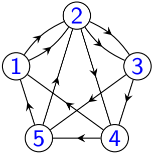
$Dih_4 \ltimes W(A_1^{(1)})$



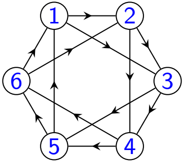
$W(A_1^{(1)})$



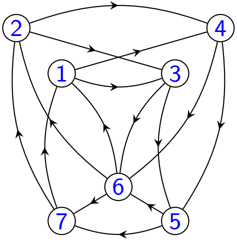
$\tilde{W}((A_1 + A_1)^{(1)})$



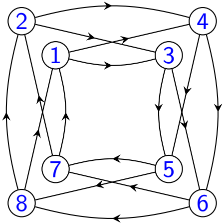
$\tilde{W}((A_1 + A_2)^{(1)})$



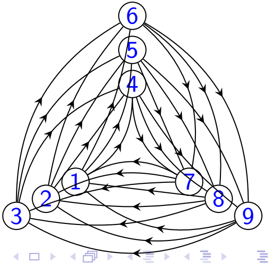
$\tilde{W}(D_4^{(1)})$



$\tilde{W}(D_5^{(1)})$



$\tilde{W}(E_6^{(1)})$



Cluster MCG for spin chains

- MCG from Newton polygon $\mathcal{G}_\mathcal{Q} \supset W\left(A^{(1)}_{N-1} \times A^{(1)}_{N-1} \times A^{(1)}_{M-1} \times A^{(1)}_{M-1}\right)$;
- Spin chains (of XXZ-type), obvious $N \leftrightarrow M$ duality (fiber-base?).

In special cases:

- If $M = 2$ (or $N = 2$!)

$$\mathcal{G}_\mathcal{Q} \supset W\left(A^{(1)}_{2N-1} \times A^{(1)}_{1} \times A^{(1)}_{1}\right) \supset W\left(A^{(1)}_{N-1} \times A^{(1)}_{N-1} \times A^{(1)}_{1} \times A^{(1)}_{1}\right)$$

  by the Gaiotto transform.
- If $M = N = 2$

$$\mathcal{G}_\mathcal{Q} = W(D^{(1)}_5) \supset W\left(A^{(1)}_{3} \times A^{(1)}_{1} \times A^{(1)}_{1}\right) \supset W\left(A^{(1)\times 4}_{1}\right)$$

Tau-functions: spin chains

For distinguished set of the generators generators $\{s_{i,i+1}\}$ and $\{s_{a,a+1}\}$ for the subgroups $W(A^{(1)}_{N-1}) \subset \mathcal{G}_{\mathcal{Q}}$ and $W(A^{(1)}_{M-1}) \subset \mathcal{G}_{\mathcal{Q}}$:

- 'Monomial action' on the Casimir functions;
- For the tau-variables: $\{\tau^{\times}_{ia}, \tau^{+}_{jb}\}$, $i, j \in \mathbb{Z}/N\mathbb{Z}$, $a, b \in \mathbb{Z}/M\mathbb{Z}$ generate the set of *bilinear* relations;
- Solutions to be given in terms of Nekrasov functions for the quiver gauge theories with (fundamental and bifundamental) matter.

'In principle' exist for Dynkin (e.g. linear) gauge quivers ...

Solutions: $q = 1$

Cf. with autonomous Hirota equations

$$\tau_{n,m+1}\tau_{n,m-1} = \tau_{n,m}^2 + z^{1/N} \cdot \tau_{n+1,m}\tau_{n-1,m}$$

with (N,k)-periodicity $\tau_{n+N,m+k} = \tau_{n,m}$.

For $(N,k) = (2,0)$

$$\tau_{0,m} = \left(\frac{\theta_3(0)}{\theta_3(U)}\right)^{m^2} \theta_3(Z + mU), \quad \tau_{1,m} = e^{i\pi/4} \left(\frac{\theta_3(0)}{\theta_3(U)}\right)^{m^2} \theta_1(Z + mU)$$

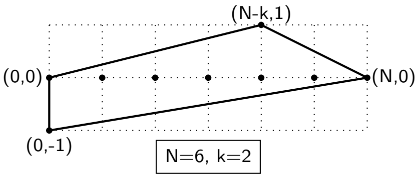
where Jacobi's

$$\theta_j(Z) = \sum_{n \in \mathbb{Z}+e_j} e^{2\pi i n Z} \mathfrak{q}^{n^2} = \sum_{n \in \mathbb{Z}+e_j} s^n \mathcal{Z}(n)$$

is just *Fourier* transform of 'classical' $\mathcal{Z}(\sigma) = \mathcal{Z}_{\rm cl}(\sigma) = \mathfrak{q}^{\sigma^2}$.
Classical factors – from tropical limit?

Solutions: $q = 1$

Almost the same for $SU(N)_k$ (hyperelliptic $\Sigma_{N,k}$) polygon



$$\tau_{n,m} = \Theta(Z + nV + mU) \left(\frac{E(x,y)E(u,v)}{E(x,v)E(u,y)} \right)^{(nk-mN)^2/2N^2}$$

where from the Fay identities

$$2U = \mathcal{A}(x) - \mathcal{A}(y) - \mathcal{A}(u) + \mathcal{A}(v), \quad 2V = \mathcal{A}(x) - \mathcal{A}(y) + \mathcal{A}(u) - \mathcal{A}(v)$$

with Abel maps $\mathcal{A}(P) = \int^P \omega$ on $\Sigma_{N,k} \in Jac(\Sigma_{N,k})$, and Prime forms $E(P, P')$.

Solutions: $q = 1$

Actually the same formula for generic Newton polygon Δ, $j = 1, \ldots, B$

$$\mathcal{T} \sim \Theta(Z + \sum_j n_j A(P_j)) \prod_{i<j} E(P_i, P_j)^{n_i n_j} \times \prod_i e^{Q_i n_i^2}$$

just to satisfy Fay identities.

Easy to write up to 'Casimir factor'

$$e^{Q_i} \sim \prod_{j \neq i} E(P_i, P_j)^{-l_j/2l_i} \sim \prod_{j \neq i} E(P_i, P_j)^{-m_j/2m_i}$$

from the divisors $(\lambda) = \sum_j l_j P_j$ and $(\mu) = \sum_j m_j P_j$ of two functions from the equation $f_\Delta(\lambda, \mu) = 0$.

Solutions: deautonomization $q \neq 1$

Again just Fourier transform, e.g. instead of Jacobi's theta

$$\tau_j(u, s; q|z) = \sum_{n \in \mathbb{Z}+j/2} s^n \mathcal{Z}(uq^n; q^{-1}, q|z), \quad j \in \mathbb{Z}/2\mathbb{Z}$$

where $\mathcal{Z}(u; q_1, q_2|z)$ is pure $SU(2)$ 5d Nekrasov function (Kiev formula).

Generally for the (N, k)-theory

$$\tau_j^{N,k}(\vec{u}, \vec{s}; q|z) = \sum_{\vec{\Lambda} \in Q_{N-1}+\omega_j} s^{\Lambda} \mathcal{Z}_{N,k}(\vec{u} q^{\vec{\Lambda}}; q^{-1}, q|z) \tag{7}$$

where sum is over A_{N-1} root lattice, $\{\omega_j\}$ are fundamental weights, and $\mathcal{Z}_{N,k} = \mathcal{Z}_{\mathrm{cl}}^{N,k} \cdot \mathcal{Z}_{1-\mathrm{loop}}^{N} \cdot \mathcal{Z}_{\mathrm{inst}}^{N,k}$ are 5d Nekrasov functions.

Nekrasov functions

Here:

$$\mathcal{Z}_{\rm cl}^{N,k} = \exp\left(\log z \frac{\sum(\log u_i)^2}{-2\log q_1 \log q_2} + k\frac{\sum(\log u_i)^3}{-6\log q_1\log q_2}\right)$$

$$\mathcal{Z}_{1-\rm loop}^{N} = \prod_{1\le i\neq j\le N} (u_i/u_j; q_1, q_2)_\infty$$

$$\mathcal{Z}_{\rm inst}^{N,k} = \sum_{\vec\lambda} \frac{z^{|\vec\lambda|}\prod_{i=1}^N \mathsf{T}_{\lambda^{(i)}}(u; q_1, q_2)^k}{\prod_{i,j=1}^N \mathsf{N}_{\lambda^{(i)},\lambda^{(j)}}(u_i/u_j; q_1, q_2)}$$

with

$$\mathsf{N}_{\lambda,\mu}(u, q_1, q_2) = \prod_{s\in\lambda}(1 - uq_2^{-a_\mu(s)-1}q_1^{\ell_\lambda(s)})\prod_{s\in\mu}(1 - uq_2^{a_\lambda(s)}q_1^{-\ell_\mu(s)-1})$$

$$\mathsf{T}_\lambda(u; q_1, q_2) = u^{|\lambda|}q_1^{\frac12(\|\lambda^t\|-|\lambda^t|)}q_2^{\frac12(\|\lambda\|-|\lambda|)} = \prod_{(i,j)\in\lambda} uq_1^{i-1}q_2^{j-1},$$

and $\vec\lambda = (\lambda^{(1)},\dots,\lambda^{(N)})$, $|\vec\lambda| = \sum|\lambda^{(i)}|$, $|\lambda| = \sum\lambda_j$, $\|\lambda\| = \sum\lambda_j^2$.

Conclusion

- Cluster varieties and integrable systems;
- Deautonomization: solutions from Nekrasov functions of 5d SYM;
- Many open questions ...

Merci beaucoup!