Rigorous approach to the XXZ chain at finite temperature

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Outline

Typical setting of condensed matter physics

- Hilbert space $\mathfrak{h} = \mathfrak{h}_1 \otimes \cdots \otimes \mathfrak{h}_L$
- Basis of operators $\mathrm{o}^{(\alpha)}$ on $\mathfrak{h}_0$ $\rightsquigarrow$ operators $\mathrm{o}_{\ell}^{(\alpha)} = \underbrace{\mathrm{id} \otimes \dots \mathrm{id}}_{\ell-1 \mathrm{times}} \otimes \mathrm{o}^{(\alpha)} \otimes \underbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{N-\ell-1}$.
- H_L lattice Hamiltonian in 1D with L sites and periodic bc $\rightsquigarrow$ quasi-local in $\mathrm{o}_{\ell}^{(\alpha)}$

$$\mathrm{H}_L = \sum_{n=1}^{L} \boldsymbol{F}\big(\mathrm{o}_{\ell}^{(\alpha)}, \mathrm{o}_{\ell+1}^{(\beta)}\big) \quad , \quad \mathrm{o}_{n+L}^{(\alpha)} \equiv \mathrm{o}_{n}^{(\alpha)}$$

What one would like to know?

i) Fully characterise the finite temperature per-site free energy

$$f = -T \lim_{L \to +\infty} \Big\{ \frac{1}{L} \ln \mathcal{Z}_L \Big\} \qquad \text{with} \qquad \mathcal{Z}_L = \mathrm{tr}_{\mathfrak{h}} \Big[\mathrm{e}^{-\frac{1}{T}\mathrm{H}_L} \Big]$$

ii) Fully characterise the correlators in infinite volume at finite temperature

$$\big\langle \mathrm{o}_1^{(\alpha)} \mathrm{o}_{1+x}^{(\beta)} \big\rangle \equiv \lim_{L \to +\infty} \Big\{ \mathrm{tr}_{\mathfrak{h}} \Big[\frac{\mathrm{e}^{-\frac{1}{T}\mathrm{H}_L}}{\mathcal{Z}_L} \mathrm{o}_1^{(\alpha)} \mathrm{o}_{1+x}^{(\beta)} \Big] \Big\}$$

✔ The limits exist for a wide class of models (**’60’s Ruelle**)

The free energy of the Bose gas I

⊛ The non-linear Schrödinger model

$$\mathrm{H}_{NLS} = \int_0^L \Big\{ \partial_y \Psi^\dagger(y) \partial_y \Psi(y) + c \Psi^\dagger(y) \Psi^\dagger(y) \Psi(y) \Psi(y) - h \Psi^\dagger(y) \Psi(y) \Big\} \mathrm{d}y$$

L: length of circle, $c > 0$ coupling constant (repulsive regime), $h > 0$ chemical potential.

- Eigenfunctions & spectrum (**'63 Lieb, Liniger**), (**'64 Brezin, Pohil, Finkelberg**) .

$$\mathrm{e}^{iL\lambda_j} = \prod_{\substack{a=1 \\ \neq j}}^{N} \frac{\lambda_j - \lambda_a + ic}{\lambda_j - \lambda_a - ic} \quad \rightsquigarrow \quad \Big(\boldsymbol{\Phi}(\{\lambda_a\}_1^N), \mathcal{E}(\{\lambda_a\}_1^N) = \sum_{k=1}^{N} \big(\lambda_k^2 - h \big) \Big)$$

- Heuristic derivation of the per-site free energy (**'69 Yang, Yang**):

$$f = -T \int_{\mathbb{R}} \ln\Big(1 + \mathrm{e}^{-\frac{\varepsilon(\mu)}{T}} \Big) \cdot \frac{\mathrm{d}\mu}{2\pi}$$

in terms of the solution ε to a non-linear integral equation

$$\varepsilon(\lambda) = \lambda^2 - h - T \int_{\mathbb{R}} \frac{2c}{(\lambda-\mu)^2 + c^2} \ln\Big(1 + \mathrm{e}^{-\frac{\varepsilon(\mu)}{T}} \Big) \cdot \frac{\mathrm{d}\mu}{2\pi}$$

The free energy of the Bose gas II

$$\mathcal{Z}_L = \sum_{N \geq 0} \sum_{\{\lambda_a\}_1^N} \mathrm{e}^{-\frac{1}{T}\mathcal{E}(\{\lambda_a\}_1^N)}$$

- Estimate large N, L behaviour of $\mathcal{E}(\{\lambda_a\}_1^N)$ for a fixed "density" of Bethe roots;
- Estimate number of solutions to Bethe equation for a fixed "density" ⇝ entropic term;
- Summand $\mathrm{e}^{-LI[\varrho]}$ $\quad f = T\inf_{\varrho} I[\varrho]$.

⊛ **'93** **Dorlas** : Completeness of Bethe Ansatz for NLSM;

⊛ **'89** **Dorlas, Lewis, Pulé** : LDP for free gas & Varadhan's lemma to estimate $\mathcal{Z}_L$;

⇝ Proof of Yang-Yang's results.

The XXZ chain

- The XXZ spin-1/2 chain on $\mathfrak{h}_{XXZ} = \otimes_{n=1}^{L} \mathbb{C}^2$, σ^α Pauli matrices

$$\mathrm{H} = \sum_{n=1}^{L} \left\{ \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \cos(\zeta) \sigma_n^z \sigma_{n+1}^z - h \sigma_n^z \right\} \quad , \quad \sigma_{n+L}^\alpha \equiv \sigma_n^\alpha$$

- ('60s **Ruelle**) $\mathtt{f} = -T \lim_{L \to +\infty} \left\{ \frac{1}{L} \ln \mathrm{tr}_{\mathfrak{h}_{XXZ}} \left[\mathrm{e}^{-\frac{1}{T}\mathrm{H}} \right] \right\}$
- Model solvable by Bethe Ansatz (**'31 Bethe** , **'58 Orbach**)

✓ Yang-Yang approach to $\mathtt{f}$ (**'71 Gaudin** , **'71 Takahashi**):

- Bethe equations admit complex solutions forming strings when $L \to +\infty$
- ∞ system of coupled NLIE for string centre densities $\varrho_1, \varrho_2, \ldots,$
- $\mathtt{f} \equiv$ 1D integral involving ϱ_1

BUT

- Completeness of the Bethe Ansatz is subtle (**'09 Mukhin, Tarasov, Varchenko**)
- Counting of "allowed" strings is very subtle (**'72 Takahashi**)
- String hypothesis is wrong (**'82 Destri, Lowenstein**)
- ∞ system of NLIE "bad" for numerics
- Method does not seem extendable to compute the correlators

Koma's transfer matrix

- Main idea ⇝ Trotter-Suzuki decomposition (**'87-'89** **Koma**)

$$\mathrm{e}^{-\frac{1}{T}\mathrm{H}} = \lim_{N\to+\infty} \mathrm{O}_{N;L} \qquad \text{so that} \qquad \mathrm{tr}_{\mathfrak{h}_{XXZ}}\Big[\mathrm{e}^{-\frac{1}{T}\mathrm{H}}\Big] = \lim_{N\to+\infty} \mathrm{tr}_{\mathfrak{h}_{\mathfrak{q}}}\Big[\mathcal{T}^{L}_{N,T}\Big]$$

- $\mathcal{T}_{N,T}$ transfer matrix diagonalisable by Bethe Ansatz
- exchangeability of N and $L \to +\infty$ limits (**'85** **Suzuki**)
- $\mathcal{T}_{N,T}$ admits a real, non-degenerate maximal in modulus Eigenvalue $\Lambda_{\max}\big(\mathcal{T}_{N,T}\big)$

$$\mathfrak{f} = -T \lim_{L\to+\infty}\lim_{N\to+\infty}\Big\{\frac{1}{L}\ln \mathrm{tr}_{\mathfrak{h}_{\mathfrak{q}}}\Big[\mathcal{T}^{L}_{N,T}\Big]\Big\} = \lim_{N\to+\infty}\lim_{L\to+\infty}\Big\{\frac{1}{L}\ln \mathrm{tr}_{\mathfrak{h}_{\mathfrak{q}}}\Big[\mathcal{T}^{L}_{N,T}\Big]\Big\} = -T \lim_{N\to+\infty}\Big\{\ln \Lambda_{\max}\big(\mathcal{T}_{N,T}\big)\Big\}$$

- Which Bethe vectors give the dominant Eigenvalue?
- Numerical study of the limit from Bethe Ansatz equations
- Formal Trotter limit of Bethe Ansatz equations (**'91** **Takahashi**)

Akutsu-Wadati quantum transfer matrix

- $\mathrm{tr}_{\mathfrak{h}_{XXZ}}\Big[\mathrm{e}^{-\frac{1}{T}\mathrm{H}}\Big] = \lim_{N\to+\infty} \mathrm{tr}_{\mathfrak{h}_{\mathfrak{q}}}\Big[\mathrm{t}_{\mathfrak{q}}^{L}\Big]$ ('**90** **Akutsu, Wadati**)
- Quantum transfer matrix: $\mathrm{t}_{\mathfrak{q}} = \mathrm{tr}_{\mathfrak{h}_0}\Big[\mathrm{T}_{\mathfrak{q};0}(0)\Big]$
- Staggered monodromy matrix

$$\mathrm{T}_{\mathfrak{q};0}(\xi) = \mathrm{R}^{\mathrm{t}_{2N}}_{2N,0}\Big(-\frac{\beta}{N}-\xi\Big)\mathrm{R}_{0,2N-1}\Big(\xi-\frac{\beta}{N}\Big)\cdots\mathrm{R}^{\mathrm{t}_2}_{2,0}\Big(-\frac{\beta}{N}-\xi\Big)\mathrm{R}_{0,1}\Big(\xi-\frac{\beta}{N}\Big)\cdot\mathrm{e}^{\frac{h}{2T}\sigma_0^z} \quad \text{with} \quad \beta = \frac{-\mathrm{i}J}{T}\sin(\zeta)$$

- Six-vertex R-matrix $\mathrm{R}(\lambda) = \dfrac{1}{\sinh(-\mathrm{i}\zeta)}\begin{pmatrix} \sinh(\lambda-\mathrm{i}\zeta) & 0 & 0 & 0 \\ 0 & \sinh(\lambda) & \sinh(-\mathrm{i}\zeta) & 0 \\ 0 & \sinh(-\mathrm{i}\zeta) & \sinh(\lambda) & 0 \\ 0 & 0 & 0 & \sinh(\lambda-\mathrm{i}\zeta) \end{pmatrix}$
- *If* commutativity of limits & existence of non-degenerate, real, Eigenvalue hold

$$\mathfrak{f} = -T\lim_{N\to+\infty}\Big\{\ln\widehat{\Lambda}_{\max}\Big\}$$

The Bethe Ansatz approach

⊛ Bethe Ansatz ⇝ Eigenvectors $\mathtt{t}_{\mathfrak{q}} \cdot \boldsymbol{\Psi}\big(\{\lambda_a\}_{a=1}^{M}\big) = \tau\big(\{\lambda_k\}_1^M\big)\boldsymbol{\Psi}\big(\{\lambda_a\}_{a=1}^{M}\big)$

$$\mathrm{e}^{-\frac{h}{T}}(-1)^{N-M}\prod_{k=1}^{M}\left\{\frac{\sinh(\mathrm{i}\zeta-\lambda_a+\lambda_k)}{\sinh(\mathrm{i}\zeta+\lambda_a-\lambda_k)}\right\}\cdot\left\{\frac{\sinh(\lambda_a-\frac{\beta}{N})\sinh(\mathrm{i}\zeta+\lambda_a+\frac{\beta}{N})}{\sinh(\lambda_a+\frac{\beta}{N})\sinh(\mathrm{i}\zeta-\lambda_a+\frac{\beta}{N})}\right\}^{N} = -1$$

- $\lambda_a \neq \lambda_b \;,\; \lambda_a \neq \lambda_b \pm \mathrm{i}\zeta \mod \mathrm{i}\pi\mathbb{Z}$ for any a, b
- $\lambda_a \notin \left\{\pm\frac{\beta}{N}, \pm\frac{\beta}{N} \pm \mathrm{i}\zeta\right\}$ for any a

♦ No dense distribution of Bethe roots when $N \to +\infty$.

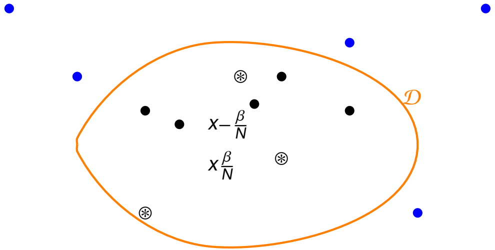
⊛ ('92 **Klümper** , '92 **Destri, de Vega**)

$$\mathrm{e}^{\widehat{\mathfrak{A}}(\xi)} = \mathrm{e}^{-\frac{h}{T}}(-1)^{N-M}\prod_{k=1}^{M}\left\{\frac{\sinh(\mathrm{i}\zeta-\xi+\lambda_k)}{\sinh(\mathrm{i}\zeta+\xi-\lambda_k)}\right\}\cdot\left\{\frac{\sinh(\xi-\frac{\beta}{N})\sinh(\mathrm{i}\zeta+\xi+\frac{\beta}{N})}{\sinh(\xi+\frac{\beta}{N})\sinh(\mathrm{i}\zeta-\xi+\frac{\beta}{N})}\right\}^{N}$$

- $\mathrm{e}^{\widehat{\mathfrak{A}}(\lambda_a)} = -1$ by construction
- Alternative characterisation of $\widehat{\mathfrak{A}}$ ⇝ tool to compute the Bethe roots

The non-linear integral equation

- Pick a domain $\mathcal{D} \subset \mathbb{C}$
- $\{\lambda_a\}_1^M \cap \mathcal{D}^c = \widehat{\mathcal{Y}} = \{\widehat{y}_a\}_1^{|\widehat{\mathcal{Y}}|}$ Bethe roots outside of $\mathcal{D}$ "particles" $\quad e^{\widehat{\mathfrak{A}}(\widehat{y}_a)} = -1$
- $\lambda \in \{\lambda_a\}_1^M \setminus \widehat{\mathcal{Y}}$ Bethe roots inside of $\mathcal{D}$ $\quad e^{\widehat{\mathfrak{A}}(\lambda)} = -1$
- $\widehat{\mathfrak{X}} = \{\widehat{x}_a\}_1^{|\widehat{\mathfrak{X}}|}$ non-Bethe roots inside of $\mathcal{D}$ "holes" $\quad e^{\widehat{\mathfrak{A}}(\widehat{x}_a)} = -1$



● : particle roots ⊛ :hole root ●:inner Bethe root

The non-linear problem

- Monodromy condition

$$\oint_{\partial\mathcal{D}} \frac{\widehat{\mathfrak{A}}'(u)}{1+\mathrm{e}^{-\widehat{\mathfrak{A}}(u)}} \cdot \frac{\mathrm{d}u}{2\mathrm{i}\pi} = M - N - |\widehat{\mathcal{Y}}| + |\widehat{\mathfrak{X}}|$$

- Non-linear integral equation

$$\widehat{\mathfrak{A}}(\xi) = -\frac{h}{T} + \mathfrak{w}_N(\xi) + \mathrm{i}\pi(M-N) + \mathrm{i}\sum_{y\in\widehat{\mathcal{Y}}}\theta(\xi-y) - \mathrm{i}\sum_{x\in\widehat{\mathfrak{X}}}\theta(\xi-x) + \oint_{\partial\mathcal{D}} \frac{\sin(2\zeta)\cdot\mathcal{L}\mathrm{n}\big[1+\mathrm{e}^{\widehat{\mathfrak{A}}}\big](u)}{\sinh(\xi-u-\mathrm{i}\zeta)\sinh(\xi-u+\mathrm{i}\zeta)} \cdot \frac{\mathrm{d}u}{2\pi}$$

  - Driving terms

$$\mathfrak{w}_N(\xi) = N\ln\left(\frac{\sinh(\xi-\frac{\beta}{N})\sinh(\xi+\frac{\beta}{N}-\mathrm{i}\zeta)}{\sinh(\xi+\frac{\beta}{N})\sinh(\xi-\frac{\beta}{N}-\mathrm{i}\zeta)}\right) \quad , \quad \theta(\lambda) = \mathrm{i}\ln\left(\frac{\sinh(\mathrm{i}\zeta+\lambda)}{\sinh(\mathrm{i}\zeta-\lambda)}\right)$$

- Auxiliary conditions on $\widehat{\mathcal{Y}}$ and $\widehat{\mathfrak{X}}$

$$\widehat{x}\in\widehat{\mathfrak{X}} \; : \; \mathrm{e}^{\widehat{\mathfrak{A}}(\widehat{x})} = -1 \qquad\qquad \widehat{y}\in\widehat{\mathcal{Y}} \; : \; \mathrm{e}^{\widehat{\mathfrak{A}}(\widehat{y})} = -1$$

- Integral representation for the Eigenvalues

$$\tau\big(\widehat{\mathfrak{A}}\big) = \prod_{y\in\widehat{\mathcal{Y}}}\left\{\frac{\sinh(y-\mathrm{i}\zeta)}{\sinh(y)}\right\}\prod_{x\in\widehat{\mathfrak{X}}}\left\{\frac{\sinh(x)}{\sinh(x-\mathrm{i}\zeta)}\right\}\cdot\left(\frac{\sinh(\frac{\beta}{N}+\mathrm{i}\zeta)}{\sinh(\mathrm{i}\zeta)}\right)^{2N}\exp\left\{\frac{h}{2T} - \oint_{\partial\mathcal{D}}\frac{\sin(\zeta)\,\mathcal{L}\mathrm{n}\big[1+\mathrm{e}^{\widehat{\mathfrak{A}}}\big](u)}{\sinh(u-\mathrm{i}\zeta)\sinh(u)}\cdot\frac{\mathrm{d}u}{2\pi}\right\}$$

Trotter limit for dominant Eigenvalue

- Dominant Eigenvalue's $\widehat{\mathfrak{A}}$-function:

$$\mathcal{D} = \Big\{ -\mathbb{R} + \mathrm{i}\frac{\min(\zeta, \pi - \zeta)}{2} \Big\} \cup \Big\{ \mathbb{R} - \mathrm{i}\frac{\min(\zeta, \pi - \zeta)}{2} \Big\} \quad , \qquad \widehat{\mathfrak{X}} = \widehat{\mathcal{Y}} = \emptyset \quad , \qquad N = M$$

- pointwise Trotter limit of driving term

$$\mathfrak{w}_N(\xi) \underset{N \to +\infty}{\longrightarrow} \mathfrak{w}_\infty(\xi) = \frac{2J \sin^2(\zeta)}{T \sinh(\xi) \sinh(\xi - \mathrm{i}\zeta)}$$

- $\mathfrak{A}$ solves NLIE with $\mathfrak{w}_N \hookrightarrow \mathfrak{w}_\infty$

- $\widehat{\mathfrak{A}} \underset{N \to +\infty}{\longrightarrow} \mathfrak{A}$ and $\tau(\widehat{\mathfrak{A}}) \underset{N \to +\infty}{\longrightarrow} \tau(\mathfrak{A})$

The open problems

Conjectures

- Exchangeability of the Trotter and thermodynamic limits
- $\mathtt{t}_{\mathfrak{q}}$ has a non-degenerate maximal in modulus real Eigenvalue $\widehat{\Lambda}_{\max}$
- Which domain $\mathcal{D}$ & solution to NLIE describes $\mathfrak{A}$-function associated with $\widehat{\Lambda}_{\max}$
- Existence and uniqueness of solutions to NLIEs
- Continuity for $N \to +\infty$ of the solutions

Other Questions

- Sub-dominant Eigenvalues (correlation lengths)

Integral representation for the per-site free energy

Theorem

There exist $T_0 > 0$, $\epsilon > 0$ and $\mathfrak{c} > 0$ such that for any $T > T_0$,

$$-\frac{1}{T}\mathrm{f} = \frac{h}{2T} - \frac{2J}{T}\cos(\zeta) - \oint_{\partial\mathcal{D}_{0,\epsilon}} \frac{\sin(\zeta)\,\ln\left[1+\mathrm{e}^{\mathfrak{A}}\right](u)}{\sinh(u-\mathrm{i}\zeta)\,\sinh(u)} \cdot \frac{\mathrm{d}u}{2\pi}$$

with $\mathfrak{A}$ the unique solution to the non-linear integral equation on $\mathcal{B}_{\mathfrak{c}/T}$

$$\mathfrak{A}(\xi) = -\frac{1}{T}\left\{h - \frac{2J\sin^2(\zeta)}{\sinh(\xi)\,\sinh(\xi-\mathrm{i}\zeta)}\right\} + \oint_{\partial\mathcal{D}_{0,\epsilon}} \frac{\sin(2\zeta)\cdot\ln\left[1+\mathrm{e}^{\mathfrak{A}}\right](u)}{\sinh(\xi-u+\mathrm{i}\zeta)\,\sinh(\xi-u-\mathrm{i}\zeta)} \cdot \frac{\mathrm{d}u}{2\pi}$$

with $\mathcal{B}_r = \left\{f \in \mathcal{O}(\mathcal{S}) \;:\; \|f\|_{L^\infty(\mathcal{S})} \leq r\right\}$.

⊛ Similar results are established for the subdominant Eigenvalues ratios $\lim_{N\to+\infty}\left\{\widehat{\Lambda}_k/\widehat{\Lambda}_{\max}\right\}$

Large-T behaviour of Eigenvalue ratios

Theorem

Fix integers n_x, n_y and pick some $\varrho > 0$ small enough. Let $h_1, \dots, h_{n_x} \in \mathbb{Z}$ be pairwise distinct.

Let $\{y_a\}_1^{n_y}$ solve the system

$$(-1)^{n_x-n_y+1}\prod_{b=1}^{n_y}\Big\{\sinh(i\zeta+y_b-y_a)\Big\}\cdot\Big(\sinh(i\zeta+y_a)\Big)^{n_x} = \prod_{b=1}^{n_y}\Big\{\sinh(i\zeta+y_a-y_b)\Big\}\cdot\Big(\sinh(i\zeta-y_a)\Big)^{n_x}$$

and fulfill the three subsidiary constraints

- $y_a \neq y_b \pm i\zeta, \quad y_a \neq y_b \mod i\pi\mathbb{Z}$ for $a,b \in [\![1\,;\,n_y]\!]$
- $\Big|(-1)^{n_x-n_y}\prod_{b=1}^{n_y}\dfrac{\sinh(i\zeta+y_b)}{\sinh(i\zeta-y_b)} + 1\Big| > \varrho$
- $y_a \in \Big\{z \in \mathbb{C} \;:\; |\Im(z)| \leq \dfrac{\pi}{2}\,,\; z \notin \mathcal{D}_{\pm i\zeta_{\mathrm{m}},\varrho} \cup \{0\}\Big\}$ *with* $\zeta_{\mathrm{m}} = \min\{\zeta, \pi-\zeta\}$

Then, there exists an Eigenvalue $\widehat{\Lambda}_k$ of $\mathtt{t}_{\mathfrak{q}}$ with large-T asymptotics

$$\lim_{N\to+\infty}\widehat{\Lambda}_k = \frac{1}{2T^{n_x}}\prod_{a=1}^{n_x}\Big\{\frac{-2iJ}{(2h_a+1+n_x-n_y)\pi-\sum_{a=1}^{n_y}\theta_+(-y_a)}\Big\}\prod_{a=1}^{n_y}\frac{\sinh(y_a-i\zeta)}{\sinh(y_a)}\cdot\Big(1+\mathrm{o}(1)\Big)$$

⊛ $\exp\{-\frac{1}{\xi}_k\} = \lim_{N\to+\infty}\Big\{\widehat{\Lambda}_k/\widehat{\Lambda}_{\max}\Big\}$ control the exponential decay of correlation functions

An operator High-T expansion

- Monodromy matrix

$$\mathrm{T}_{\mathfrak{q};0}(0) = \mathrm{R}^{t_{2N}}_{2N,0}\Big(-\frac{\beta}{N}\Big)\mathrm{R}_{0,2N-1}\Big(-\frac{\beta}{N}\Big)\cdots\mathrm{R}^{t_2}_{2,0}\Big(-\frac{\beta}{N}\Big)\mathrm{R}_{0,1}\Big(-\frac{\beta}{N}\Big)\cdot e^{\frac{h}{2T}\sigma_0^z} \quad \text{with} \quad \beta = \frac{-iJ}{T}\sin(\zeta)$$

- Local large-T expansion

$$\mathrm{R}_{ab}\Big(-\frac{\beta}{N}\Big) = \mathrm{P}_{ab} + \mathrm{N}_{ab} \quad \rightsquigarrow \quad \mathrm{R}^{t_{2\ell}}_{2\ell\,0}\Big(-\frac{\beta}{N}\Big)\cdot\mathrm{R}_{0\,2\ell-1}\Big(-\frac{\beta}{N}\Big) = \underbrace{\Pi_\ell}_{\mathrm{P}^{t_{2\ell}}_{2\ell\,0}\mathrm{P}_{0\,2\ell-1}} + \underbrace{\mathrm{W}_\ell}_{\mathrm{O}(\frac{1}{TN})}$$

- Monodromy matrix expansion

$$\mathrm{T}_{\mathfrak{q};0}(0) = \Omega_{N;0} + \sum_{n=1}^{N}\sum_{\boldsymbol{\ell}\in\mathcal{L}_N^{(n)}}\boldsymbol{O}_{\boldsymbol{\ell}} \quad \text{with} \quad \boldsymbol{O}_{\boldsymbol{\ell}} = \Omega_{N;\ell_n}\cdot\mathrm{W}_{\ell_n}\cdot\Omega_{\ell_n-1;\ell_{n-1}}\cdots\mathrm{W}_{\ell_1}\cdot\Omega_{\ell_1-1;0}$$

- Transfer matrix expansion $\mathrm{t}_{\mathfrak{q}} = \omega_{N;0} + \delta\mathrm{t}_{\mathfrak{q}}$

$$\omega_{N;0} = \mathrm{Tr}_0\big[\Omega_{N;0}\big] \qquad \text{and} \qquad \delta\mathrm{t}_{\mathfrak{q}} = \sum_{n=1}^{N}\sum_{\boldsymbol{\ell}\in\mathcal{L}_N^{(n)}}\underbrace{\mathrm{tr}_0\big[\boldsymbol{O}_{\boldsymbol{\ell}}\big]}_{=\mathrm{O}_{\boldsymbol{\ell}}}$$

Some estimates

- The "leading" operator

$$\omega_{N;0} = \boldsymbol{v}\cdot\boldsymbol{w}^{t} \qquad \big(\boldsymbol{w},\boldsymbol{v}\big) = 2\cosh\Big(\frac{h}{2T}\Big) \quad , \qquad \|\boldsymbol{v}\|^2 = 2^N\cosh\Big(\frac{h}{T}\Big) \quad , \qquad \|\boldsymbol{w}\|^2 = 2^N$$

- The "sub-leading" operators

$$|||\mathrm{O}_{\boldsymbol{\ell}}||| \le 2^N\cdot\Big(\frac{C}{NT}\Big)^n \qquad , \quad \boldsymbol{\ell} = (\ell_1,\dots,\ell_n)$$

$$|||\mathrm{O}_{\boldsymbol{\ell}^{(1)}}\cdots\mathrm{O}_{\boldsymbol{\ell}^{(M)}}||| \le 2^N\cdot C_1^M\cdot\prod_{s=1}^{M}\Big\{\frac{C_2}{NT}\Big\}^{n_s} \qquad , \quad \boldsymbol{\ell}^{(s)} = (\ell_1^{(s)},\dots,\ell_{n_s}^{(s)})$$

- Uniform in N control on spectral radius and perturbed traces

$$r_S(\delta\mathrm{t}_{\mathfrak{q}}) \le \frac{C}{T} \qquad , \qquad \Big|\mathrm{tr}_{\mathfrak{h}_{\mathfrak{q}}}\Big[\omega_{N;0}\cdot\big(\delta\mathrm{t}_{\mathfrak{q}}\big)^{\ell_1}\omega_{N;0}\cdots\omega_{N;0}\cdot\big(\delta\mathrm{t}_{\mathfrak{q}}\big)^{\ell_n}\Big]\Big| \le \big(C'\big)^n\cdot\prod_{a=1}^{n}\Big(\frac{C}{T}\Big)^{\ell_a}$$

$$\big(\delta\mathrm{t}_{\mathfrak{q}}\big)^M = \sum_{\substack{\{n_s\}_{s=1}^M \\ n_s=1}}^{N}\ \sum_{\substack{\{\boldsymbol{\ell}^{(s)}\}_{s=1}^M \\ \boldsymbol{\ell}^{(s)}\in\mathcal{L}_N^{(n_s)}}} \mathrm{O}_{\boldsymbol{\ell}^{(1)}}\dots\mathrm{O}_{\boldsymbol{\ell}^{(M)}} \qquad \& \qquad \sum_{n=1}^{N}\sum_{\boldsymbol{\ell}\in\mathcal{L}_N^{(n)}} \le \frac{N^n}{n!}$$

♦ $|||(\delta\mathrm{t}_{\mathfrak{q}})^M||| \le 2^N\Big(\frac{C}{T}\Big)^M \qquad \& \qquad r_S(\delta\mathrm{t}_{\mathfrak{q}}) = \lim_{M\to+\infty}|||(\delta\mathrm{t}_{\mathfrak{q}})^M|||^{\frac{1}{M}}$

Spectral estimates

Lemma

⊛ $\widehat{\Lambda}_{\mathrm{max}} = 2 + \mathrm{O}(T^{-1}) \quad , \quad \widehat{\Lambda}_a = \mathrm{O}(T^{-1})$ uniformly in N

♦ control on zeroes of characteristic polynomial

$$\det\big[\lambda - \mathfrak{t}_{\mathfrak{q}}\big] = \det\big[\lambda - \delta\mathfrak{t}_{\mathfrak{q}}\big] \cdot \det\big[\mathrm{id} - (\lambda - \delta\mathfrak{t}_{\mathfrak{q}})^{-1}\omega_{N;0}\big] = \det\big[\lambda - \delta\mathfrak{t}_{\mathfrak{q}}\big] \cdot \Big\{1 - \mathrm{tr}_{\mathfrak{h}_{\mathfrak{q}}}\big[(\lambda - \delta\mathfrak{t}_{\mathfrak{q}})^{-1}\omega_{N;0}\big]\Big\}$$

- $\det\big[\lambda - \delta\mathfrak{t}_{\mathfrak{q}}\big]$ zeroes inside of $\mathcal{D}_{0,\frac{C}{T}}$
- Rouché & perturbed traces bound ⇝ other zero inside of $\mathcal{D}_{2,\frac{C}{T}}$

Commutativity of limits

Proposition

There exists $T_0 > 0$ such that, for any $T \geq T_0$,

$$\lim_{L\to+\infty}\lim_{N\to+\infty}\frac{1}{L}\ln \mathrm{tr}_{\mathfrak{h}_q}\big[\mathrm{t}_q^L\big] = \lim_{N\to+\infty}\lim_{L\to+\infty}\frac{1}{L}\ln \mathrm{tr}_{\mathfrak{h}_q}\big[\mathrm{t}_q^L\big]$$

⊛ (**'85, Suzuki**) Let $a_{N,L}$ be a sequence in $\mathbb{C}$ such that

$$\lim_{N\to+\infty} a_{N,L} = \alpha_L \quad , \quad \lim_{L\to+\infty} \alpha_L = \alpha \quad , \quad \lim_{L\to+\infty} a_{N,L} = z_N \qquad \text{uniformly in N}$$

Then $\lim_{N\to+\infty} = \alpha$.

⊛ Construction of quantum transfer matrix

$$\tau_{N,L} = \frac{1}{L}\ln \mathrm{tr}_{\mathfrak{h}_q}\big[\mathrm{t}_q^L\big] \qquad \lim_{N\to+\infty}\tau_{N,L} = \frac{1}{L}\ln \mathrm{tr}_{\mathfrak{h}_{XXZ}}\big[\mathrm{e}^{-\frac{1}{T}\mathrm{H}}\big]$$

⊛ (**'60s, Ruelle**) $\lim_{L\to+\infty}\frac{1}{L}\ln \mathrm{tr}_{\mathfrak{h}_{XXZ}}\big[\mathrm{e}^{-\frac{1}{T}\mathrm{H}}\big]$ exists

⊛ $\ln \mathrm{tr}_{\mathfrak{h}_q}\big[\mathrm{t}_q^L\big] = \widehat{\Lambda}_{\max}^L + \sum_{a=1}^{2^{2N}-1}\widehat{\Lambda}_a^L \quad \rightsquigarrow \quad \lim_{L\to+\infty}\tau_{N,L} = \ln\big[\widehat{\Lambda}_{\max}\big]$

⊛ Uniformness in N: $\sum_{a=1}^{2^{2N}-1}\widehat{\Lambda}_a^L = \mathrm{tr}_{\mathfrak{h}_q}\Big[\big(\mathfrak{P}\mathrm{t}_q\mathfrak{P}\big)^L\Big] \qquad \mathfrak{P} = \oint_{\partial\mathcal{D}_{0,1}}\frac{\mathrm{d}\lambda}{2i\pi}\frac{1}{\lambda-\mathrm{t}_q}$

The Bethe roots from NLIE

- Non-linear problem $\rightsquigarrow$ fixed point of a strictly contractive operator $0_{T,N}$ continuous in $N \to +\infty$
- Solution $e^{\mathfrak{A}}$ gives rise to N roots $\{\lambda_a\}_1^N$
  - pairwise distinct
  - solve Bethe Ansatz equations
  - $\boldsymbol{\Psi}\big(\{\lambda_a\}_1^N\big) \neq 0$ by determinant representation for norms (**'82, Korepin**)
- $T \to +\infty$ asymptotics of integral representation for the associated Eigenvalue

$$\widehat{\Lambda}\big(\{\lambda_a\}_1^N\big) = 2 + \mathrm{O}(T^{-1})$$

- $\widehat{\Lambda}\big(\{\lambda_a\}_1^N\big) = \widehat{\Lambda}_{\max}$ by form of spectrum
- $\lim_{N\to+\infty} \widehat{\Lambda}\big(\{\lambda_a\}_1^N\big)$ exists by continuity in N of fixed point

Conclusion and perspectives

Review of the results

- ✔ Proof of spectral properties of the quantum transfer matrix;
- ✔ Existence and uniqueness of solutions to NLIE;
- ✔ Commutativity of thermodynamic and Trotter limits ;
- ✔ Sub-dominant Eigenvalues described by spin-1 XXZ chain Bethe Ansatz;

Further developments

- Extend to all values of T;
- Extend to correlation functions (static and dynamic);
- Surface free energy;