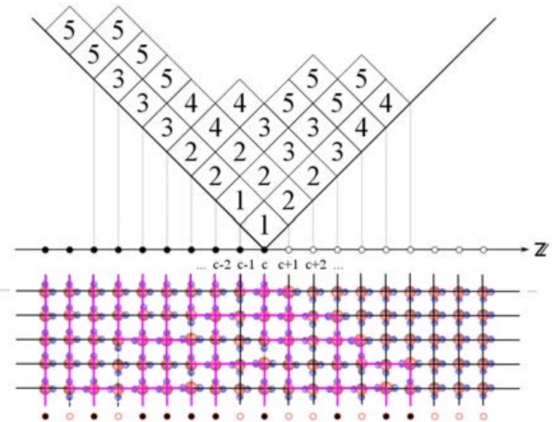
Cylindric symmetric functions: from integrability to positivity

The asymmetric six-vertex model, cylindric symmetric functions and virtual Hecke characters



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MOTIVATION: symmetric polynomials @ roots of unity

$\Lambda_k = \mathbb{C}[y_1, \dots, y_k]^{S_k}$ ring of symmetric polynomials in k variables

particle number

Basis ①: Schur polynomials $s_\lambda(y_1,\dots,y_k) = \dfrac{\det\left(y_j^{\lambda_i+k-i}\right)_{1\le i,j\le k}}{\det\left(y_j^{k-i}\right)_{1\le i,j\le k}} = \sum_T y^T$

partition

tableaux of shape λ

Bethe wave function

$$s_\mu(y)\, s_\nu(y) = \sum_\lambda c^\lambda_{\mu\nu}\, s_\lambda(y), \qquad c^\lambda_{\mu\nu} \in \mathbb{Z}_{\ge 0} \text{ LR numbers}$$

Basis ②: monomial symmetric "functions": $m_\lambda(y_1,\dots,y_k) = \sum_{w\in S^\lambda} y^{\lambda.w}$

Bethe wave function

$$m_\mu(y)\, m_\nu(y) = \sum_\lambda f^\lambda_{\mu\nu}\, m_\lambda(y), \quad f^\lambda_{\mu\nu} = \#\{(w,w')\in S^\lambda \times S^\mu \mid \lambda.w + \mu.w' = \nu\}$$

min length coset reps of $S_\lambda \backslash S_k$

lattice size

QUESTION: What happens if we evaluate $y_1,\dots,y_k$ at n^{th} (complex) roots of unity?

Bethe roots

① all k-subsets of $\{1, e^{2\pi i/n}, \dots, e^{\frac{2\pi i}{n}(n-1)}\}$

② all k-multisets of $\{1, e^{2\pi i/n}, \dots, e^{\frac{2\pi i}{n}(n-1)}\}$

Positivity?

Outline

1. Cyclic elements & Schur-Weyl duality
2. Geometric interpretation : quantum cohomology
3. Representation theory of the generalised symmetric group
4. Integrable lattice models, cylindric Hecke characters, cylindric RPP
5. Open problems

MAIN REFERENCES : C.K. & D. Palazzo, arxiv 1804.05647 (accepted in Algebraic Combinatorics)

C.K., Cylindric Hecke characters & GW invariants via the asymm 6v model
arxiv. 1906.02565

Schur - Weyl duality

Let $\mathfrak{g} = \mathfrak{gl}_n$ and consider its loop algebra $\mathfrak{gl}_n[z, z^{-1}] = \mathfrak{gl}_n \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$

Define $P_r = \sum_{j-i=r} E_{ij} + z \sum_{i-j=n-r} E_{ij}$, $r = 1, 2, \ldots, n-1$ $\quad P_1 = \sum_{i=1}^{n-1} E_{i\,i+1} + z\, E_{n1}$

'cyclic element' Y

and set $P_{r+n} = z\, P_r$ and P_0 - central element

The P_r span a maximal <u>abelian</u> subalgebra $[P_r, P_s] = 0$, $r, s \in \mathbb{Z}_{\geq 0}$ in $\mathfrak{gl}_n[z]$

Extended affine symmetric group $\hat{S}_k = S_k \ltimes P$

generators & relations : $s_i^2 = 1$, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, $y_i s_i = s_i y_{i+1}$, $y_i y_j = y_j y_i$

Consider the vector rep $V \cong \mathbb{C}^n$ of $\mathfrak{gl}_n$ and set $V^{\otimes k}[z^{\pm 1}] = V^{\otimes k} \otimes_{\mathbb{C}} \mathbb{C}[z^{\pm 1}]$

<u>**Prop**</u> The map $(v, y^\lambda w) \mapsto Y^{-\lambda_1} \otimes \cdots \otimes Y^{-\lambda_k}\, v.\, w$ defines a right $\hat{S}_k$-action

and $\Delta^{k-1}(P_r) = \sum_{i=1}^{n} 1 \otimes \cdots \otimes \underset{i}{Y^r} \otimes \cdots \otimes 1$

power sums in the cyclic element $Y = \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & & & 1 \\ z & 0 & \cdots & 0 \end{pmatrix}_{n \times n}$

↳ co-product of $\mathcal{U}(\mathfrak{gl}_n[z^{\pm 1}])$

REMARKS

① This $\hat{S}_k$-action factors through the following quotient:

skew group ring $\mathcal{H}_k = \mathbb{C} S_k \otimes \mathbb{C}[y_1, \dots, y_k] / \langle y_1^n - z \rangle$ ‵roots of unity′

‵Classical limit′ of a cyclotomic Hecke / Ariki-Koike algebra

② The symmetric polynomials in the $Y_i = 1 \otimes \dots \otimes \underset{i}{Y} \otimes \dots \otimes 1$

form the centre $\mathcal{Z}[\mathbb{C}\hat{S}_k]$ and, hence, commute with the natural

S_k-action on $V^{\otimes k}$

Consider the projections of $S_\lambda = \sum_\mu \chi^\lambda(\mu) P_\mu$ and $M_\lambda = \sum_{w \in S^\lambda} Y^{\lambda . w}$

on the subspaces of alternating $\Lambda^k V$ and symmetric tensors $S^k V$

eigenvalue problem of S_λ, M_λ $\rightsquigarrow$ symmetric polynomials @ roots of unity

Geometric Interpretation

alternating tensors

vector rep $V=\mathbb{C}^n$ of $\mathfrak{gl}_n$

(small) quantum cohomology

Chern roots

$$v_{\alpha_1}\otimes\cdots\otimes v_{\alpha_k}\otimes f(z) \mapsto y_1^{\alpha_1}\cdots y_k^{\alpha_k}\, f(\tilde q)$$

$$\hat S_k = S_k \ltimes P_k \circlearrowright$$

$$\begin{tikzcd}
V^{\otimes k}\otimes\mathbb{C}[z] \arrow[r, "\sim"] \arrow[d, two heads] & qH^*(\overbrace{\mathbb{P}^{n-1}\times\cdots\times\mathbb{P}^{n-1}}^{k \text{ factors}}) \cong \bigotimes_{i=1}^{k}\mathbb{C}[\tilde q, y_i]/\langle y_i^n-\tilde q\rangle \arrow[d, two heads] \\
\Lambda^k V\otimes\mathbb{C}[z] \arrow[r, "\sim"] & qH^*(Gr_k(\mathbb{C}^n)) \cong \mathbb{C}[q, y_1,\dots,y_k]^{S_k}/\langle y_i^n-(-1)^k q\rangle
\end{tikzcd}$$

$$e^- = \frac{1}{k!}\sum_{w}(-1)^{\ell(w)}\, w$$

'quantum' Satake correspondence

$$\tilde q = (-1)^k q$$

$$\mathcal{Z}[\mathbb{C}\hat S_k] \circlearrowright$$

centre

$$v_{\lambda_k}\wedge\cdots\wedge v_{\lambda_1}\otimes f(z) \longmapsto f((-1)^k q)\; s_\lambda(y_1,\dots,y_k)$$

Schur polynomial / Schubert class

Product structure:

$$v_\lambda * v_\mu \overset{\text{def}}{=} S_\lambda(Y)\, v_\mu \longmapsto s_\lambda(y)s_\mu(y) = \sum_{\nu\subset\square_{n-k}^{k}} q^d\, C^{\nu,d}_{\lambda,\mu}\, s_\nu(y)$$

3 pt genus 0 Gromov-Witten invariants

THE GENERALISED SYMMETRIC GROUP $\rightsquigarrow$ symmetric tensors

complex reflection group

$$G(n,1,k) = (\mathbb{Z}/n\mathbb{Z})^k \rtimes S_k \qquad 1 \to (\mathbb{Z}/n\mathbb{Z})^k \hookrightarrow G(n,1,k) \twoheadrightarrow S_k \to 1$$

finite - dimn'l irreducible reps : $L(\underline{\lambda})$ induced by a irred rep of $\mathcal{N} \cong (\mathbb{Z}/n\mathbb{Z})^k$ and its stabiliser group $S_\lambda \subset S_k$.

n - multi partition

Explanation: any partition $\lambda = (1^{m_1} 2^{m_2} \cdots n^{m_n})$ with $\sum_i m_i = k$ fixes a character $\prod_j e^{\frac{2\pi i}{n}\lambda_j}$ of $\mathcal{N}$

Fix irreducible rep for its stabiliser group $S_\lambda \cong S_{m_1} \times S_{m_2} \times \cdots \times S_{m_n}$

The irred reps of S_{m_i} (Specht modules) are indexed by partitions $\lambda^{(i)} \vdash m_i$

$\Rightarrow$ $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(n)})$ n-multipartition with $\sum_i |\lambda^{(i)}| = k$ and $|\lambda^{(i)}| = m_i(\lambda)$

One calls $\lambda = (1^{m_1} 2^{m_2} \cdots n^{m_n})$ the 'type of $\underline{\lambda}$'.

Denote by $\operatorname{Rep} G(n,1,k)$ the Grothendieck ring of finite dimn'l $G(n,1,k)$ modules.

Generalised Fusion Rings

$$L(\underline{\lambda})\otimes L(\underline{\mu}) = \sum_{\underline{\nu}} g^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}\, L(\underline{\nu})$$

tensor multiplicities

'generalised Kronecker coefficients'

$$\begin{tikzcd}
\hat{S}_k \circlearrowright V^{\otimes k} \arrow[r, "\sim"] \arrow[d, "e^+ = \frac{1}{k!}\sum_w w"', "z=1"] & \bigoplus_{\lambda \subset \square^k_{n-1}} \mathbb{C}L(\overbrace{\square\cdots\square}^{m_1},\ldots,\overbrace{\square\cdots\square}^{m_n}) \subset \mathrm{Rep}\; G(n,1,k) \arrow[d] \\
\mathbb{Z}[\hat{S}_k] \circlearrowright S^k V \arrow[r, "\sim"] & \mathrm{Char}(\mathcal{N})
\end{tikzcd}$$

$L(\underline{\lambda}) \sim L(\underline{\mu})$ iff $\mathrm{type}(\underline{\lambda}) = \mathrm{type}(\underline{\mu})$

$\mathcal{N} \cong (\mathbb{Z}/n\mathbb{Z})^k$

normal subgroup

$$V_\lambda = V_{\lambda_1} V_{\lambda_2} \cdots V_{\lambda_k} \mapsto [L(\underline{\lambda})]$$

$$V_\lambda * V_\mu \overset{\text{def}}{=} M_\lambda(Y)\, V_\mu$$

fusion product

$$[L(\underline{\lambda})][L(\underline{\mu})] = \sum_{\mathrm{type}(\underline{\nu})=\nu} N^{\nu}_{\lambda\mu}\, [L(\underline{\nu})]$$

$$\boxed{N^{\lambda}_{\mu\nu} = \langle v^\lambda, M_\mu(Y)\, v_\nu \rangle = \sum_{\mathrm{type}(\underline{\nu})=\nu} g^{\underline{\lambda}}_{\underline{\mu}\underline{\nu}} \prod_i \frac{f_{\lambda^{(i)}}}{f_{\mu^{(i)}} f_{\nu^{(i)}}}}$$

$f_{\lambda^{(i)}} = \dim \mathcal{S}_{\lambda^{(i)}}$

Specht module

'classical limit' of the fusion ring / TQFT defined in C.K. CMP 318 (2013) 173-246

'Cauchy identities'

Consider the following Cauchy identities for elements in $\mathbb{Z}[\mathbb{C}\hat{S}_k]$:

complete symmetric fctns

Spectral parameters

$$\sum_\lambda S_\lambda(Y)\, s_\lambda(x_1,x_2,\dots) = \sum_\lambda M_\lambda(Y)\, h_\lambda(x_1,x_2,\dots) = \prod_{i\geq 1}\prod_{j=1}^{k}\frac{1}{1-Y_i x_j} = \prod_{i\geq 1} H(x_i)$$

① ② power series transfer matrices

Taking matrix elements with alternating / symmetric tensors the RHS yields

partition functions of integrable lattice models $\hat{=}$ cylindric symmetric fcts $\in \Lambda = \varprojlim \mathbb{C}[x_1,\dots,x_\ell]^{S_\ell}$

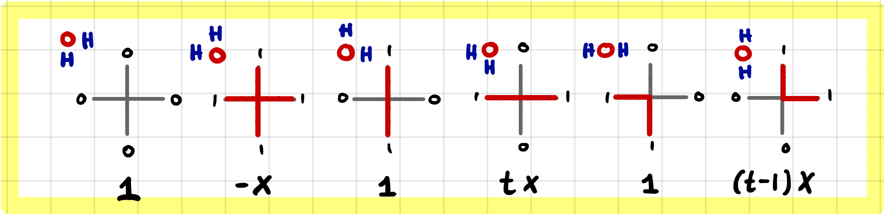
∞-dimn'l sub-coalgebra $\subset$ Hopf algebra

① The asymmetric six-vertex model and cylindric Hecke characters C.K. ARXIV 1906.02565

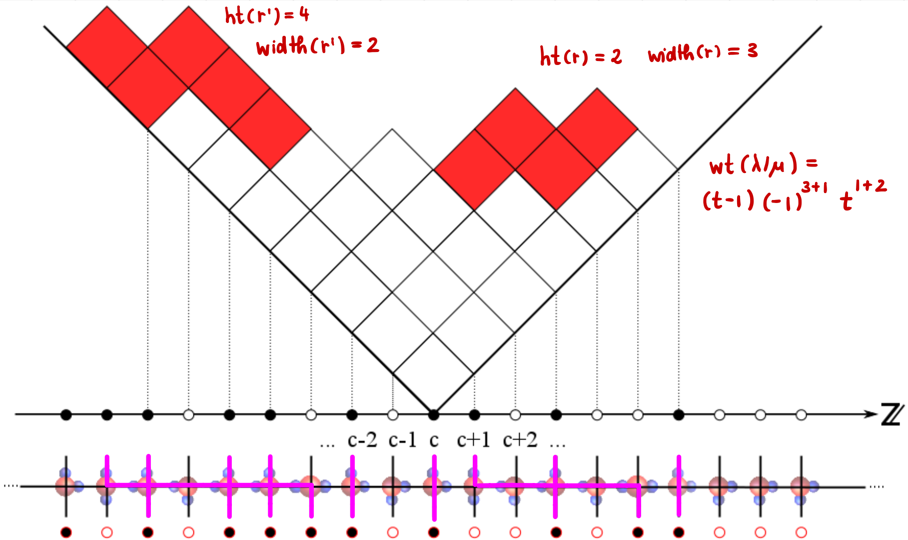
② The classical limit of the Q^--operator for the q-boson model C.K. JPA 49 (2016) 104001

C.K., D. Palazzo, Alg. Comb.

① The asymmetric six-vertex model & cylindric Hecke characters



six vertex configurations & Boltzmann weights



$$\mathrm{wt}(\lambda/\mu) = (t-1)^{\#(\lambda/\mu)-1} \prod_{\substack{\text{rim hooks}\\ r}} (-1)^{\mathrm{ht}(r)-1}\, t^{\mathrm{width}(r)-1}$$

THM [Ram]

Hecke characters are given as weighted sums over broken rim hook tableaux.

$$\chi_t^\lambda(T_{w_\mu}) = \sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T})$$

$$\mathrm{wt}(\mathcal{T}) = \prod_i \mathrm{wt}(\lambda^{(i)}/\lambda^{(i+1)})$$

w_μ representative of conjugacy class fixed by μ

A Hecke version (t-deformation) of the boson-fermion correspondence

Fermions | **Bosons**

$$\begin{tikzcd}
\text{Grothendieck ring} \quad \mathcal{R}_t = \bigoplus_{m\geq 0} \mathcal{R}_t^m \arrow[d, "\sim"'] \arrow[drr, "\sim", "ch_t"'] & & \mathcal{Z}_t = \bigoplus_{m\geq 0} \mathcal{Z}(\mathcal{H}_m(t)) \quad \text{centres of Hecke algebras} \arrow[ll, "\sim"'] \arrow[d, "\sim", "F_t"'] \\
\underbrace{\Lambda^{\frac{\infty}{2},c} V \otimes \mathbb{C}(t)}_{\Lambda^{\frac{\infty}{2},c} V_t} \arrow[rr, "\sim"] & & \Lambda \otimes \mathbb{C}(t)
\end{tikzcd}$$

quantum characteristic map

F_t 'quantum' Frobenius map

fermionic Fock space

ring of symm functions

$$s_\lambda[X] = \sum_\mu \chi_t^\lambda(T_{w_\mu})(t-1)^{\ell(\mu)} m_\mu\left[\frac{X}{t-1}\right]$$

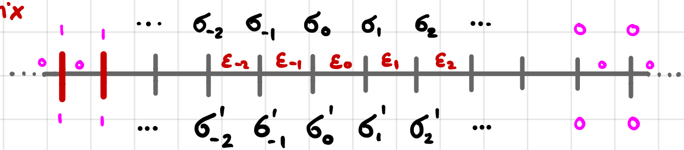
plethystic change of variables

Maya diagram

PROP $\sigma(\lambda,c) \mapsto \langle \sigma(\lambda,c), A(x_1)A(x_2)\cdots \sigma(\emptyset,c)\rangle = \langle \sigma(\lambda,c), e^{H_t[p]}\sigma(\emptyset,c)\rangle = s_\lambda[(t-1)X]$

6v - transfer matrix

$$\langle \sigma', A(x)\sigma\rangle = \sum_{\varepsilon_i}$$

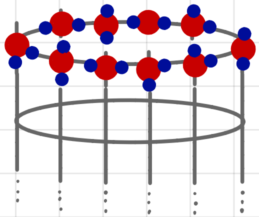


fixed boundary conditions at $\pm\infty$

Periodic boundary conditions

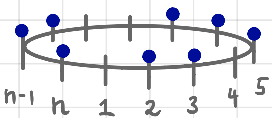
$$\Lambda^{\frac{\infty}{2}} V \otimes \mathbb{C}(t) \longrightarrow \Lambda\, \mathbb{C}^n(t) = \bigoplus_{k=0}^{n} \Lambda^k \mathbb{C}^n(t)$$

∞ wedge finite wedge



Denote by $T(x)$ the six-vertex transfer matrix with quasi-periodic boundary conditions and twist parameter q

$$T(x) = \sum_{r=0}^{n} x^r\, T_r\,, \quad T_r : \bigoplus_{d\in\mathbb{Z}} q^d \otimes \Lambda^k \mathbb{C}^n(t) \to \bigoplus_{d\in\mathbb{Z}} q^d \otimes \Lambda^k \mathbb{C}^n(t)$$



$\longmapsto v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k}$, i_j positions of H-atoms $\bullet$ on the circle

Define a 'renormalised' transfer matrix $H(x;t) : \bigoplus_{d\in\mathbb{Z}} q^d \otimes \Lambda^k \mathbb{C}^n(t) \to \bigoplus_{d\in\mathbb{Z}} q^d \otimes \Lambda^k \mathbb{C}^n(t)$

$$H(x;t) = \frac{T(x)}{1+(-1)^k q\, x^n t^n} = \sum_{r\geq 0} x^r H_r(t)$$

t-deformed

Cauchy identity:

$$\prod_{i\geq 1} H(x_i;t) = \sum_{\lambda} s_\lambda[(t-1)X]\; S_\lambda(Y)\,\Big|_{\Lambda^k\mathbb{C}^n(t)\otimes\mathbb{C}[q^{\pm 1}]}$$

Cylindric Hecke characters

Cauchy identity: $\prod_{i\geq 1} H(x_i; t) = \sum_\lambda s_\lambda[(t-1)x]\, S_\lambda(Y)\Big|_{\Lambda^k\mathbb{C}^n(t)\otimes\mathbb{C}[q^{\pm 1}]}$

$$= \sum_\lambda \underbrace{m_\lambda(x)}_{\text{monomial symmetric function}} \quad \underbrace{h_\lambda[(t-1)Y]}_{\text{complete symmetric polynomial } h_\lambda = h_{\lambda_1} h_{\lambda_2}\cdots}\Big|_{\Lambda^k\mathbb{C}^n(t)\otimes\mathbb{C}[q^{\pm 1}]}$$

cylindric skew Hecke character

DEF $\quad q^d \chi_t^{\lambda/d/\mu}(\nu)\,(t-1)^e \overset{\text{def}}{=} \langle v_\lambda, H_{\nu_1}\cdots H_{\nu_e}\, v_\mu \rangle, \qquad d = \dfrac{|\mu|+|\nu|-|\lambda|}{n}$

alternating tensor $\quad v_\mu = v_{\mu_k}\wedge\cdots\wedge v_{\mu_2}\wedge v_{\mu_1}$

$$H(x;t) = \sum_{r\geq 0} x^r H_r(t), \qquad H_r = h_r[(t-1)Y]\Big|_{\Lambda^k\mathbb{C}^n(t)\otimes\mathbb{C}[q^{\pm 1}]}$$

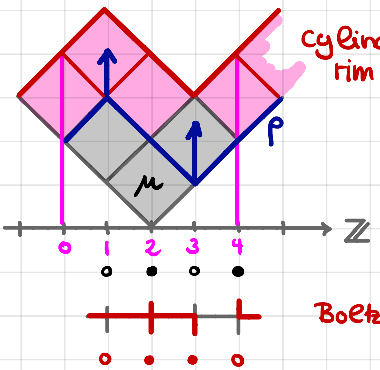
renormalised 6-vertex transfer matrix $\rightsquigarrow$ power sums P_r in $t\to 1$ limit

LEMMA (Cylindric Murnaghan-Nakayama rule)

(i) $\chi_t^{\lambda/d/\mu}(\dots, \nu_i + n, \dots) = (-1)^{k-1} t^n \, \chi_t^{\lambda/d-1/\mu}(\dots, \nu_i, \dots)$

(ii) $\chi_t^{\lambda/d/\mu}(\nu, r) = \sum_\rho \chi_t^{\lambda/d/\rho}(\nu)\, \chi_t^{\rho/\mu}(r) + \underbrace{\sum_\rho \chi_t^{\lambda/d-1/\rho}(\nu)\, \chi_t^{\rho/1/\mu}(r)}_{\text{Cylindric part}}$

The rules (i), (ii) allow one to compute $\chi_t^{\lambda/d/\mu}$.



$\rho/1/\mu$

$n = 4$, $k = 2$ $\qquad \chi_t^{(2,0)/1/(2,1)}(3) = (-1)^{2-1} t^{2-1}$

Boltzmann weight $(t-1)(-1)t$

The coalgebra of cylindric Hecke characters

Main Theorem (C.K.) The cylindric Hecke characters $\chi_t^{\lambda/d/\emptyset}$ with $\lambda \subset \square^{k}_{n-k}$, $d \geq 0$ span an ∞-dimn'l sub-coalgebra of $\mathcal{R}_t$ (Grothendieck ring) with

$$\operatorname{Res}^{H_{m'+m''}}_{H_{m'}\otimes H_{m''}} \chi_t^{\lambda/d/\emptyset} = \sum_{d'+d''\leq d} \sum_{\substack{\mu \vdash m' \\ \nu \vdash m''}} C_{\mu\nu}^{\lambda, d-d'-d''} \chi_t^{\mu/d'/\emptyset} \otimes \chi_t^{\nu/d''/\emptyset}$$

where $C_{\mu\nu}^{\lambda,d} \in \mathbb{Z}_{\geq 0}$ are the 3-point genus 0 Gromov-Witten invariants of the Grassmannian $Gr_k(\mathbb{C}^n)$.

The cylindric Hecke characters $\chi_t^{\lambda/d/\emptyset} = \sum_{\rho} c_\rho \chi_t^{\rho}$ are virtual characters, $c_\rho = 0, \pm 1$

For $d=0$ one recovers $H^*(Gr_k(\mathbb{C}^n))$ as coalgebra with $C_{\mu\nu}^{\lambda,0} = c_{\mu\nu}^{\lambda}$ being the Littlewood-Richardson coefficients.

Connection with cylindric Schur functions

$$
\begin{tikzcd}
\mathcal{R}_t = \bigoplus_{m\geq 0} \mathcal{R}_t^m \arrow[d, "\sim"'] \arrow[dr, "\sim", "ch_t"'] & \mathcal{Z}_t = \bigoplus_{m\geq 0} \mathcal{Z}(\mathcal{H}_m(t)) \arrow[l, "\sim"'] \arrow[d, "\sim", "F_t"'] \\
\Lambda^{\frac{\infty}{2},c} V \otimes \mathbb{C}(t) \arrow[r, "\sim"] \arrow[d, two heads] & \Lambda \otimes \mathbb{C}(t) \arrow[d, two heads] \\
\bigoplus_{d\geq 0} q^d \otimes \Lambda^k \mathbb{C}^n \otimes \mathbb{C}(t) \arrow[r, "\text{Satake}"'] \arrow[ur] & qH^*(Gr_k(\mathbb{C}^n)) \otimes \mathbb{C}(t)
\end{tikzcd}
$$

Grothendieck ring ($\mathcal{R}_t$)

centres of Hecke algebras ($\mathcal{Z}_t$)

quantum characteristic map (ch_t)

F_t 'quantum' Frobenius map

fermionic Fock space ($\Lambda^{\frac{\infty}{2},c} V \otimes \mathbb{C}(t)$)

ring of symm functions ($\Lambda \otimes \mathbb{C}(t)$)

rim-hook algorithm

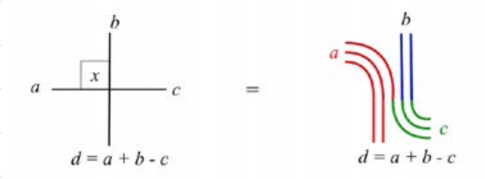
Satake

cylindric boson-fermion correspondence

$$v_{\lambda_k} \wedge \cdots \wedge v_{\lambda_2} \wedge v_{\lambda_1} \mapsto \langle V_\lambda, H(x_1;t)H(x_2;t)\cdots, V_\emptyset \rangle = \sum_{d\geq 0} q^d s_{\lambda/d/\emptyset}[(t-1)X]$$

t-deformed cylindric Schur fctns

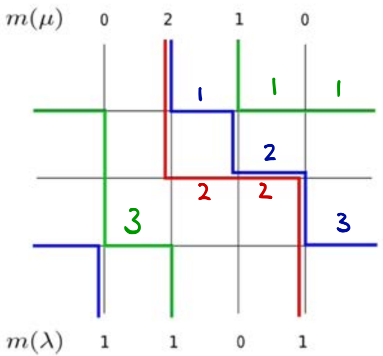
② Cylindric Reverse Plane Partitions & free bosons



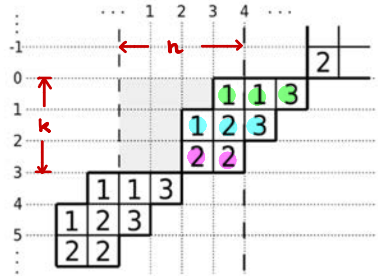
$$x^a \binom{a+b}{b}$$

These Boltzmann weights define an integrable lattice model with transfer matrix Q^- ⇝ soln of YBE

The corresponding quantum system are free bosons on a circle.



cylindric RPP



$$Q^-(x) = \sum_{r\geq 0} Q^-_r x^r$$

$$h_r[Y]\Big|_{S^kV\otimes\mathbb{C}[z]} = \begin{cases} Q^-_r, & 0\leq r<n \\ Q^-_r - z\,Q^-_{r-n}, & r\geq n \end{cases}$$

~~**Proposition 2.13**~~. *The partition function of the Q^- lattice model has the expansion*

$$\langle\lambda|Q^-(x_1)Q^-(x_2)\cdots Q^-(x_l)|\mu\rangle \overset{\text{def}}{=} \prod_{i=1}^{l}\frac{1}{1-zx_i^n}\sum_{d\geq 0} z^d h_{\lambda/d/\mu}(x_1,x_2,\ldots,x_l) \qquad (21)$$

cylindric complete symmetric polynomial

Cylindric complete symmetric functions

coproduct in the ring of symmetric fctns : $\Delta : \Lambda \to \Lambda \otimes \Lambda$, $f[X] \longmapsto f[X,Y]$

coproduct of complete symmetric fctns : $\Delta h_\lambda = \sum_\mu h_{\lambda/\mu} \otimes h_\mu$

$$f^\lambda_{\mu\nu} = \#\{ (w,w') \in S^\lambda \times S^\mu \mid \lambda.w + \mu.w' = \nu \}$$

min length coset reps of $S_\lambda \backslash S_\infty$

$$h_{\lambda/\mu} = \sum_\nu f^\lambda_{\mu\nu} h_\nu = \sum_{\substack{\pi \\ \text{RPP of shape } \lambda/\mu}} \theta_\pi x^\pi$$

$$\theta_{\rho/\sigma} = \{ w \in S^\sigma \mid \sigma.w \subset \rho \} \qquad \theta_\pi = \prod_i \theta_{\lambda^{(i)}/\lambda^{(i+1)}}$$

THM (C.K., D. Palazzo) coalgebra of cyl. compl. symm. fctns

(i) $$h_{\lambda/d/\mu} = \sum_{d'=0}^{d+k} \sum_{\nu \subset \square^k_{n-1}} N^\lambda_{\mu\nu} h_{\nu/d-d'/\emptyset} ,$$

translation by weight α

$$N^\lambda_{\mu\nu} = \#\{ (w,w') \in S^\lambda \times S^\mu \mid \exists\, \alpha \in P_k \;\; \lambda.w + \mu.w' = y^\alpha \nu \}$$

(ii) $$\Delta h_{\lambda/d/\emptyset} = \sum_{d'} \sum_{\mu \subset \square^k_{n-1}} h_{\lambda/d'/\mu} \otimes h_{\mu/d-d'/\emptyset}$$

Open Problems

- Extension to generalised cohomology theories, equivariant / K-theory
- general flag varieties
- connection with puzzles (Knutson-Tao, Knutson - Zinn Justin, Vakil, Wheeler Zinn-Justin, Buch et al, ...)
- crystals & combinatorial R-matrix (in preparation) C.K. FPSAC 2017; arxiv 1702.07162
- geometric interpretation of the bosonic case: deformation of the Verlinde algebra

Bethe roots of t-bosons

$$P_\mu(y;t)\,P_\nu(y;t)=\sum_\lambda N_{\mu\nu}^\lambda(t)\,P_\lambda(y;t)$$ 2D TQFT

'deformed' fusion coefficients

<u>CONJ</u> (C.K. CMP 2013) $N_{\mu\nu}^\lambda(t)/(1-t)\in\mathbb{Z}_{\geq 0}[\![t]\!]$

$N_{\mu\nu}^\lambda(0)\,\hat{=}\,\widehat{su}(n)_k$ fusion coefficients

$N_{\mu\nu}^\lambda(1)=N_{\mu\nu}^\lambda$ cylindric RPP