Schubert calculus and quiver varieties

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Abstract

Schubert calculus, the intersection theory of homogeneous spaces such as Grassmannians (or "1-step flag manifolds"), is famously a problem for which we have easy alternating-sum formulæ but know in advance that the intersection numbers will be nonnegative. We've had positive rules (i.e. counting a set, such as Young tableaux) for the Grassmannian case since 1934, but 2-step and 3-step rules only came in 2009 and 2017.

I'll explain how connecting these "puzzle" rules to quantum integrable systems made them easy to derive and prove, and how further connection to quiver varieties has brought about several more advances. This work is joint with Paul Zinn-Justin and Iva Halacheva (both of Melbourne).

An intersection theory problem.

Let L_1, L_2 be two different, but crossing, lines in 3-space.
Let Y_1, Y_2 be the set of lines touching L_1, L_2 respectively. Then

$$Y_1 \cap Y_2 = \{\text{lines in the } L_1L_2 \text{ plane}\} \bigcup_{\{\text{lines doing both}\}} \{\text{lines through } L_1 \cap L_2\}$$

Let $Gr(1, \mathbb{P}^3) \cong Gr(2, \mathbb{C}^4)$ be the **Grassmannian** of lines in projective 3-space. Although $Y_1 \neq Y_2$ as sets, they are homologous in $Gr(2, \mathbb{C}^4)$, so define the same element "S_{0101}" in cohomology (or K-theory).

More generally, consider lines in $\mathbb{P}^{n-1}$ that touch a fixed j-plane and are contained in a fixed k-plane. Make a length n binary string λ with two zeros, in positions $n-k, n-j$, and let S_λ denote the cohomology (or K-theory) class.

Then the above lets us compute

$$(S_{0101})^2 = S_{1001} + S_{0110} \quad \text{in } H^*(Gr(2, \mathbb{C}^4)) \quad \left(\text{or that minus } S_{1010}, \text{ in } K(Gr(2, \mathbb{C}^4))\right)$$

Cohomology and K-theory of Grassmannians.

To a length n binary string λ with k zeroes, consider the **Schubert cell**

$$X^\circ_\lambda := \left\{ \text{row span} \begin{bmatrix} 0 & 1 & * & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \atop \text{the } k \text{ pivot columns at } \lambda\text{'s zeroes} \right\} \qquad \subseteq Gr(k, \mathbb{C}^n)$$

Using Gaussian elimination, we see these cells give a paving of $Gr(k, \mathbb{C}^n)$ by affine spaces, so their closures give bases $\{S_\lambda\}$ of cohomology and K-theory called **Schubert classes**. When we have a ring with basis $\{S_\lambda\}$, we want to understand the structure constants $c^\nu_{\lambda\mu}$ of its multiplication $S_\lambda S_\mu = \sum_\nu c^\nu_{\lambda\mu} S_\nu$.

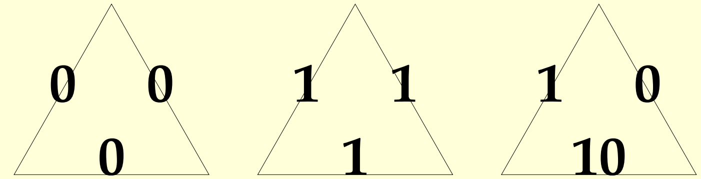
Theorem [Littlewood-Richardson 1934, made correct in 1970s]
The H^* structure constants count a set (of Young tableaux), so are ≥ 0.

Theorem [Kleiman 1973]. There's a geometric reason for this, and it applies to other homogeneous spaces G/P as well, but gives no formula. (Indeed, there is a Galois group *obstruction* to enumerating points of intersection [Harris 1979].)

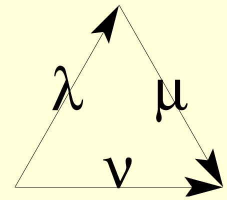
The corresponding results in K-theory are [Buch '02], followed by [Brion '02].

A first formula for the structure constants of $H_T^*(Gr(k, \mathbb{C}^n))$.

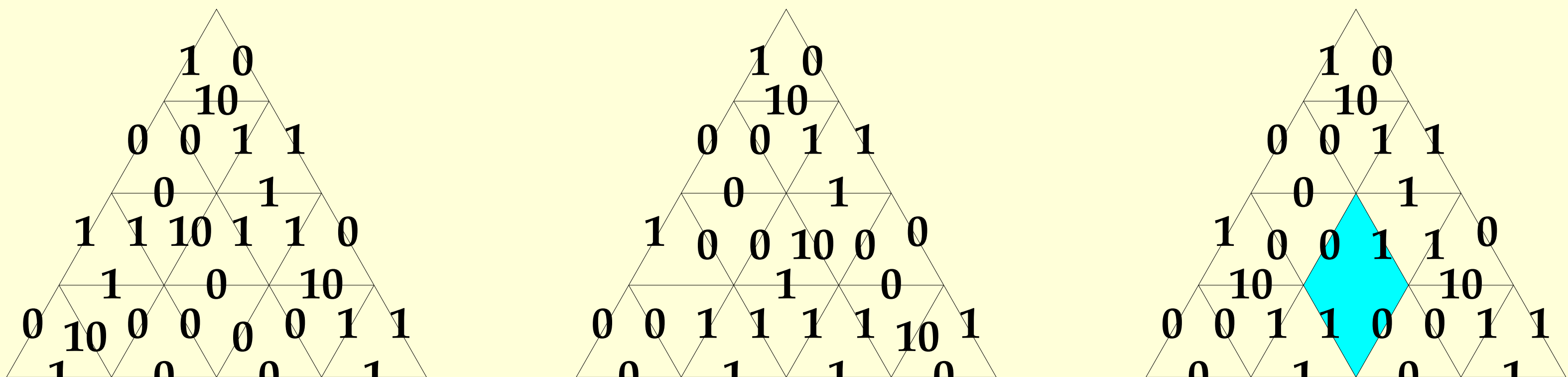
Theorem [K-Tao, '03]. Glue these **puzzle pieces** (which may be rotated) into **puzzles**, which aren't permitted 10-labels on the boundary.



Then in H^*, $c_{\lambda\mu}^{\nu}$ is the number of puzzles with boundary conditions λ, μ, ν like so:



In fact our result is in *torus-equivariant* cohomology, with structure constants $c_{\lambda\mu}^{\nu}$ now in $H_T^*(pt) \cong \mathbb{Z}[y_1, \ldots, y_n]$:



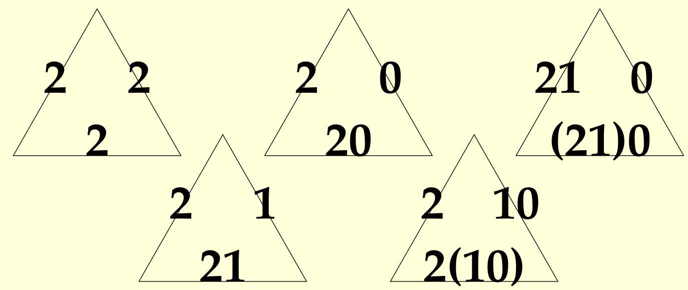
$$(S_{0101})^2 = S_{1001} + S_{0110} + (y_2 - y_3) S_{0101}$$

The **equivariant piece** doesn't break into triangles, *can't be rotated,* and contributes a factor of $y_i - y_j$ according to its position.

Puzzles for multistep flag manifolds.

A d**-step flag manifold** $Fl(n_1, n_2, \ldots, n_d; \mathbb{C}^n)$ is the space of chains $\{0 \le V^{n_1} \le V^{n_2} \le \ldots \le V^{n_d} \le \mathbb{C}^n\}$ of subspaces with a fixed list of dimensions, the $d = 1$ case being Grassmannians. This manifold too comes with a decomposition into Schubert cells, now indexed by strings in $\{0, 1, \ldots, d\}$ with multiplicities given by the differences $n_{i+1} - n_i$ (where $n_0 = 0$, $n_{d+1} = n$).

Conjecture [K 1999], Theorem [Buch-Kresch-Purbhoo-Tamvakis '16].
The same puzzle count computes structure constants in $H^*(Fl(n_1, n_2; \mathbb{C}^n))$, requiring only these new puzzle pieces (& rotations):

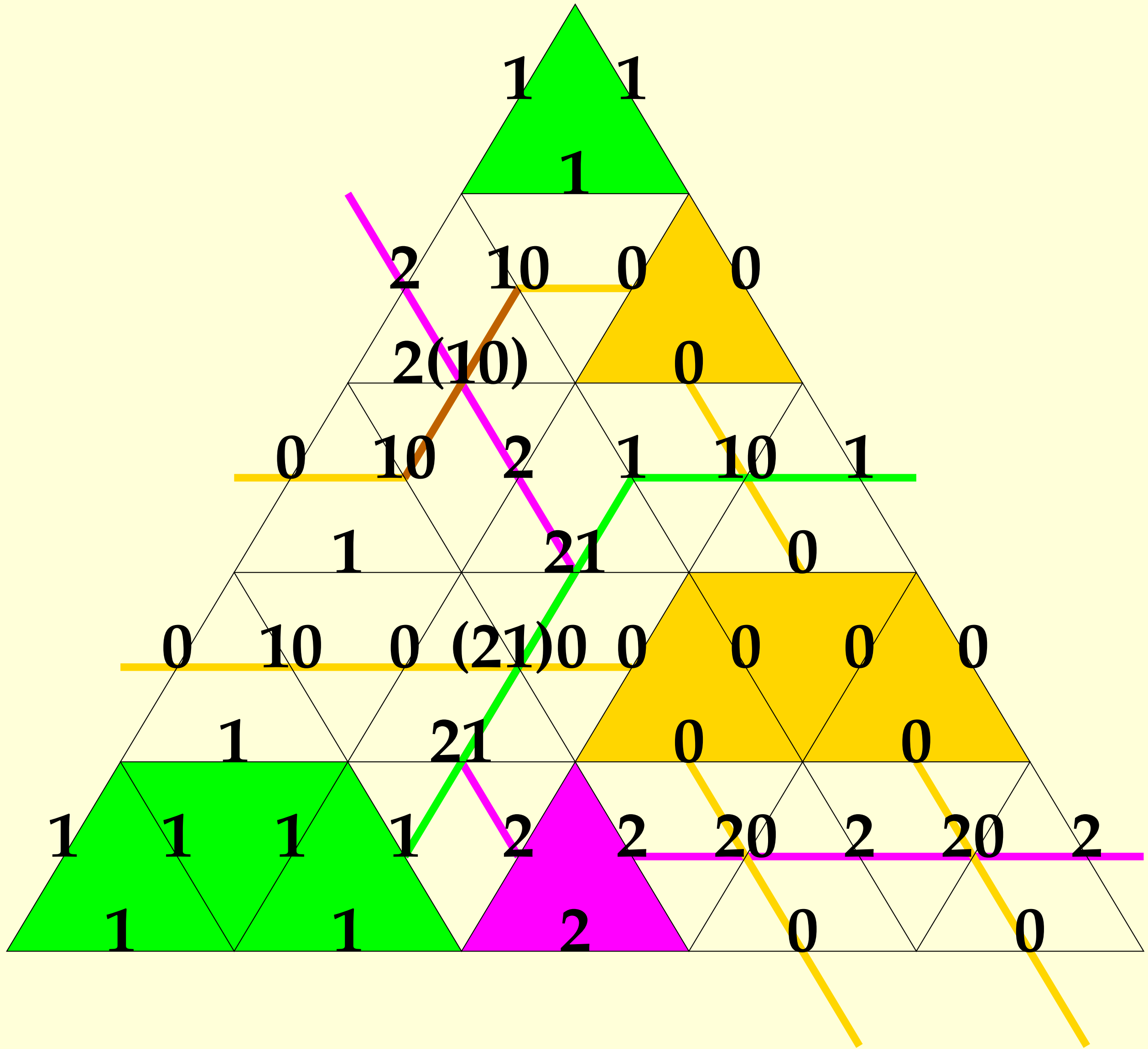


Their lengthy and delicate proof is that my puzzle rule is *associative*. It's relatively easy to check that it gives the correct multiplication by generators.

So, apparently one wants numbers $0, 1, 2$ around the outside of the puzzle plus on the inside, "multinumbers" (XY) where all $X >$ all Y. I found that the analogous 3-step multinumbers gave 23 labels and didn't quite work.

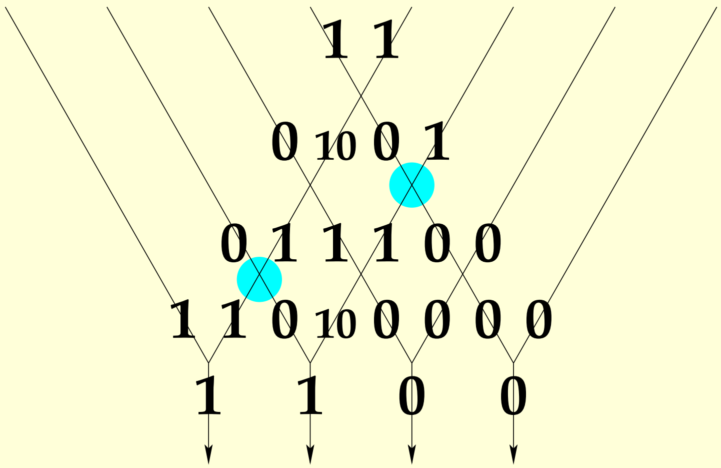
Corrected conjecture [Buch '06], Theorem [K–Zinn-Justin].
The same puzzle count computes $d = 3$ structure constants, but one needs 27 labels, the ones I missed being $(3(21))(10)$, $(32)((21)0)$, $3(((32)1)0)$, $(3(2(10)))0$.

Example. A 2-step puzzle in which all 8 labels appear.



A dual picture: scattering diagrams and a surprise.

The n triangles on the bottom of a puzzle shape are different from the others: they can't occur in an equivariant piece. Let's pair up the other triangles into vertical rhombi.
Now, let's look at the graph-theory dual of an equivariant puzzle, an overlay of n Ys.

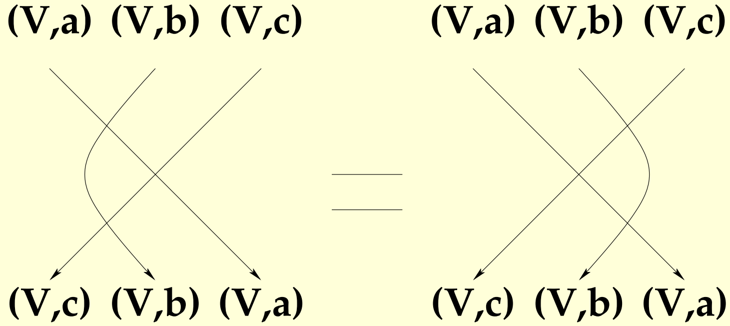


This one is worth $(y_1 - y_2)(y_2 - y_4)$:

If V is the 3-d space with basis $\vec{0}, \vec{1}, \vec{10}$, then we can regard the options at a crossing as giving a matrix $R : V \otimes V \to V \otimes V$; at a trivalent vertex as a matrix $U : V \otimes V \to V^*$; and the puzzle formula as a matrix coefficient $V^{\otimes 2n} \to (V^*)^{\otimes n}$.

That's not quite right because of the $y_i - y_j$ coefficients; we need the tensor factors V to "carry" these parameters in some sense, (V, y_i).

Observation [Zinn-Justin '05].
Rotating the nonrotatable equivariant pieces appropriately (!?), the equivariant puzzle R-matrix satisfies the **Yang-Baxter equation:**

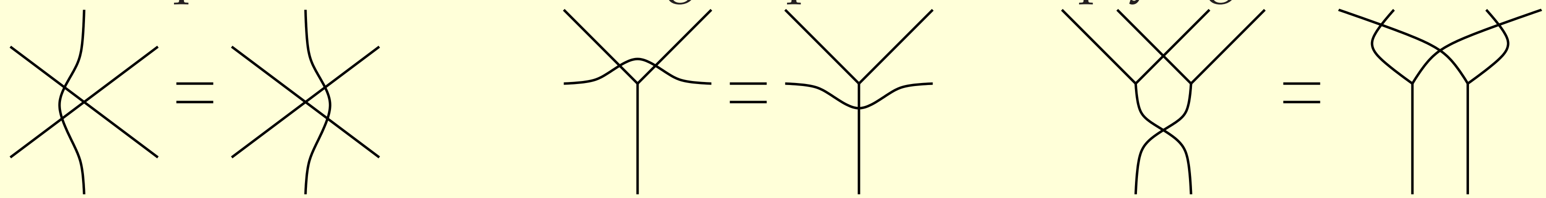


YBE *and bootstrap* proof that puzzles compute $c^{\nu}_{\lambda\mu} \in H^*_T$.

Enough to check restrictions to fixed points: $[X_\lambda]|_\sigma\, [X_\mu]|_\sigma = \sum_\nu c^\nu_{\lambda\mu} [X_\nu]|_\sigma$.

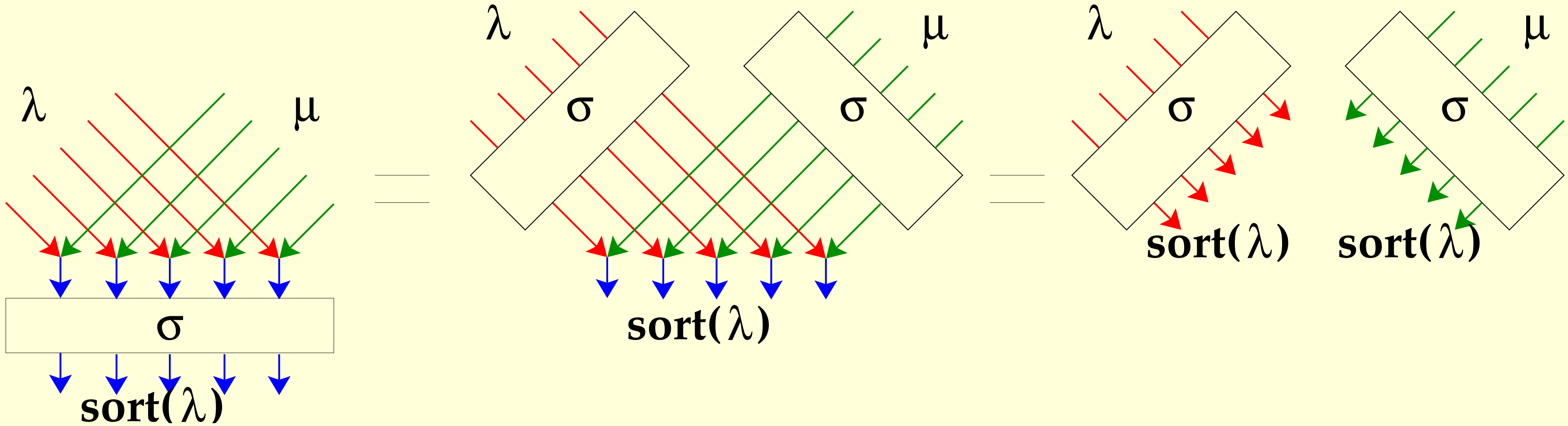
Theorem [K-ZJ '17 ?].

1. With any choice of orientations, colors, and boundary conditions, we have the first two equations on scattering amplitudes, implying the third:



2. If a puzzle has the identity on the bottom, it must also have it on the NW and NE sides, and have scattering amplitude = 1.

Hence



so there's our $[X_\lambda]|_\sigma\, [X_\mu]|_\sigma$. Of course proposition #1 above is a big case check.

Where do solutions to Yang-Baxter (typically) come from?

Let $U_q(\mathfrak{g}[z^{\pm}])$ be the **quantized loop algebra**; it comes with many "evaluation representations" $(V_\delta, c \in \mathbb{C}^\times)$ taking $z \mapsto c$ then using the usual irrep V_δ of $\mathfrak{g}$.

Drinfel′d and Jimbo observed that $(V_\gamma, a) \otimes (V_\delta, b)$ is irreducible for generic a/b, but $\cong$ to $(V_\delta, b) \otimes (V_\gamma, a)$, and these isos are "R-matrices" (solution to YBE).

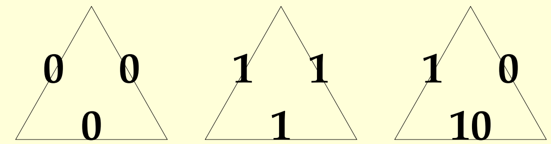
Theorem [K-ZJ]. 1. The $d = 1$ puzzle R-matrix, acting on the $\otimes^2$ of the 3-space with basis $\{\vec{0}, \vec{1}, \vec{10}\}$, is a $q \to \infty$ limit of the R-matrix for $\mathfrak{sl}_3 \circlearrowright \mathbb{C}^3 \otimes \mathbb{C}^3$.
2. For the $d = 2$ case and its 8 edge labels $\vec{0}, \vec{1}, \vec{2}, \vec{10}, \vec{20}, \vec{21}, 2(\vec{10}), (\vec{21})0$, we need a $q \to \infty$ limit of the R-matrix for $\mathfrak{d}_4 \circlearrowright spin_+ \otimes spin_-$.
3. For the $d = 3$ case and its 27 edge labels, we need a $q \to \infty$ limit of the R-matrix for $\mathfrak{e}_6 \circlearrowright \mathbb{C}^{27} \otimes \mathbb{C}^{27}$ (which one can find in the 1990s physics literature).
4. For the $d = 4$ case, the same technology led us to a **nonpositive** 249-label rule based on $\mathfrak{e}_8 \circlearrowright (\mathfrak{e}_8 \oplus \mathbb{C})^{\otimes 2}$.

In each case, the Yang-Baxter equation (and similar "bootstrap" equation to deal with trivalent vertices) is used in a quick proof [K-ZJ '17] of the puzzle rule, and the nonzero matrix entries in the $q \to \infty$ limit tell us the valid puzzle pieces.

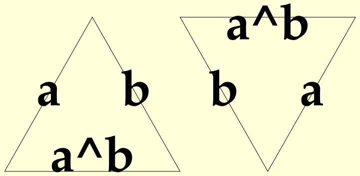
There was even no *conjecture* for K-theory in 2- or 3-step until 2017 (which arrived with our YBE-based proof, and in 3-step requires 151 new pieces).

Revisiting associativity at $d = 1$.

The $\mathfrak{sl}_3$-equivariant map $\mathbb{C}^3 \otimes \mathbb{C}^3 \to \mathrm{Alt}^2\mathbb{C}^3$ only allows fermionic mixing, i.e. of *different* basis vectors, which is not what we saw in the first two pieces:

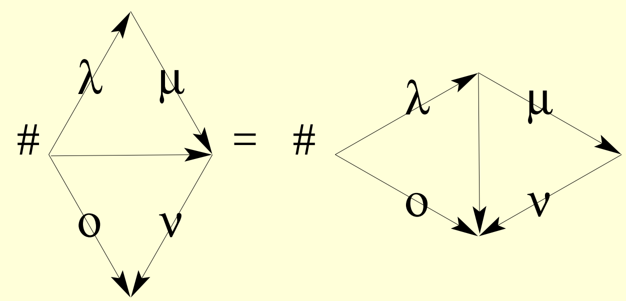


Reformulation of the Grassmannian puzzle rule. Consider the puzzle pieces



where $a, b \in \{0, 1, 2\}$, $a \neq b$, and excluding $a = 0, b = 2$ for some weird reason. Then we can compute $c_{\lambda\mu}^{\nu}$ using puzzles with μ on the Northeast, λ on the Northwest but written in 1s and 2s, and ν on the South but written in $0 \wedge 1$s and $1 \wedge 2$s.

Associativity says that the coefficients of S_o in $(S_\lambda S_\mu)S_\nu$ and $S_\lambda(S_\mu S_\nu)$ are the same. In puzzle terms, we label the front or back of a tetrahedron with bipuzzles, and should be able to biject them:

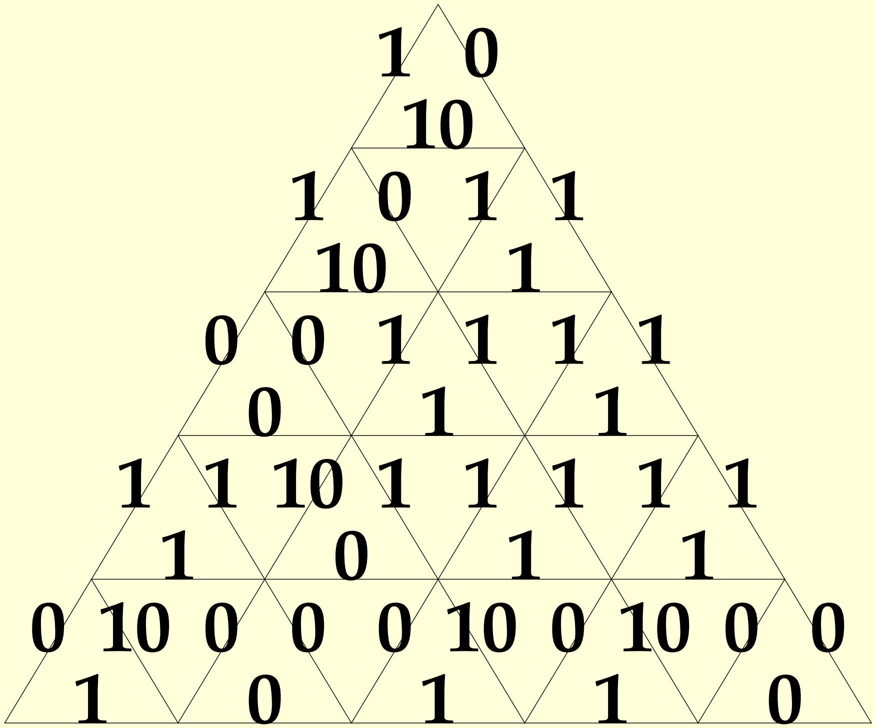


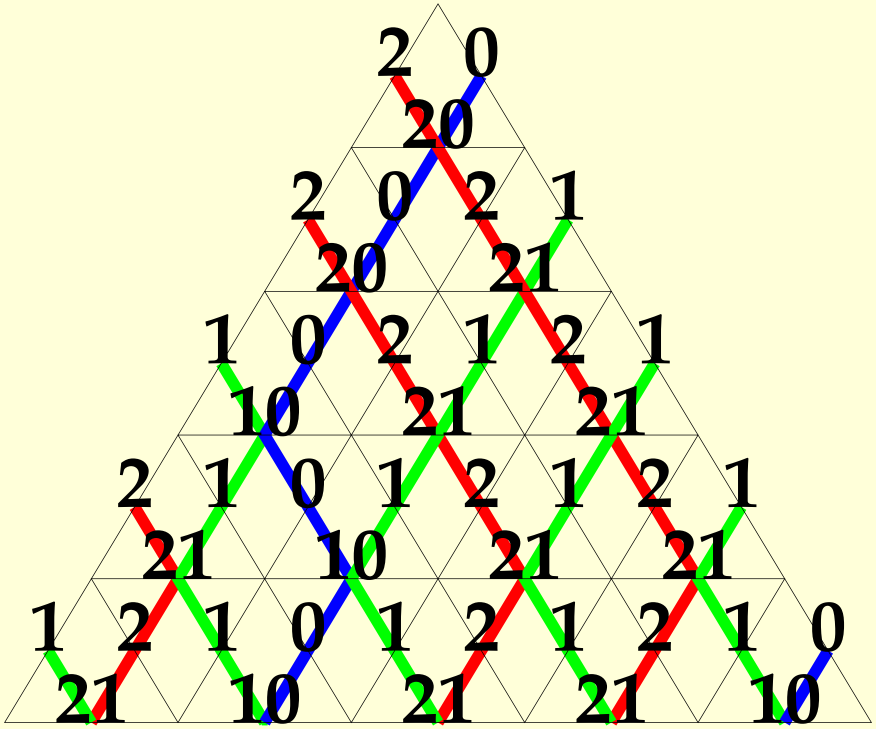
Theorem [Henriques ~'04]. One can compute $c_{\lambda\mu\nu}^{o}$ using any lattice surface Σ in the tetrahedron with $\partial\Sigma$ this same (λ, μ, ν, o) boundary.

Proof: $\exists$ 3-d puzzle pieces giving correspondences between Σ- and Σ'-puzzles.

[Halacheva-Perry-ZJ] All is much simpler and extends to K-theory if ν is written in 0s and 1s, while μ is written in 1s and 2s, and λ in 2s and 3s.

Example. Before and after reformulation.





Nakajima's geometry of some $U_q(\mathfrak{g}[z^\pm])$ representations.

But why *should* such representations come up in studying $Fl(n_1, n_2, \ldots, n_d; \mathbb{C}^n)$?

Given an oriented graph (Q_0, Q_1), with some vertices declared "gauged" and the others "framed", double it by adding a backwards arrow for every arrow. Attach a vector space W_i to each framed vertex and V_j to each gauged vertex.

Definition. A point in the **quiver variety** $\mathcal{M}(Q_0, Q_1, W, V)$ is a choice of linear transformation for every edge,

- such that $\sum \pm$ (go out) $\circ$ (come back in) is zero at each gauged vertex;
- every $\vec{v}$ in each V_i can leak into some W_j via *some* path;
- all is considered up to $\prod_i GL(V_i)$ change-of-bases at the gauged vertices.

Let $\mathcal{M}(Q_0, Q_1, W) := \coprod_W \mathcal{M}(Q_0, Q_1, W, V)$ be the **quiver scheme**.

Theorem [Nakajima '01]. If Q is ADE, then $U_q(\text{its } \mathfrak{g}[z^\pm]) \circlearrowright K(\mathcal{M}(Q_0, Q_1, W))$.

Main example. $$\mathcal{M}\begin{pmatrix} \boxed{n} & & & & \\ \uparrow & & & & \\ n_d & \leftarrow & n_{d-1} & \leftarrow \ldots \leftarrow & n_1 \end{pmatrix} \cong T^*Fl(n_1, \ldots, n_d; \ \mathbb{C}^n).$$

For this framing the $U_q(\mathfrak{sl}_{d+1}[z^\pm])$-action appears already in [Ginzburg-Vasserot 1993], and the rep is $K(\mathcal{M}(Q_0, Q_1, n\omega_1)) \cong (\mathbb{C}^{d+1})^{\otimes n}$, whose weight multiplicities are $(d+1)$-nomial coefficients.

Some Lagrangian relations of quiver varieties.

Recall that we decided that the puzzle labels should be $0^k, 1^{n-k}$ on NE but $1^k, 2^{n-k}$ on NW, suggesting we work with "2-step" $Fl(k, n;\ \mathbb{C}^n)$ and $Fl(0, k;\ \mathbb{C}^n)$.

On $\mathbb{C}^n \oplus \mathbb{C}^n$ we put a $\mathbb{C}^\times$-action with weights $0, 1$, extending to an action on $\mathcal{M}\begin{pmatrix}\boxed{n+n} & \\ n+k & k\end{pmatrix}$; then $\mathcal{M}\begin{pmatrix}\boxed{n} & \\ k & 0\end{pmatrix} \times \mathcal{M}\begin{pmatrix}\boxed{n} & \\ n & k\end{pmatrix}$ is a fixed-point component. Let $attr$ be the **(closed!)** attracting set, the Morse/Białynicki-Birula stratum.

Now let $\Phi_N^{-1}(\mathbf{1}) := \{$the composite $(\mathbb{C}^n \oplus 0) \searrow \mathbb{C}^{n+k} \nearrow (0 \oplus \mathbb{C}^n)$ is the identity$\}$. Points (reps) in that set enjoy splittings of $\mathbb{C}^{n+k}$, plus coordinates on the $\mathbb{C}^n$.

Imprecisely stated theorem [K-ZJ]. The Lagrangian relations

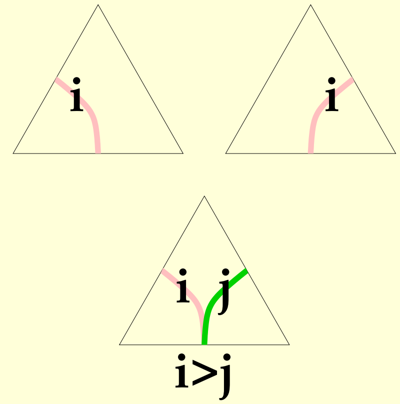
$$\mathcal{M}\begin{pmatrix}\boxed{n} & \\ k & 0\end{pmatrix} \times \mathcal{M}\begin{pmatrix}\boxed{n} & \\ n & k\end{pmatrix} \overset{attr}{\longleftrightarrow} \mathcal{M}\begin{pmatrix}\boxed{n+n} & \\ n+k & k\end{pmatrix} \overset{\Phi_N^{-1}(\mathbf{1})}{\longleftrightarrow} \mathcal{M}\begin{pmatrix} & \boxed{n} \\ k & k\end{pmatrix}$$

induce the usual multiplication map on $H^*_{T\times\mathbb{C}^\times}(T^*Gr(k, \mathbb{C}^n))$, up to a scale, and by following the natural (analogues of Schubert) bases (and taking q, or really $\hbar$, to ∞) we recover Grassmannian puzzles.

Changing the left k to j gives $H^*(Gr(j, \mathbb{C}^n)) \otimes H^*(Gr(k, \mathbb{C}^n)) \to H^*(Fl(j, k;\ \mathbb{C}^n))$, i.e. all this time the 1-step puzzle pieces were already enough to do some 2-step!

The newest Schubert calculus: separated descents.

Theorem [K-ZJ]. Consider the puzzle pieces at right, and their $180°$ rotations. Make size n puzzles with $1, \ldots, k$ and $n-k$ blanks on NE side, $k+1, \ldots, n$ and k blanks on NW side. Then these compute the structure constants of $H^*(Fl(k, \ldots, n; \mathbb{C}^n)) \otimes H^*(Fl(1, \ldots, k; \mathbb{C}^n)) \to H^*(Fl(\mathbb{C}^n))$, and with two more pieces we get the K_T-version.



[Kogan '01], the previous state-of-the-art for general $H^*(Fl(\mathbb{C}^n))$ calculations (extended to K-theory in [K-Yong '04]), assumed that one of the two factors was a Grassmannian (and was algorithmic, and nonequivariant).

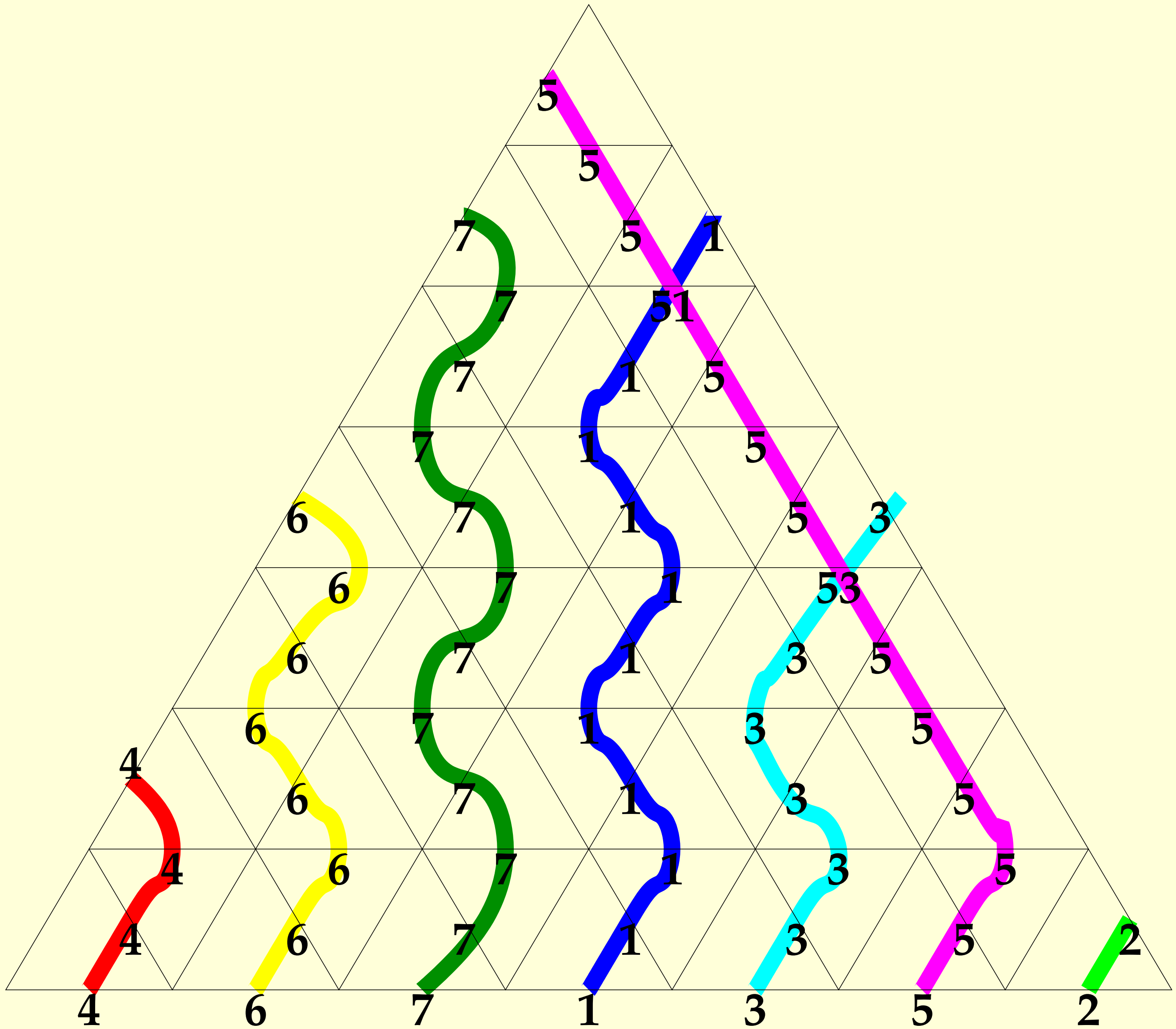
"Proof". Same recipe as last slide, using the Lagrangian relations

$$\mathcal{M}\begin{pmatrix} \boxed{n} & & & & & \\ n & n\ldots n & k & k-1 & \ldots & 1 \end{pmatrix} \times \mathcal{M}\begin{pmatrix} \boxed{n} & & & & & \\ n-1 & n-2\ldots k & 0 & 0 & \ldots & 0 \end{pmatrix}$$

$$\overset{attr}{\longleftrightarrow} \mathcal{M}\begin{pmatrix} \boxed{n+n} & & & & & & \\ 2n-1 & 2n-2 & \ldots & n+k & k & k-1 & \ldots & 1 \end{pmatrix}$$

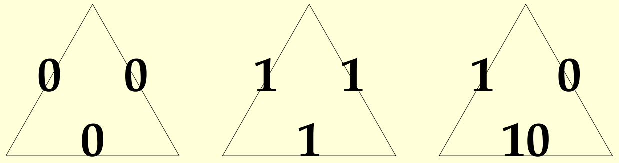
$$\overset{\Phi_N^{-1}(1)}{\longleftrightarrow} \mathcal{M}\begin{pmatrix} & \boxed{n} & & & & & & \\ n-1 & 2n-2 & \ldots & n+k & k & k-1 & \ldots & 1 \end{pmatrix} \cong T^*Fl(\mathbb{C}^n)$$

Example. A separated-descents puzzle.



Beyond quiver varieties and R-matrices.

The original puzzle rule based on these three pieces enjoys **Grassmannian duality**: flip a puzzle left-right while exchanging 0s and 1s, comparing computations on $Gr(k, \mathbb{C}^n)$ and on $Gr(n-k, \mathbb{C}^n)$.



This prompts the question: what are *self*-dual puzzles good for?
It turns out there are almost none of them, unless we allow equivariant pieces down the centerline, so we build that into the definition of "self-dual puzzle".

Theorem [Halacheva-K-ZJ]. Let J be an antidiagonal symplectic form on $\mathbb{C}^{2n}$, and $SpGr(k, \mathbb{C}^{2n}) := \left\{ V \in Gr(k, \mathbb{C}^{2n}) \ : \ V \leq V^\perp \right\}$ the symplectic Grassmannian. Index its $2^k \binom{n}{k}$ Schubert classes by strings μ of length n in $0, 1, 10$ with $n-k$ 10s.

Then the restriction $\iota^*(S_\lambda) = \sum_\mu d_\lambda^\mu S_\mu$ of a Grassmannian Schubert class $S_\lambda \in H^*(Gr(k, \mathbb{C}^{2n}))$, along the inclusion $\iota : SpGr(k, \mathbb{C}^{2n}) \hookrightarrow Gr(k, \mathbb{C}^{2n})$, has coefficients $d_\lambda^\mu = \#\{$self-dual puzzles with λ on the NW and $\mu\mu^*$ on the bottom$\}$.

If we allow equivariant pieces off the centerline (so, in pairs), but only compute $\prod \{(y_i - y_j) :$ left piece in such a pair$\}$, we get the H_T^* restriction formula.

Previously known formulæ [Pragacz 1998, Coşkun '13] were algorithmic, and didn't admit equivariant extensions. Our proof requires the analogues of YBE for R-matrices *and* K-*matrices*. Note: $T^*SpGr(k, \mathbb{C}^{2n})$ is not a quiver variety!

BONUS: a cluster connection?

Consider two parameters: d the number of steps in the flag manifold, and e the dimension of a puzzle simplex $(1, 2, 3$ in the above).

The labels on an e-dimensional puzzle have, to date, corresponded to weights of a representation of some Lie algebra:

	$d = 1$	$d = 2$	$d = 3$	$d = 4$	$d = 5$	...
$e = 1$	A_1	A_2	A_3	A_4	A_5	...
$e = 2$	A_2	D_4	E_6	E_8		
$e = 3$	A_3					

Mysterious pattern: the *finite-type cluster varieties* associated with these Dynkin diagrams are quite familiar: $Gr(d + 1, \mathbb{C}^{d+e+2})$.

To exploit this insight, we'd need to know not just how to assign a cluster variety to a Dynkin diagram, but what to do with a representation. Any ideas?

Multiplying Segre-Schwartz-MacPherson classes.

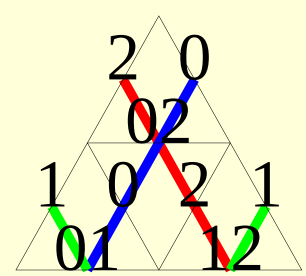
If we keep q around, instead of taking it to ∞, we get classes in $K_{\mathbb{C}^\times}(T^*Fl(j,k;\mathbb{C}^n))$ associated to certain conical-Lagrangian-supported sheaves. Puzzles then compute the products of a related set: those classes, but divided by the class of the zero section (also Lagrangian). These puzzles also compute (in the $K \dashrightarrow H^*$ limit) the *comultiplication* of Chern-Schwarz-MacPherson classes.

The Grassmannian rule has puzzle pieces for *all* nonzero matrix entries of $\mathbb{C}^3 \otimes \mathbb{C}^3 \to Alt^2\mathbb{C}^3$; unlike as in ordinary puzzles, this rule doesn't forbid the 02 label (those entries are suppressed only in the $q \to 0$, $K \dashrightarrow H^*$ limit).

Theorem [K-ZJ]. The CSM result lets one compute compactly supported Euler characteristics of intersections of generically translated Bruhat cells:

$$\chi_c\left(\bigcap_{i=1}^{3}(g_i \cdot X^\circ_{\lambda_i})\right) = (-1)^{k(n-k)-\sum_{i=1}^3 \ell(\lambda_i)} \#\left\{\text{puzzles now including } 02 \text{ labels}\right\}$$

Example. Intersect three open Bruhat cells on $\mathbb{CP}^1$ transversely, resulting in $\mathbb{CP}^1 \setminus \{3 \text{ points}\}$. That has $\chi_c = 2 - 3(1) = -1^{1(2-1)}$, and indeed there is one puzzle, using the 02 label in the interior.



Recognizing quiver varieties that are just $T^*Fl(n_1, \ldots, n_d;\ \mathbb{C}^n)$.

Obviously if the V dimension vector is supported on a type A subdiagram $S \subseteq Q$, and W on a single $\boxed{\text{vertex}}$ at one end of S, then by the last slide $\mathcal{M}(Q_0, Q_1, \boxed{W}, V) \cong T^*Fl(n_1, \ldots, n_d;\ \mathbb{C}^n)$. Say that these (V, W) are of **flag type**.

Nakajima defined "reflections" $\mathcal{M}(Q_0, Q_1, \boxed{W}, V, \theta) \cong \mathcal{M}(Q_0, Q_1, \boxed{W}, r_\alpha \cdot V, r_\alpha \cdot \theta)$ but they involve θ-stability, in general more subtle than our "each $\vec{v} \in V_i$ leaks into some $\boxed{W_j}$" stability condition (which corresponds to $\forall \langle \theta_i, \alpha_j \rangle > 0$).
If $\langle \theta_i, \alpha_j \rangle > 0$ for all $V_j > 0$, though, our naïve notion of stability is still correct.

The action of $r_\alpha \cdot V$ replaces the α label by the sum of the neighbors **including the $\boxed{\textbf{framed}}$ neighbor in $\boxed{\mathbf{W}}$**, minus the original label. In particular the new dimension is a linear combination of the original dimensions.

Theorem [K-ZJ]. Assume $(Q_0, Q_1, \boxed{W}, V)$ is of flag type, and that the dimensions in $\pi \cdot V$ are nonnegative combinations of the dimensions in V. Then $\mathcal{M}(Q_0, Q_1, \boxed{W}, \pi \cdot V) \cong T^*Fl(n_1, \ldots, n_d;\ \mathbb{C}^n)$, steps coming from $\dim V$.

Some D_4 *examples.*

$$\begin{pmatrix} & & \boxed{n} \\ 0 & j & k \\ & 0 & \end{pmatrix} \to \begin{pmatrix} & & \boxed{n} \\ \mathbf{j} & j & k \\ & \mathbf{j} & \end{pmatrix} \to \begin{pmatrix} & & \boxed{n} \\ j & \mathbf{j+k} & k \\ & j & \end{pmatrix}$$

$$\to \begin{pmatrix} & & \boxed{n} \\ \mathbf{k} & j+k & \mathbf{n+j} \\ & \mathbf{k} & \end{pmatrix} \to \begin{pmatrix} & & \boxed{n} \\ k & \mathbf{n+k} & n+j \\ & k & \end{pmatrix} \to \begin{pmatrix} & & \boxed{n} \\ \mathbf{n} & n+k & n+j \\ & k & \end{pmatrix}$$

Quiver varieties that recover $d = 2, 3$ puzzles.

Each of the below reflects to a flag type quiver variety, which is fun to verify.

$$d = 2: \quad \begin{pmatrix} \boxed{n} & & \\ k & j & 0 \\ & 0 & \end{pmatrix} \times \begin{pmatrix} & & \boxed{n} \\ n & n+k & n+j \\ & k & \\ \end{pmatrix} \xleftrightarrow[]{\text{sum, then split using } \boxed{n} \xrightarrow{1} \boxed{n}} \begin{pmatrix} k & k+j & j \\ & k & \\ & \boxed{n} & \end{pmatrix}$$

remember me from slide 10?

$$d = 3: \quad \begin{pmatrix} \boxed{n} & & & & \\ l & k & j & 0 & 0 \\ & & 0 & & \end{pmatrix} \times \begin{pmatrix} \boxed{n} & & & & \\ 2n & 2n+l & 2n+l+k & n+l+j & l \\ & & n+k & & \end{pmatrix}$$

this Lagrangian relation involves *two* matrix equations

$$\leftrightarrow \begin{pmatrix} & & & & \boxed{n} \\ l & l+k & l+k+j & l+j & l \\ & & k & & \end{pmatrix}$$

We know some E_8 quiver varieties giving $d = 4$, but the corresponding reps $\mathfrak{e}_8 \oplus \mathbb{C}$ are not multiplicity-free, and don't lead to a positive rule. (It's a *mostly* positive rule, and surely the most efficient known, but definitely not positive.)

Quiver varieties that recover $d = 1$ associativity.

$$\begin{pmatrix} \boxed{n} & & \\ k & 0 & 0 \end{pmatrix} \times \begin{pmatrix} \boxed{n} & & \\ n & k & 0 \end{pmatrix} \times \begin{pmatrix} \boxed{n} & & \\ n & n & k \end{pmatrix} \to \begin{pmatrix} & \boxed{3n} & \\ 2n+k & n+k & k \end{pmatrix} \to \begin{pmatrix} & & \boxed{n} \\ k & k & k \end{pmatrix}$$

In rep theory terms, this studies the map on particular weight spaces inside

$$(\mathbb{C}^4)^{\otimes n} \otimes (\mathbb{C}^4)^{\otimes n} \otimes (\mathbb{C}^4)^{\otimes n} \to (\mathbb{C}^4)^{\otimes 3n} \to (\mathrm{Alt}^3 \mathbb{C}^4)^{\otimes n}$$

Properly speaking, this would lead to labeling some 3-d puzzle edges with basis elements of $\mathbb{C}^4$ (a colored line), some (the horizontal ones) with basis elements of $\mathrm{Alt}^2\mathbb{C}^4$ (two different colors), some with basis elements of $\mathrm{Alt}^3\mathbb{C}^4$ (three different colors). For visibility one works with $(\mathbb{C}^4)^*$ instead (the missing color), the cost being that at each horizontal edge the two colors coming from one side **anti**-match the two from the other.