Representations of the two-boundary Temperley-Lieb algebras

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Joint work with Arun Ram;
work in progress additionally with
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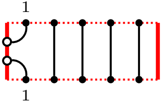
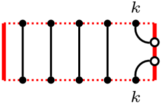
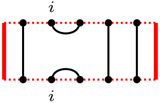
September 4, 2019

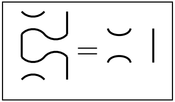
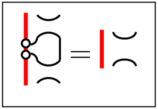
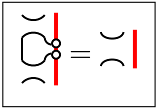
Two-boundary Temperley-Lieb algebras

Mitra, Nienhuis, De Gier, Batchelor (2004), De Gier, Nichols (2009):
Fix $z, \delta_0, \delta_k \in \mathbb{C}$. The *two-boundary Temperley-Lieb algebra* TL_k is a diagram algebra generated over $\mathbb{C}$ by diagrams

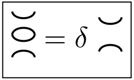
$e_0 =$ , $e_k =$ , and $e_i =$ 

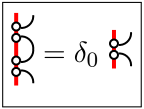
for $i = 1, \ldots, k-1$, with relations $e_i e_j = e_j e_i$ for $|i-j| > 1$,

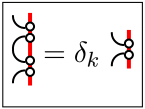
$e_i e_{i\pm1} e_i = e_i$
for $1 \leq i \leq k-1$,

 or  or 

$e_i^2 = \delta_i e_i.$

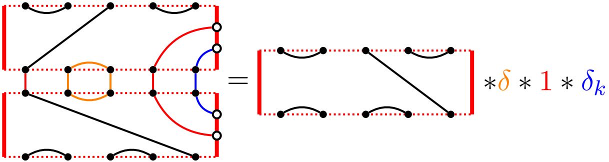


or


or


(Side loops are resolved with a 1 or a δ_i depending on whether there are an even or odd number of connections below their lowest point.)

Diagram multiplication:



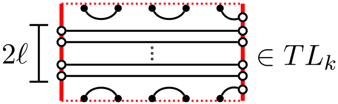
In short, TL_k has basis given by non-crossing diagrams with

(1) k connections to the top and to the bottom,

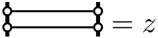
(2) an even number of connections to the right and to the left, and

(3) no edges with both ends on the left or both ends on the right.

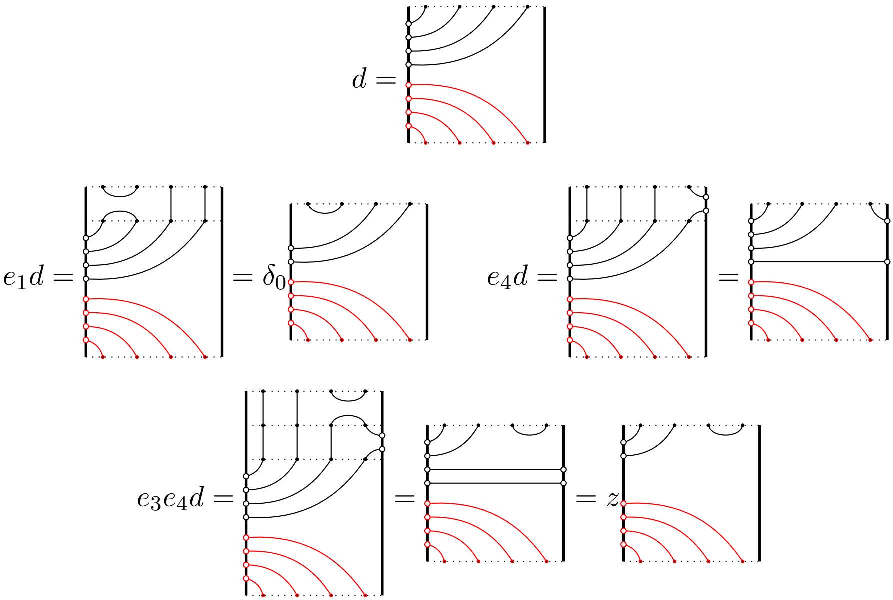
However,



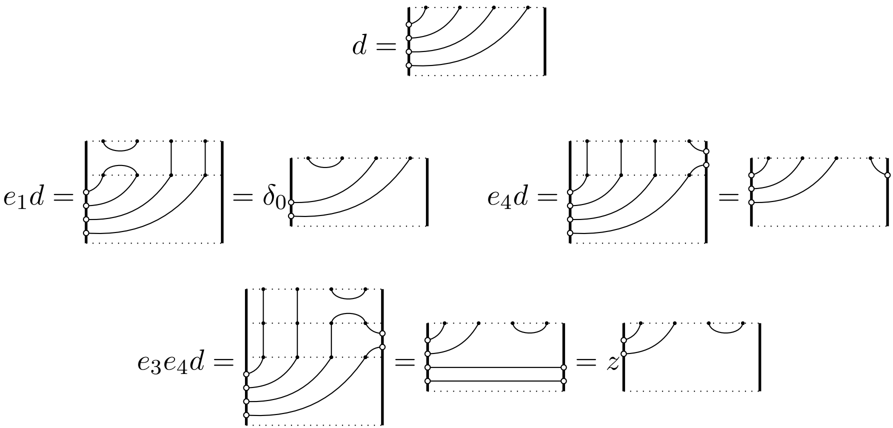
So unlike the classical T-L algebras, TL_k is not finite dimensional!
Take quotient giving



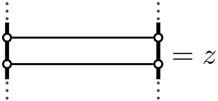
Representation theory of TL_k: action on diagrams



Representation theory of TL_k: half diagrams



You can tell when to use

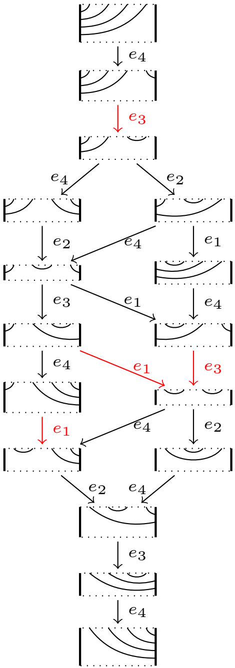


or not by the parity of connections to the left/right walls.

Standard module:

(act by e_i, don't make loops)

Red arrows indicate coef of z.

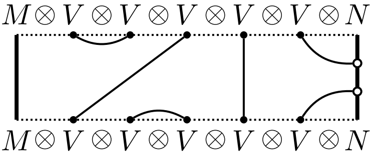


For what z does this module split?

Actions on tensor space

Two-boundary Temperley-Lieb diagrams have a natural action on special tensor products of $U_q\mathfrak{sl}_2$-modules...

Let $V = L(\square) = \mathbb{C}^2$, $M = L(a)$, $N = L(b)$ be highest-weight $U_q\mathfrak{sl}_2$-modules. Then TL_k acts on



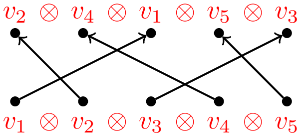
via factor permutation and projection operators (the parameters depend on q, a, and b). Further, this action centralizes the action of $U_q\mathfrak{sl}_2$

Schur-Weyl Duality

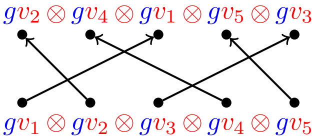
$\mathrm{GL}_n(\mathbb{C})$ acts on $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n = (\mathbb{C}^n)^{\otimes k}$ diagonally.

$$g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_k.$$

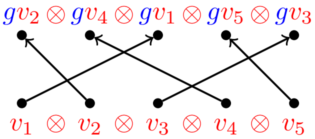
S_k also acts on $(\mathbb{C}^n)^{\otimes k}$ by place permutation.



These actions commute!



vs.



Schur-Weyl Duality

$$\underbrace{\mathrm{End}_{\mathrm{GL}_n}\left((\mathbb{C}^n)^{\otimes k}\right)}_{\substack{\text{(all linear maps that} \\ \text{commute with } \mathrm{GL}_n\text{)}}} = \underbrace{\pi(\mathbb{C}S_k)}_{\substack{\text{(img of } S_k \\ \text{action)}}} \quad \text{and} \quad \mathrm{End}_{S_k}\left((\mathbb{C}^n)^{\otimes k}\right) = \underbrace{\rho(\mathbb{C}\mathrm{GL}_n)}_{\substack{\text{(img of } \mathrm{GL}_n \\ \text{action)}}}.$$

Powerful consequence: a duality between representations

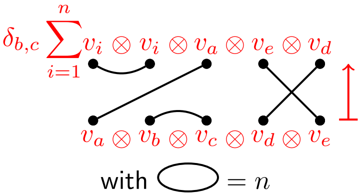
The double-centralizer relationship produces

$$(\mathbb{C}^n)^{\otimes k} \cong \bigoplus_{\lambda \vdash k} G^\lambda \otimes S^\lambda \quad \text{as a } \mathrm{GL}_n\text{-}S_k \text{ bimodule,}$$

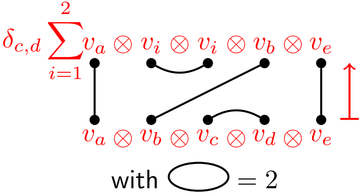
where
G^λ are distinct irreducible GL_n-modules
S^λ are distinct irreducible S_k-modules

More centralizer algebras

Brauer (1937)
Orthogonal and symplectic groups (and Lie algebras) acting on $(\mathbb{C}^n)^{\otimes k}$ diagonally centralize the **Brauer algebra**:



Temperley-Lieb (1971)
GL_2 and SL_2 (and $\mathfrak{gl}_2$ and $\mathfrak{sl}_2$) acting on $(\mathbb{C}^2)^{\otimes k}$ diagonally centralize the **Temperley-Lieb algebra**:

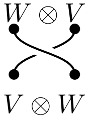


(Diagrams encode maps $V^{\otimes k} \to V^{\otimes k}$ that commute with the action of some classical algebra.)

Quantum groups and braids

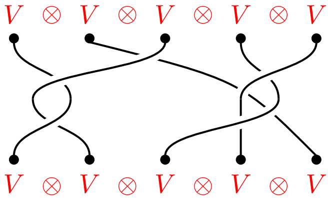
Fix $q \in \mathbb{C}$, and let $\mathcal{U} = \mathcal{U}_q\mathfrak{g}$ be the Drinfeld-Jimbo quantum group associated to Lie algebra $\mathfrak{g}$.
$\mathcal{U} \otimes \mathcal{U}$ has an invertible element $\mathcal{R} = \sum_{\mathcal{R}} R_1 \otimes R_2$ that yields a map

$$\check{\mathcal{R}}_{VW} : V \otimes W \longrightarrow W \otimes V$$



that (1) satisfies braid relations, and
(2) commutes with the $\mathcal{U}$-action on $V \otimes W$
for any $\mathcal{U}$-modules V and W.

The braid group shares a commuting action with $\mathcal{U}$ on $V^{\otimes k}$:



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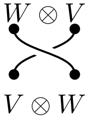
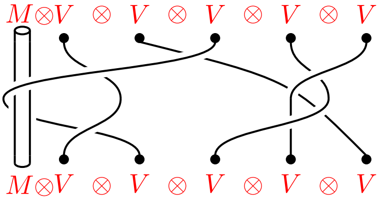
$$\check{\mathcal{R}}_{VW} \colon V \otimes W \longrightarrow W \otimes V$$

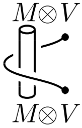


that (1) satisfies braid relations, and
(2) commutes with the $\mathcal{U}$-action on $V \otimes W$
for any $\mathcal{U}$-modules V and W.

The one-pole/affine braid group shares a commuting action with $\mathcal{U}$ on $M \otimes V^{\otimes k}$:



Around the pole:



$= \check{R}_{MV} \check{R}_{VM}$

Quantum groups and braids

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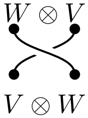
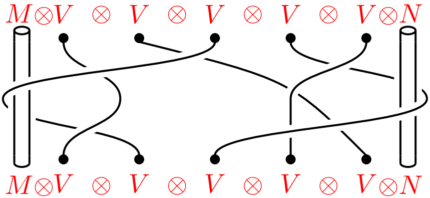
$$\check{\mathcal{R}}_{VW} \colon V \otimes W \longrightarrow W \otimes V$$

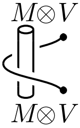


that (1) satisfies braid relations, and
(2) commutes with the $\mathcal{U}$-action on $V \otimes W$
for any $\mathcal{U}$-modules V and W.

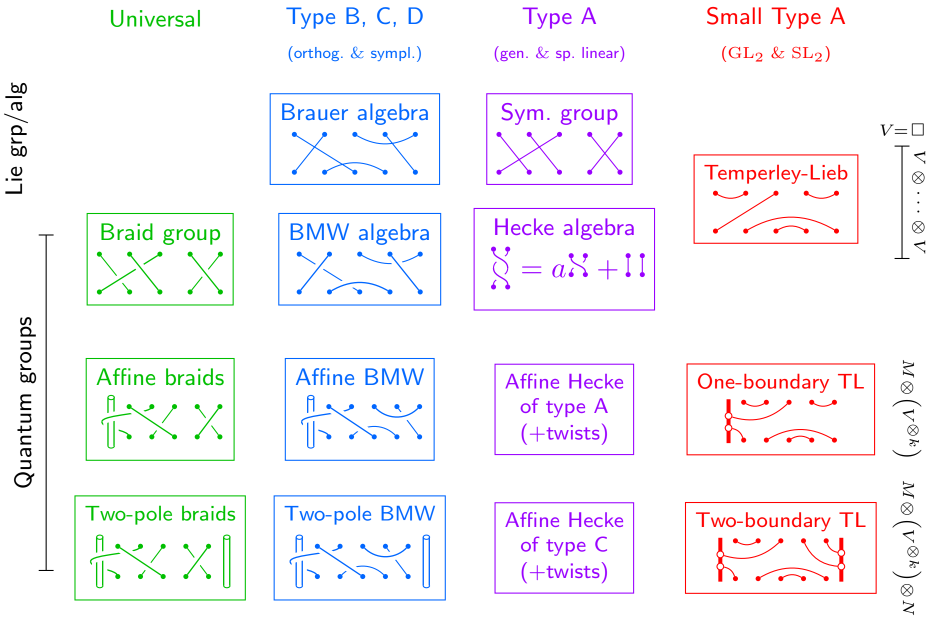
The two-pole braid group shares a commuting action with $\mathcal{U}$ on $M \otimes V^{\otimes k} \otimes N$:



Around the pole:

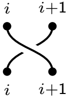


$$= \check{R}_{MV} \check{R}_{VM}$$



Universal
Type B, C, D
(orthog. & sympl.)
Type A
(gen. & sp. linear)
Small Type A
(GL_2 & SL_2)
Lie grp/alg
Quantum groups
Brauer algebra
Sym. group
Temperley-Lieb
$V = \square$
$V \otimes \cdots \otimes V$
Braid group
BMW algebra
Hecke algebra
$= a \quad + \quad$
Affine braids
Affine BMW
Affine Hecke
of type A
(+twists)
One-boundary TL
$\left(V^{\otimes k}\right) \otimes M$
Two-pole braids
Two-pole BMW
Affine Hecke
of type C
(+twists)
Two-boundary TL
$N \otimes \left(V^{\otimes k}\right) \otimes M$

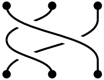
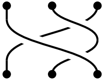
The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

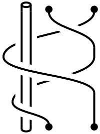
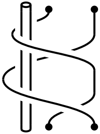
$T_k =$ , $T_0 =$  and $T_i =$ 

for $1 \leq i \leq k-1$,

subject to relations

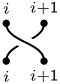
i.e.

$T_i T_{i+1} T_i =$  $=$  $= T_{i+1} T_i T_{i+1}$,

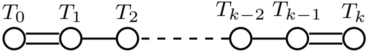
$T_1 T_0 T_1 T_0 =$  $=$  $= T_0 T_1 T_0 T_1$,

and, similarly, $T_{k-1} T_k T_{k-1} T_k = T_k T_{k-1} T_k T_{k-1}$.

(1) The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

$T_k =$ , $T_0 =$  and $T_i =$ 

for $1 \le i \le k-1$,

subject to relations 

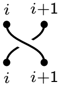
.

(2) Fix constants $t_0, t_k, t \in \mathbb{C}$.
The affine type C Hecke algebra $\mathcal{H}_k$ is the quotient of $\mathbb{C}\mathcal{B}_k$ by the relations

$$(T_0 - t_0^{1/2})(T_0 + t_0^{-1/2}) = 0, \quad (T_k - t_k^{1/2})(T_k + t_k^{-1/2}) = 0$$

$$\text{and} \quad (T_i - t^{1/2})(T_i + t^{-1/2}) = 0 \quad \text{for } i = 1, \ldots, k-1.$$

(1) The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

$T_k =$ , $\quad T_0 =$  and $\quad T_i =$ 

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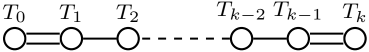
subject to relations 

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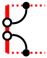
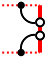
(3) Set

 $= t_0^{1/2}$  $-$  $\qquad (e_0 = t_0^{1/2} - T_0)$

 $= t_k^{1/2}$  $-$  $\qquad (e_k = t_k^{1/2} - T_k)$

 $= t^{1/2}$  $-$  $\qquad (e_i = t^{1/2} - T_i)$

so that $e_j^2 = z_j e_j$ (for good z_j).
The two-boundary Temperley-Lieb algebra is the quotient of $\mathcal{H}_k$ by the relations $e_i e_{i\pm1} e_i = e_i$ for $i = 1, \ldots, k-1$.

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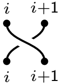
$T_k =$ , $T_0 =$  and $T_i =$ 

for $1 \le i \le k-1$.

(2) Fix constants $t_0, t_k, t = t_1 = t_2 = \cdots = t_{k-1} \in \mathbb{C}$.
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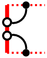
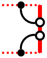
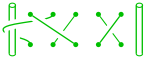
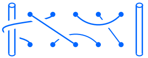
(3) Set

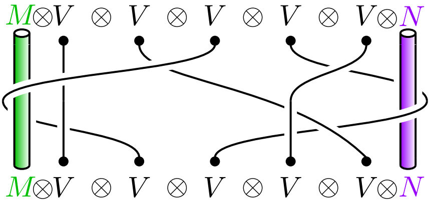
 $= t_0^{1/2}$  $-$ ,  $= t_k^{1/2}$  $-$  and  $= t^{1/2}$  $-$ 

so that $e_j^2 = z_j e_j$. The two-boundary Temperley-Lieb algebra is the quotient of $\mathcal{H}_k$ by the relations $e_i e_{i\pm 1} e_i = e_i$ for $i = 1, \ldots, k-1$.

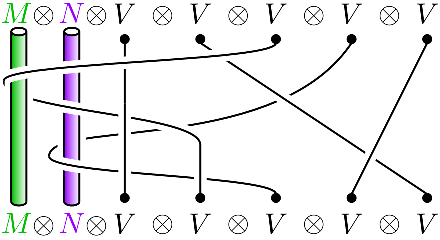
Universal	Type B, C, D	Type A	Small Type A
Two-pole braids	Two-pole BMW	Affine Hecke of type C (+twists)	Two-boundary TL



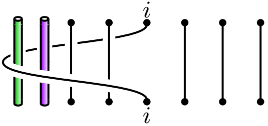




Move both poles to the left ↓



Jucys-Murphy elements:

$Z_i =$ 

- Pairwise commute
- $Z(\mathcal{H}_k)$ is (type-C) symmetric Laurent polynomials in Z_i's
- Central characters indexed by $\mathbf{c} \in \mathbb{C}^k$ (modulo signed permutations)

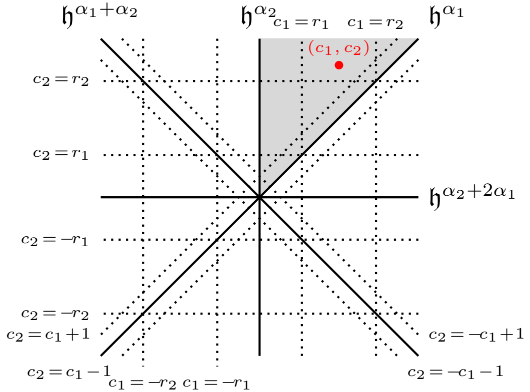
Representation theory of $\mathcal{H}_k$

The representations of $\mathcal{H}_k$ are indexed by pairs $(\mathbf{c}, J)$, where

$\mathbf{c}$ is a point in the fundamental chamber of the (finite) type C hyperplane system, and

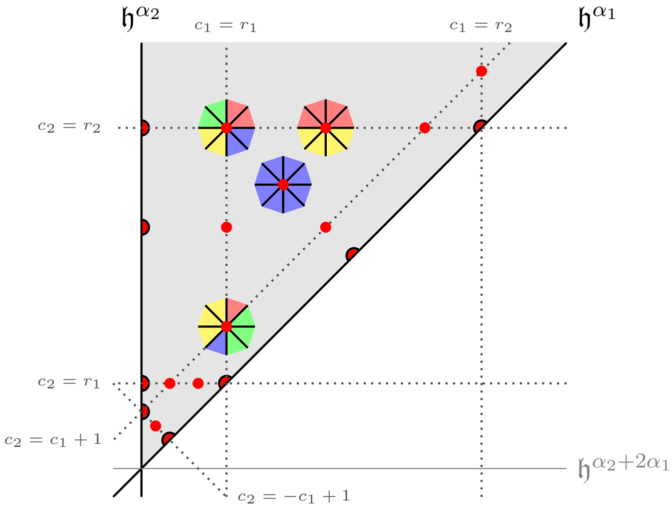
J is a set of choices of positive/negative sides of other distinguished hyperplanes intersecting $\mathbf{c}$

Example: $k = 2$



The r_is depend on $\mathcal{H}_k$'s parameters t_0 and t_k: $r_1 = \log_t(t_0/t_k)$, $r_2 = \log_t(t_0 t_k)$

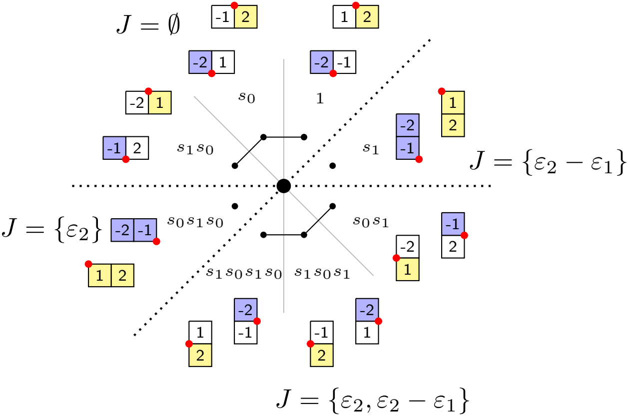
Representation theory of $\mathcal{H}_k$

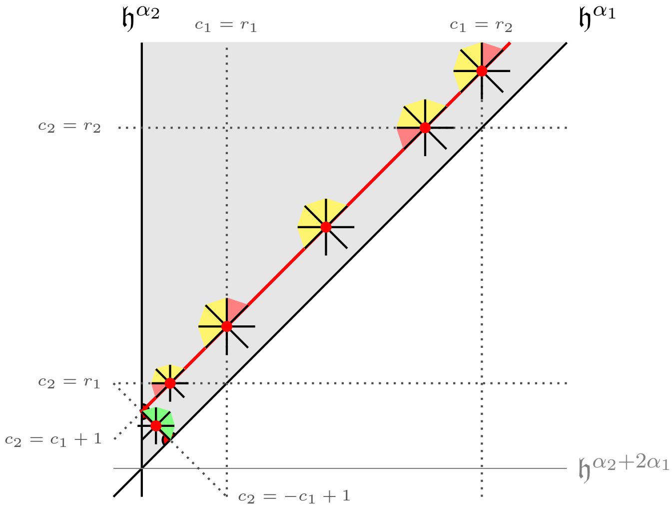


The r_is depend on $\mathcal{H}_k$'s parameters t_0 and t_k: $r_1 = \log_t(t_0/t_k)$, $r_2 = \log_t(t_0 t_k)$

A little more detail

- J is determined by a set of positive roots (corresp. to hyperplanes).
- For “nice” characters, there is a bijection between alcoves and marked type-C generalized Young tableaux.
- “Intertwining operators” τ_i move between alcoves;
  dotted lines correspond to $\tau_i = 0$.

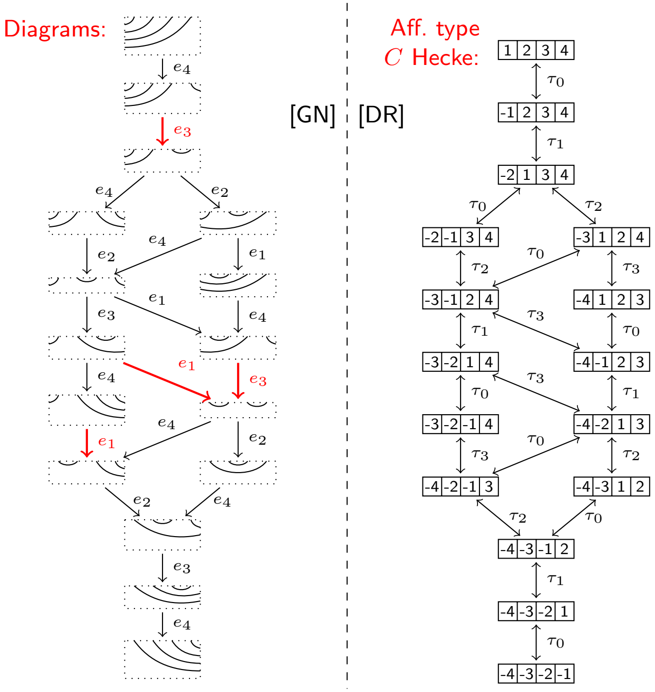




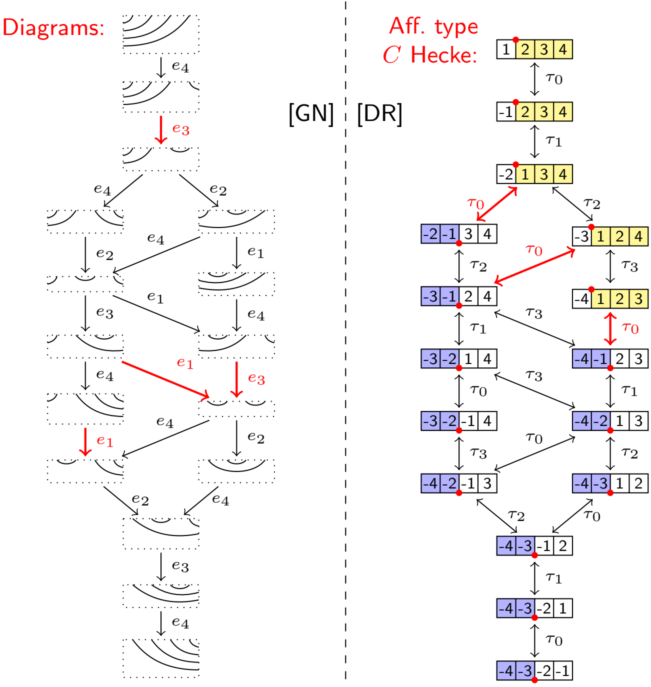
Thm. (D.-Ram)

(1) Representations of $\mathcal{H}_k$ are indexed by pairs $(\mathbf{c}, J)$.

(2) The representations of $\mathcal{H}_k$ that factor through the Temperley-Lieb quotient are as above.



Diagrams:
e_4
e_3
e_4
e_2
e_2
e_4
e_1
e_3
e_1
e_4
e_4
e_1
e_3
e_1
e_4
e_2
e_2
e_4
e_3
e_4
[GN]
[DR]
Aff. type
C Hecke:
1 2 3 4
τ_0
-1 2 3 4
τ_1
-2 1 3 4
τ_0
τ_2
-2 -1 3 4
-3 1 2 4
τ_2
τ_0
τ_3
-3 -1 2 4
-4 1 2 3
τ_1
τ_3
τ_0
-3 -2 1 4
-4 -1 2 3
τ_0
τ_3
τ_1
-3 -2 -1 4
-4 -2 1 3
τ_3
τ_0
τ_2
-4 -2 -1 3
-4 -3 1 2
τ_2
τ_0
-4 -3 -1 2
τ_1
-4 -3 -2 1
τ_0
-4 -3 -2 -1


Diagrams:
[GN]
[DR]
Aff. type
C Hecke:
e4
e3
e4
e2
e2
e4
e1
e3
e1
e4
e4
e1
e3
e1
e4
e2
e2
e4
e3
e4
1 2 3 4
τ0
-1 2 3 4
τ1
-2 1 3 4
τ0
τ2
-2 -1 3 4
-3 1 2 4
τ2
τ0
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-3 -1 2 4
-4 1 2 3
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τ0
-3 -2 1 4
-4 -1 2 3
τ0
τ3
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-3 -2 -1 4
-4 -2 1 3
τ3
τ0
τ2
-4 -2 -1 3
-4 -3 1 2
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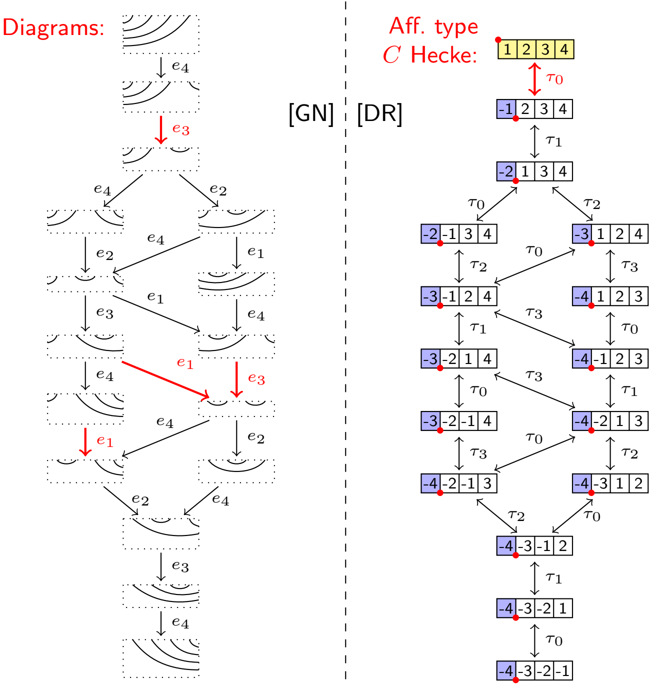
τ0
-4 -3 -1 2
τ1
-4 -3 -2 1
τ0
-4 -3 -2 -1


Diagrams:
e4
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e2
e1
e2
e4
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e4
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τ2
τ0
τ3
-3 -1 2 4
-4 1 2 3
τ1
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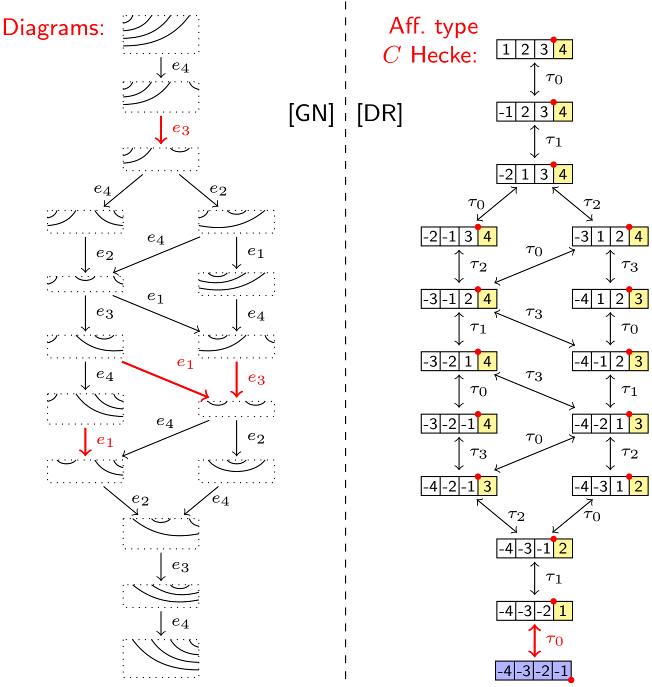
τ0
-3 -2 1 4
-4 -1 2 3
τ0
τ3
τ1
-3 -2 -1 4
-4 -2 1 3
τ3
τ0
τ2
-4 -2 -1 3
-4 -3 1 2
τ2
τ0
-4 -3 -1 2
τ1
-4 -3 -2 1
τ0
-4 -3 -2 -1



Diagrams:
e4
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e4
e2
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e4
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e1
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e4
e1
e3
e1
e4
e2
e2
e4
e3
e4
[GN]
[DR]
Aff. type
C Hecke:
1 2 3 4
τ0
-1 2 3 4
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τ0
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τ2
τ0
τ3
-3 -1 2 4
-4 1 2 3
τ1
τ3
τ0
-3 -2 1 4
-4 -1 2 3
τ0
τ3
τ1
-3 -2 -1 4
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τ3
τ0
τ2
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τ2
τ0
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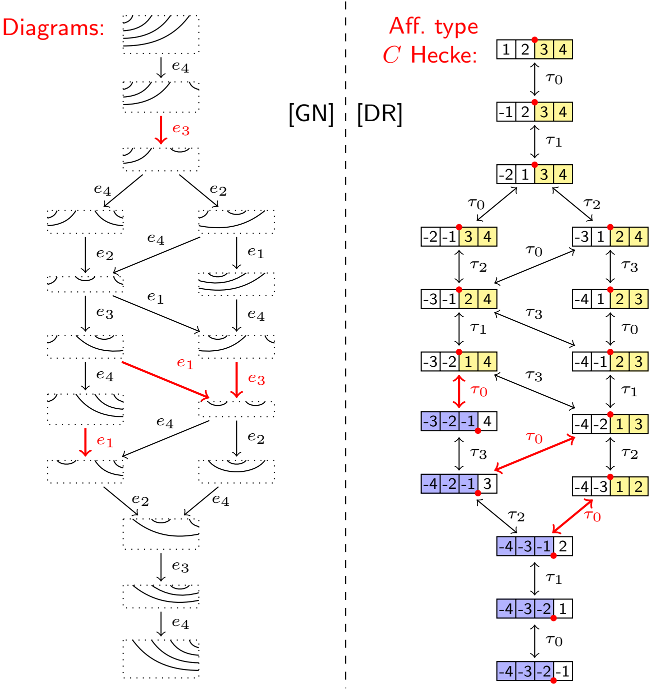
τ1
-4 -3 -2 1
τ0
-4 -3 -2 -1


Diagrams:
[GN]
[DR]
Aff. type C Hecke: