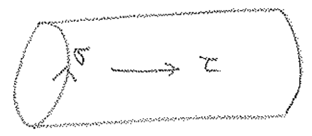
String Interactions

Up till now, we described the propagation of free closed/open strings in flat space:



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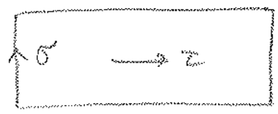


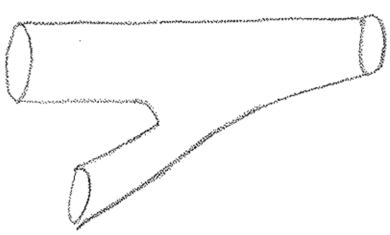
In conformal gauge, this is described by a free 2D CFT with action

$$S_p = \frac{1}{2\pi l_s^2} \int d^2z \; \partial X^\mu \bar\partial X_\mu \; + \; \frac{1}{2\pi} \int d^2z \; b \bar\partial c \; + \bar b \partial c$$

$b \equiv b_{zz}, c^z$ Faddeev - Popov ghosts for Diff & Weyl sym.

$$T = -\frac{1}{l_s^2} \partial X^\mu \partial X_\mu + 2(\partial c) b + c \partial b$$

Strings interact by splitting and joining interactions, incorporated by allowing non-trivial worldsheet topologies:



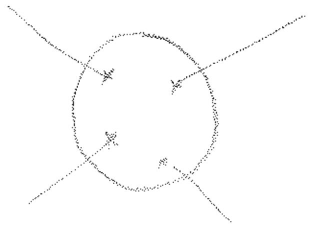
- Still, local properties of the CFT are unaffected!
- Overall power $g_s^{\#\text{ext} - \chi}$

Naively, one might want to compute transition amplitudes between arbitrary set of loops at initial / final time.

However, unlike ordinary field theory such "off-shell closed string amplitudes" are ill-defined (which is hardly surprising since gravity has no local gauge invariant observables; the situation is different for open strings in fixed gravitational backgrounds)

Instead, we should focus on S-matrix elements between on-shell asymptotic states. Using the state/operator correspondence, they can be represented by integrated vertex operators on the sphere

$$\boxed{\langle \text{out} | S | \text{in} \rangle = \int DX^\mu \, Db \, Dc \, \prod_i \int dz_i d\bar{z}_i \, \Phi_i(z,\bar{z}) \, e^{-S_p}}$$



Recall that physical states $|\varphi\rangle$ $\left|\begin{array}{l}\text{annihilated by all } L_{m>0} \, , \, \tilde{L}_{m>0} \\ (L_0 - a)|\varphi\rangle = (\tilde{L}_0 - \tilde{a})|\varphi\rangle = 0\end{array}\right.$

are mapped to primary operators Φ of conformal dimension $(\Delta, \tilde{\Delta}) = (a, \tilde{a})$

In order for $\int dz d\bar{z} \, \Phi$ to be conformally invariant, one needs $(a, \tilde{a}) = (1,1)$

Eg: . tachyon $|p\rangle \longleftrightarrow \Phi = :e^{ipX}:$, $\Delta = p^2 \ell_s^2/4 = 1$ $(-+++)$

. massless states

$$\zeta_{\mu\nu} \alpha^\mu_{-1} \tilde{\alpha}^\nu_{-1} \longleftrightarrow \Phi = \zeta_{\mu\nu} :\partial X^\mu \bar{\partial} X^\nu e^{ipX}:$$

primary if $p^\mu \zeta_{\mu\nu} = p^\nu \zeta_{\mu\nu} = 0$

dimension $(1,1)$ if $(p^\mu)^2 = 0$

Rk: normalization: each closed string vertex should be multiplied by g_s, and the overall amplitude by $g_s^{-\chi}$. This ensures that g_s can be interpreted as vev of $e^{2\phi}$ where ϕ is the dilaton.

Rk: the physical state condition is equivalent to asking that the operator $\mathcal{O} = c(z)\,\bar{c}(\bar{z})\,\Phi$ is annihilated by the BRST operator

$$Q_B = \frac{1}{2\pi i}\oint j_B\,dz = \sum_n c_n L^X_{-n} + \sum_{m,n} \frac{m-n}{2} :c_m c_n b_{-m-n}:$$

$$j_B = c\left(T^X + \frac{1}{2} T^{ghost}\right)$$

It is a fermionic symmetry of the gauged fixed action, which is nilpotent iff the Virasoro symmetry is non anomalous:

$$Q_B^2 = \frac{1}{2}\sum_{-\infty}^{\infty}\left([L_m, L_n] - (m-n) L_{m+n}\right) c_{-m} c_{-n}$$

Restricting to operators $\mathcal{O}$ such that $Q_B \cdot \mathcal{O} = 0$ ensures that results are gauge invariants, and that spurious states of the form $\mathcal{O} = Q_B \mathcal{O}'$ decouple.

If Φ does not depend on the ghosts,

$$Q_B \mathcal{O} = 0 \iff Q_B \Phi = \underbrace{i\partial_\alpha(c^\alpha \Phi)}_{\text{total derivative}}$$

$$Q_B \int d^2z\, \Phi = 0$$

The previous formula however assumed that the "conformal gauge slice" $\gamma_{\alpha\beta} = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$ intersects each orbit of Diff $\ltimes$ Weyl once and only once.

This is true on the cylinder, but in general not on other topologies.

Eg on the sphere, the conformal gauge is invariant under $SL(2,\mathbb{C})$

$$z \to \frac{\alpha z + \beta}{\gamma z + \delta}$$

so one is still left with an (∞) volume $\dfrac{1}{V(SL(2,\mathbb{C}))}$

On surfaces of higher genus, the conformal gauge is too restrictive, and one must integrate over the moduli space of conformal structures (ie metrics modulo Weyl), a space of complex dimension

$$\begin{cases} 3h-3 & h>1 \\ 1 & h=0 \\ 0 & h=1 \end{cases}.$$

For now we restrict to tree-level scattering amplitudes of closed string states.

For 3 external closed string states:

$$S = \frac{g_c}{V(Sl(2,\mathbb{C}))} \left\langle \int dz_1 d\bar{z}_1\, \phi_1 \int dz_2 d\bar{z}_2\, \phi_2 \int dz_3 d\bar{z}_3\, \phi_3 \right\rangle$$

Conformal invariance determines

$$\langle \phi_1 \phi_2 \phi_3 \rangle = \frac{C_{123}}{z_{12}^{\Delta_1+\Delta_2-\Delta_3}\, z_{23}^{\Delta_2+\Delta_3-\Delta_1}\, z_{31}^{\Delta_3+\Delta_1-\Delta_2}} \times c.c.$$

$$\Delta_i = 1 \Rightarrow \quad = C_{123} \Big/ |z_{12}|^2 |z_{23}|^2 |z_{31}|^2$$

$Sl(2,\mathbb{C})$ allow to fix z_1, z_2, z_3 arbitrarily, eg $0, 1, \infty$

The Jacobian must cancel the denominator, leaving

$$\boxed{S = g_c\, C_{123}}$$

<u>Rk</u> instead of using integrated operators $\int \Phi_i\, d^2 z$, one can use local operators $O_i = c(z_i)\bar{c}(\bar{z}_i)\, \Phi_i(z,\bar{z})$ at fixed locations and <u>not</u> divide by $Sl(2,\mathbb{C})$. The above result is reproduced using $\langle c(z_1) c(z_2) c(z_3) \rangle = z_{12}\, z_{13}\, z_{23}$ on the sphere

More generally, for n closed string tachyons,

$$S = \frac{g_s^{n-2}}{V(SL(2,\mathbb{C}))} \left\langle \prod_{i=1}^{n} \int dz_i d\bar{z}_i \, e^{i p_i \cdot X(z_i)} \right\rangle \qquad p_i^2 \ell_s^2 = 4$$

$$= \frac{g_s^{n-2}}{V(SL(2,\mathbb{C}))} \int d^2 z_1 \cdots d^2 z_n \exp\left(\frac{\ell_s^2}{2} \sum_{i \neq j} p_i p_j \ln |z_i - z_j| \right)$$
$$\times \, \delta^{(26)}\left(\sum p_i\right)$$

The vertices 1, 2, 3 can be fixed to any desired value, at the cost of introducing the Jacobian $|z_{12} z_{23} z_{13}|^2$

Eg $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$

For $n=4$, in terms of Mandelstam variables $(z = z_4)$

$$s = -(p_1+p_2)^2 \qquad s+t+u = -\sum_i p_i^2 = -\frac{16}{\ell_s^2}$$
$$t = -(p_1+p_3)^2 \qquad s = -\frac{8}{\ell_s^2} - 2 p_1 p_2$$
$$u = -(p_1+p_4)^2$$

we get (up to normalization)

$$S = g_s^2 \, \delta^{26}\left(\Sigma p_i\right) \int d^2 z \, |z|^{\ell_s^2 p_1 p_4} \, |1-z|^{\ell_s^2 p_2 p_4}$$
$$= \qquad " \qquad |z|^{-\frac{\ell_s^2}{2}\left(u + \frac{8}{\ell_s^2}\right)} \, |1-z|^{-\frac{\ell_s^2}{2}\left(t + \frac{8}{\ell_s^2}\right)}$$

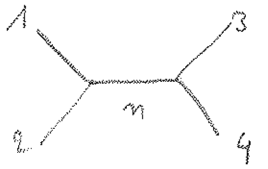
Using $\displaystyle \int d^2 z \, |z|^{2a-2} |1-z|^{2b-2} = 2\pi \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(1-a)\Gamma(1-b)\Gamma(1-c)} \Bigg|_{a+b+c=1}$

with
$$a = -\frac{\ell_s^2}{4} u - 1$$
$$b = -\frac{\ell_s^2}{4} t - 1$$
$$c = -\frac{\ell_s^2}{4} s - 1$$
;
$$\boxed{S = g_s^2 (2\pi)^{26} \delta(\Sigma p_i) \frac{\Gamma\left(-1-\frac{\ell_s^2 s}{4}\right)\Gamma\left(-1-\frac{\ell_s^2 t}{4}\right)\Gamma\left(-1-\frac{\ell_s^2 u}{4}\right)}{\Gamma\left(2+\frac{\ell_s^2 s}{4}\right)\Gamma\left(2+\frac{\ell_s^2 t}{4}\right)\Gamma\left(2+\frac{\ell_s^2 u}{4}\right)}}$$

Virasoro-Shapiro amplitude

Analyzing the result:

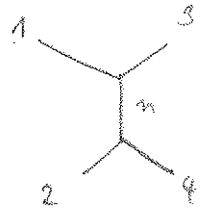
- fix t, vary s: since $\Gamma(x)$ has poles at $-x$ integer,
find poles at $l_s^2 = -4, 0, 4, 8, \ldots$ production of on shell states

$$\mathcal{A} = \sum_{n=0}^{\infty}$$  $$= \sum_{n=0}^{\infty} \frac{P_n(t)}{s - M_n^2}$$

Residue $P_n(t) \sim t^{2n}$

Consistently with level n states having max spin $J = 2n$

- fix s, vary t. [Regge limit: $s \to \infty$; t fixed]

$$\mathcal{A} = \sum$$  $$= \sum_{n=0}^{\infty} \frac{P_n(s)}{t - M_n^2};$$

exchange of on shell states

Unlike field theory, one does not need to sum over all channels!

- high energy, fixed angle scattering:

$s = E^2$

$t = (4m^2 - E^2) \sin^2 \theta/2$

$u = (4m^2 - E^2) \cos^2 \theta/2$

$E \to \infty$, θ fixed corresponds to sending $s, t, u \to \infty$ simultaneously.

Using Stirling's formula, $\Gamma(x+1) \sim x^x e^{-x} \sqrt{2\pi x}$

$$S \sim \exp\left[-\frac{l_s^2}{2}(s \log s + t \log t + u \log u)\right]$$

$$\sim \exp\left[-l_s^2 E^2 f(\theta)\right] \quad ; \quad \text{expon suppressed in UV}$$

the effective size of strings grows like $l_s^2 E$

On the other hand, at low energy:

$$S_4 \sim g_s^2 \cdot (2\pi)^{26} \delta\left(\sum p_i\right) \cdot \left(\frac{64}{stu} + \left(\frac{1}{s} + \frac{1}{t} + \frac{1}{u}\right) + \text{cte} + \dots \right)$$

The divergences are due to exchange of massless gravitons etc.

The cte part is a contribution to T^4.

Similarly, 4-graviton scattering on the sphere at low energy has poles from exchange of gravitons + irreducible contribution to $h_{\mu\nu}^4$, consistent with Einstein's equations!

From these tree level scattering amplitudes, one can reconstruct an effective action for the low lying modes of the string:

$$S = \int d^{26}x \sqrt{-g}\, e^{-2\phi}$$
$$\times \left\{ (\partial T)^2 - \frac{4}{\alpha'} T^2 + O(T^3) + R + 4(\partial\phi)^2 - \frac{1}{12} H^2 + \dots \right\}$$

An alternative way to find the effective action for the <u>massless</u> modes is to require that the non-linear σ model describing the string propagation in a curved background be conformally invariant.

$$S_P = \frac{1}{4\pi l_s^2} \int d^2\xi \sqrt{\gamma} \left(\gamma^{\alpha\beta} g_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu + \varepsilon^{\alpha\beta} B_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu + l_s^2 R(\gamma)\, \phi(X) \right)$$

For simplicity, consider $B_{\mu\nu}(X) = 0$, $\phi(X) = $ cte

Expanding $g_{\mu\nu}(X)$ around $X^\mu = x^\mu$ in Riemann normal coordinates,

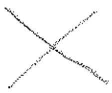
$$X^\mu = x^\mu + \ell_s Y^\mu$$

$$g_{\mu\nu}(X) = \eta_{\mu\nu} - \frac{\ell_s^2}{3} R_{\mu\lambda\nu\kappa}(x) Y^\lambda Y^\kappa + O(Y^3)$$

the Polyakov action becomes

$$S = \frac{1}{4\pi} \int d^2\sigma \left(\partial Y^\mu \partial Y_\mu - \frac{\ell_s^2}{3} R_{\mu\lambda\nu\kappa} Y^\lambda Y^\kappa \partial Y^\mu \partial Y^\nu + \dots \right)$$

In general, we have an expansion in powers of $\ell_s^2 R$, where $R \sim \frac{1}{L^2}$ is the characteristic curvature scale.

The quartic interaction  $R_{\mu\lambda\nu\kappa} (k^\mu \cdot k^\nu)$

generates a one-loop divergence  $\approx R_{\mu\lambda\nu\kappa} \langle Y^\lambda \cdot Y^\kappa \rangle \partial Y^\mu \partial Y^\nu$

In dimensional reg, $\langle Y^\ell(z)\, Y^\kappa(z') \rangle \sim \delta^{\lambda\kappa} \int \frac{d^{2+\varepsilon} k}{(2\pi)^{2+\varepsilon}} \frac{e^{ik(z-z')}}{k^2}$

$$\sim \frac{\delta^{\lambda\kappa}}{\varepsilon} \quad \text{as } z \to z'$$

So we need to add a counter term $\dfrac{R_{\mu\nu}(x)\, \partial Y^\mu \partial Y^\nu}{\varepsilon}$

Thus, the non-linear σ model is conformally invariant at 1-loop iff $R_{\mu\nu}(x) = 0$: Einstein's equations!

More generally, including $B_{\mu\nu}(x)$, $\phi(x)$ one finds

$$R_{\mu\nu} + 2 D_\mu D_\nu \phi - H_{\mu\lambda\kappa} H_\nu{}^{\lambda\kappa} = 0$$

$$\nabla^\lambda H_{\lambda\mu\nu} - \frac{1}{2} \nabla^\lambda \phi\, H_{\lambda\mu\nu} = 0$$

$$\nabla^2 \phi - \frac{1}{2} D_\mu \phi D^\mu \phi + \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} - \frac{D-26}{3\ell_s^2} = 0$$

which follow from the same action as before, as long as $D=26$ (up to overall normalization, which is ambiguous in this approach)

More generally, one finds an infinite series of corrections of order $(\ell_s^2 R)^n$.

The reason why the σ-model analysis agrees with scattering amplitude computations is complicated, but it basically follows from the fact that

$$\begin{cases} \delta S_p / \delta G_{\mu\nu} & \text{are the vertex operators for - graviton} \\ \delta S_p / \delta B_{\mu\nu} & \text{- Kalb Ramond} \\ \delta S_p / \delta \phi & \text{- Dilaton} \end{cases}$$

RK In the action for massless modes

$$S = \frac{1}{2\kappa_0^2} \int d^{26}x \sqrt{-g}\, e^{-2\phi} \left(R - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + 4 \partial\phi \partial\phi \right)$$

the Einstein Hilbert term is not canonically normalized.

Defining
$$\phi = \phi_0 + \tilde{\phi} \qquad \phi_0 = \text{cte} \quad ; \quad \frac{e^{-2\phi_0}}{2\kappa_0^2} = \frac{1}{2\kappa^2}$$
$$\tilde{g}_{\mu\nu} = e^{-4\tilde{\phi}/D-2}\, g_{\mu\nu}$$

and using $R[e^{2w} g] = e^{-2w} \left(R[g] - 2(D-1) \nabla^2 w - (D-1)(D-2)(\nabla w)^2 \right)$

one finds

$$S = \frac{1}{2\kappa^2} \int d^{26}x \sqrt{-\tilde{g}} \left(\tilde{R} - \frac{1}{12} e^{-\tilde{\phi}/3} H_{\mu\nu\lambda} H^{\mu\nu\lambda} - \frac{1}{6} \partial\tilde{\phi} \partial\tilde{\phi} \right)$$

$g_{\mu\nu}$ is called the 'string frame metric'

$\tilde{g}_{\mu\nu}$ is the 'Einstein frame metric', $G_N = \frac{\kappa^2}{8\pi}$ is the physical Newton constant

RK For $D \neq 26$, flat space with constant dilaton is not conformally invariant, but one can find solutions (to all orders in ℓ_s^2) where ϕ grows linearly in some direction:

$$\phi = \sqrt{\frac{26-D}{6\alpha'}} \cdot X$$

X = spacelike if $D < 26$

timelike if $D > 26$

One-loop amplitudes

In field theory, the vacuum-to-vacuum transition amplitude for a free scalar field of mass m in D dimensions is

$$Z = \left[\det(-\partial^2+m^2)\right]^{-\varepsilon/2} \qquad \text{with} \quad \begin{array}{l}\varepsilon = 1 \text{ for bosons} \\ \varepsilon=-1 \text{ for fermions}\end{array}$$

The connected vacuum amplitude is then

$$\log Z = -\frac{\varepsilon}{2} V_D \int \frac{d^D p}{(2\pi)^D} \log\left(p^2-m^2+i\varepsilon\right) \qquad \text{(Minkowskian)}$$

$$= -\frac{i\varepsilon}{2} V_D \int \frac{d^D p}{(2\pi)^D} \log\left(p^2+m^2\right) \qquad \text{(Euclidean)}$$

It is useful to use the Schwinger time representation:

$$\int_0^\infty \frac{dt}{t^{1+s}} \exp(-\pi t M^2) = (\pi M^2)^s\, \Gamma(-s)$$

$$= -\frac{1}{s} - \left(\log \pi M^2 + \gamma\right) + O(s)$$

hence, up to an infinite additive constant,

$$\log Z = \frac{i\varepsilon}{2} V_D \int_0^\infty \frac{dt}{t} \frac{d^D p}{(2\pi)^D} \exp\left[-\pi t (p^2+m^2)\right]$$

$$= \frac{i\varepsilon}{2} \frac{V_D}{(2\pi)^D} \int_0^\infty \frac{dt}{t^{1+D/2}} \exp\left[-\pi t m^2\right]$$

⚠ the UV divergence at $p\to\infty$ is seen as a divergence at $t=0$

For a set of free fields of mass m_i, statistics ε_i

$$\log Z_{QFT} = \frac{i}{2} \frac{V_D}{(2\pi)^D} \int_0^\infty \frac{dt}{t^{1+D/2}} \sum_i \varepsilon_i \exp\left[-\pi t m_i^2\right]$$

[i includes a sum over all physical degrees of freedom,
e.g. $D-2$ for a gauge field
$(D-2)(D-1)/2 - 1$ for a graviton, etc]

<u>Rk</u> This can be interpreted as an integral over all closed paths

$$\int_{X^\mu(0)=X^\mu(1)} DX^\mu(\tau) \exp\left[-\frac{1}{2}\int\left(\frac{(\partial X)^2}{e} + m^2 e\right)d\tau\right] \quad \text{with } e(\tau) \text{ gauged fixed to } t = \int_0^1 e(\tau)\,d\tau$$

In bosonic closed string theory, the sum over physical d.o.f is most easily carried out in light cone gauge:

$$m_i^2 = \frac{2}{\ell_s^2}\left(L_0^\perp + \bar{L}_0^\perp\right) \quad , \quad \varepsilon_i = 1$$

where
$$L_0^\perp = \sum_{m=1}^{\infty} \alpha_{-m}^i \alpha_m^i - \frac{D-2}{24} \qquad \left(= L_{0,\,\mathrm{l.c.g.}}^\perp\right)$$
$$\bar{L}_0^\perp = \sum_{m=1}^{\infty} \tilde{\alpha}_{-m}^i \tilde{\alpha}_m^i - \frac{D-2}{24}$$

subject to the level matching constraint $L_0^\perp - \bar{L}_0^\perp = 0$

Introducing a Lagrange multiplier θ for this constraint, the QFT result is

$$\log Z_{QFT} = \frac{i}{2}\frac{V_D}{(2\pi)^D}\int_0^\infty \frac{dt}{t^{1+D/2}}\int_{-1/2}^{1/2} d\theta$$
$$\times \operatorname{Tr}' \exp\left[-\frac{2\pi t}{\ell_s^2}\left(L_0^\perp + \bar{L}_0^\perp\right) + 2\pi i\theta\left(L_0^\perp - \bar{L}_0^\perp\right)\right]$$

let us define the 'complex Schwinger parameter'

$$\tau = \theta + \frac{it}{\ell_s^2} \equiv \tau_1 + i\tau_2 \quad , \quad q = e^{2\pi i\tau}$$

then

$$Z_{QFT} = \frac{i}{2}\frac{V_D}{(2\pi)^D}\iint_{\mathcal{F}_{QFT}} \frac{d\tau_1\, d\tau_2}{\tau_2^{1+D/2}} \operatorname{Tr}'\left[q^{L_0^\perp}\, \bar{q}^{\bar{L}_0^\perp}\right]$$

with $\mathcal{F}_{QFT} = \left\{\tau_2 > 0 \,,\, -\frac{1}{2} \le \tau_1 \le \frac{1}{2}\right\}$

Moreover
$$\operatorname{Tr}' q^{L_0} = q^{-\frac{D-2}{24}} \cdot \prod_{n=1}^{\infty}(1-q^n)^{-(D-2)}$$
$$= \eta^{-(D-2)}(\tau) \quad \text{where} \quad \eta(\tau) \equiv q^{\frac{1}{24}} \prod (1-q^n)$$

'Dedekind eta function'

This naive QFT result is UV divergent, due to the region $\tau_2 \to 0$.

The correct string theory result turns out to be given by the same integral, but with integration domain

$$\mathcal{F} = \mathcal{F}_{QFT} \cap \{ |\tau| > 1 \}$$

In particular, there are no UV divergences!

This can be justified as follows:

Recall the prescription

$$Z_h = \int_{\mathcal{M}_h} d^p t \int dX\, db\, dc \quad \exp[-S_\phi - S_{gh} - \chi\phi]$$ for $n \geq K$, zero otherwise

$$\left| \prod_{k=1}^{p} (b, \partial_{t_k} \hat{g}) \prod_{i=1}^{K} c(\hat{z}_i) \prod_{i=K+1}^{n} \int dz_i \right|^2 \prod_{i=1}^{n} \Phi_i(z_i)$$

Here $h = 1$, $\chi = 0$, $p = 1$, $K = 1$

The moduli space of conformal structures on the torus is

$$\mathcal{M}_1 = \{ \tau \in \mathbb{C},\ \mathrm{Im}\,\tau > 0 \} / SL(2, \mathbb{Z})$$

modular group, acting by

$$\tau \to \frac{a\tau + b}{c\tau + d} \qquad \begin{array}{l} ad - bc = 1 \\ a, b, c, d \in \mathbb{Z} \end{array}$$

For a given τ, the metric (up to Weyl rescalings) is

$$\widehat{ds^2} = |dx - \tau dy|^2 = dz\, d\bar{z} \qquad z = x - \tau y$$

invariant up to scale under $\tau \to \dfrac{a\tau + b}{c\tau + d}$ $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$$(-y \quad x) \begin{pmatrix} \tau \\ 1 \end{pmatrix} \to (-y \quad x) \underbrace{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} \tau \\ 1 \end{pmatrix}$$

$\mathcal{F}$ defined above is a fundamental domain for the action of $SL(2, \mathbb{Z})$ on τ.

To find the measure is more tricky: up to numerical factors,

$$(b, \partial_\tau g) = \int d^2z \; b_{zz} \, \partial_\tau g_{\bar z \bar z} = \frac{i}{\tau_2} \int d^2z \; b_{zz}$$

$\rightarrow$ same as inserting $b(\hat z)$ at any fixed point

Similarly, rather than fixed the position of Φ_1 at $z_1 = \hat z_1$, one can integrate over it, at the expense of dividing by the torus area:

$$Z_1 = \int_{\mathcal F} \frac{d\tau_1 d\tau_2}{\tau_2} \left\langle \left| b(\hat z) \, c(\hat z) \right|^2 \prod_{i=1}^{n} \int d^2z_i \; \phi_i(z_i) \right\rangle_{b,c,X}$$

For $n=0$, this is to be interpreted as the <u>dilaton tadpole at zero momentum</u> (since the vacuum amplitude vanishes) : $\int \sqrt{-g}\, e^{-2\phi} \Lambda$ in effective action

The path integral for the coordinates X^μ can be written in operator language as a trace

$$\begin{aligned}
Z_X(\tau) &= \mathrm{Tr} \left[\exp\left(-2\pi\tau_2 H + 2\pi i \tau_1 P \right) \right] \\
&= \mathrm{Tr} \left[\exp\left(-2\pi\tau_2 \left(L_0 + \bar L_0 - \tfrac{1}{24}(c+\bar c) \right) + 2\pi i \tau_1 \left(L_0 - \bar L_0 \right) \right) \right] \\
&= (q\bar q)^{-\frac{c}{24}} \; \mathrm{Tr} \; q^{L_0} \, \bar q^{\bar L_0} \\
&= (\eta \bar\eta)^{-D} \cdot \underbrace{\frac{i V_d}{(4\pi^2 \alpha' \tau_2)^{D/2}}}_{\text{from zero modes}}
\end{aligned}$$

For the b,c system:

$$Z_{b,c} \equiv \langle |b(\hat{z})\, c(\hat{z})|^2 \rangle$$
$$= \mathrm{Tr}\left((-1)^F\, b_0 c_0 \bar{b}_0 \bar{c}_0\, q^{L_0 - \frac{c}{24}}\, \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}\right)$$
$$= |\eta(\tau)|^4$$

where we recall that $c = -26$ and the ghost ground state $|\downarrow\rangle = c_1|0\rangle$ has $L_0 = -\frac{1}{2}$

In total,

$$Z_1 = \frac{iV_d}{(2\alpha')^D} \int_F \frac{d\tau_1 d\tau_2}{\tau_2^{1+\frac{D}{2}}} \frac{(\eta\bar{\eta})^2}{(\eta\bar{\eta})^D}$$

in agreement with the field theory answer, up to the replacement $F_{OPT} \longrightarrow F$

Importantly, the integrand is invariant under modular transformations iff $\boxed{D=26}$ - using the modular properties of the Dedekind eta function:

$$\eta(\tau+1) = e^{\frac{i\pi}{12}}\, \eta(\tau)$$
$$\eta(-1/\tau) = (-i\tau)^{1/2}\, \eta(\tau)$$

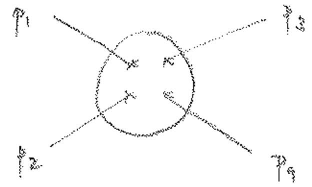
$\longrightarrow$ Modular invariance uniquely selects the critical dimension.

Rk Due to the tachyon, the amplitude is IR divergent;

At $\tau_2 \to i\infty$, $\quad \eta(\tau) \sim q^{\frac{1}{24}}(1 - q + \mathcal{O}(q^2))$

$$Z_1 \sim V_{26} \int_F \frac{d\tau_1 d\tau_2}{\tau_2^{14}} \frac{(1+24q+\dots)(1+24\bar{q}+\dots)}{q\bar{q}}$$
$$\sim V_{26} \int \frac{d\tau_2}{\tau_2^{14}} \left(\exp[4\pi\tau_2] + \dots\right) \quad \text{diverges as } \tau_2 \to \infty.$$

String theory - Exercises 5

* Consider the tree level scattering of 4 arbitrary closed string states 

: $A_4(s,t,u)$

By using general properties of the OPE, show that, for suitable normalizations, the residue

$$\operatorname*{Res}_{s=4N/\ell_s^2} A_4(s,t) = \sum_k A_3(1,2\to k)\, A_3^*(3,4\to k)$$

where k runs over all states with mass $M^2 = 4N/\ell_s^2$

* Consider the tree level scattering of n closed string tachyons, in the limit where all momenta p_i are scaled to infinity. Show that the integral over locations z_i of vertex operators is dominated by configurations which extremize the electrostatic-like potential $V = \sum_{i<j} p_i \cdot p_j \ln|z_i - z_j|^2$
For $n=4$ recover the high energy behavior of Virasoro-Shapiro amplitude. Explain why the same high energy suppression holds for any external states.

* Give an algorithm for how to map any τ in the Poincaré upper half plane into the standard fundamental domain of $PSL(2,\mathbb{Z})$ $\mathcal{F} = \{\tau / -\frac{1}{2} \le \mathrm{Re}\,\tau < \frac{1}{2},\ |\tau| > 1\}$. Compute the area $A = \int_{\mathcal{F}} \frac{d\tau_1 d\tau_2}{\tau_2^2}$