Salle G7
10h30 - 12h
13h30 - 15h30

Non perturbative effects and semi-classical methods in QFT

So far, our study of QFT has focused on small fluctuations around trivial Poincaré invariant vacua, corresponding to

- isolated minima of potential: ϕ^4, around $\phi_0 = 0$
- non isolated minima, typically related by action of some symmetry group G

. G global symmetry, $\mathrm{Stab}(\phi_0) = H \rightarrow$ massless Goldstone bosons, described by some non linear σ-model with target space G/H

. G local symmetry $\longrightarrow$ Higgs mechanism, massive gauge boson + massive scalar field in unitary gauge

In general, transition amplitudes have a perturbative expansion

$$\langle 0 | T \phi(x_1) \dots \phi(x_p) | 0 \rangle$$
$$= \mathcal{N} \int \phi(x_1) \dots \phi(x_p) \, e^{\frac{iS_2(\phi)}{\hbar} + \frac{iS_{int}(\phi)}{\hbar}}$$
$$= \mathcal{N} \sum_N \frac{1}{N!} \left(\frac{i S_{int}}{\hbar} \right)^N \phi(x_1) \dots \phi(x_p) \, e^{\frac{iS_2(\phi)}{\hbar}}$$

using Wick's theorem $\langle \phi(x) \phi(x') \rangle = \hbar \, Q^{-1}(x, x')$

if $S_2 = \int dx \, dx' \, \phi(x) Q(x,x') \phi(x')$

A diagram with E external lines, V vertices of degree n, I internal lines contributes $\left(\frac{\lambda}{\hbar} \right)^V \hbar^{E+I}$

Using $nV = E + 2I$;
$L = I - V + 1 = \#$ loops

$$\lambda^{\frac{2L+E-2}{n-2}} \hbar^{E+L-1}$$

$$\hookrightarrow = \lambda^{\frac{E-2}{n-2}} \hbar^E \left(A_0 + A_1 \left(\hbar \lambda^{\frac{2}{n-2}} \right) + A_2 \left(\quad \right)^2 + \dots \right)$$

Up to an overall tree-level factor, the answer is a Taylor series in $\hbar\lambda^{\frac{2}{n-2}}$.
If this series had finite radius of convergence, the result would be an analytic function of $\hbar$ near $\hbar = 0$.

However, due to the exponential growth in # of Feynman diagrams,

$$A_L \sim L!\,\alpha^L \qquad \alpha \text{ some constant}$$

so the radius $R = \dfrac{1}{\limsup_{L\to\infty} |A_L|^{1/L}} = 0$

(Stirling formula: $L! \sim e^{L\log L - L}$)

For fixed $\hbar$, the size of $(\alpha\hbar)^L L!$ grows as $L \to \infty$.
Assuming that the error is on the order of the last term in the sum, the optimal truncation is at $\dfrac{d}{dL}\left[L\log\alpha\hbar + \log L!\right] = 0$

$$\log(\alpha\hbar) + \log L = 0$$

So $L \sim \dfrac{1}{\alpha\hbar}$, yielding an error of order $e^{-\frac{1}{\alpha\hbar}}$

This indicates that the exact amplitude has an essential singularity at $\hbar \to 0$, which cannot be seen at any fixed order in perturb. theory

[To resum the series one may use Borel resummation:

$$A(\hbar) = \sum_{L\geq 0} A_L \hbar^L \longrightarrow B(t) = \sum_{L\geq 0} B_L t^L \qquad B_L = \frac{A_L}{L!}$$

$$L!\,\hbar^L = \frac{1}{\hbar}\int_0^\infty dt\, t^L e^{-t/\hbar}$$

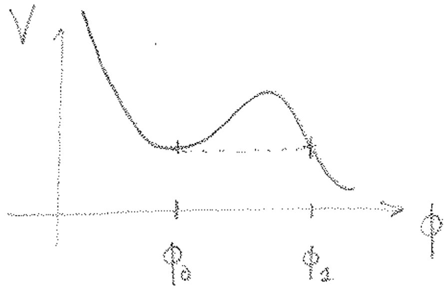
$$A(\hbar) = \frac{1}{\hbar}\int_0^\infty dt\, B(t)\, e^{-t/\hbar}$$

$A_L \sim L!\,\alpha^L \Rightarrow B(t)$ has finite radius of convergence, pole at $t = \frac{1}{\alpha}$

$\rightarrow$ ambiguity of order $e^{-1/\alpha t}$]

- Such non-perturbative effects are well-known to occur already in quantum mechanics (QFT in 1 dimension) e.g. for tunnelling amplitudes

$$H = \frac{\dot{\phi}^2}{2} + V(\phi)$$



Near a local minimum:

$$V(\phi) \sim V_0 + \frac{1}{2}\omega^2\phi^2$$

$$E_0 = V_0 + \frac{1}{2}\hbar\omega + \sum_{L\geq 2} e_L \hbar^L$$

To all orders, $\mathrm{Im}\, E_0 = 0$

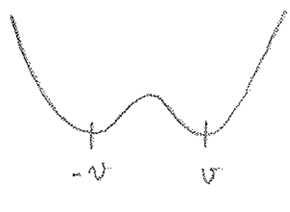
But in fact $\Gamma = 2\,\mathrm{Im}\, E_0 \sim \exp\left[-\frac{2}{\hbar}\int_{\phi_0}^{\phi_1}\sqrt{2V}\,d\phi\right]$

This can be established using standard WKB methods in Q. mechanics.

From the point of view of the path integral formulation, such contributions can be traced to non-trivial stationary points of the Euclidean action.

- To avoid subtleties associated to unstable states in QM, consider the double well potential

$$V(\phi) = \frac{\lambda(\phi^2 - v^2)^2}{8}$$



$$= \frac{1}{2}\lambda v^2(\phi \mp v)^2 \mp \frac{1}{2} v (\phi \mp v)^3 + \dots$$

To all orders, the ground state is doubly degenerate, $E_0 \sim \frac{1}{2}\hbar v\sqrt{\lambda} + O(\hbar^2)$

At non perturbative level however, the degeneracy is lifted,

$$E_0^+ - E_0^- \sim \exp\left[-\frac{S_0}{\hbar}\right] \qquad S_0 = \int_{-v}^{v}\sqrt{2V}\,d\phi$$

(ground state: symmetric wave function) Classical action of a solution approaching $\phi = \pm v$ at $t = \pm\infty$

To see this, let us evaluate the overlap

$\langle v | e^{-HT/\hbar} | -v \rangle$ between wave functions localized at $\phi = \pm v$,

in the limit $T \to \infty$

On the one hand, $\langle v | e^{-HT/\hbar} | -v \rangle = \sum_n \langle v | n \rangle \langle n | -v \rangle \, e^{-E_n T/\hbar}$

$$\sim e^{-E_0 T/\hbar} \langle v | 0 \rangle \langle 0 | -v \rangle$$

On the other hand,

$$\langle v | e^{-HT} | -v \rangle = \mathcal{N} \int_{\substack{\phi(-T/2) = -v \\ \phi(T/2) = v}} [d\phi(t)] \, e^{-S_E/\hbar}$$

$$S_E = \int_{-T/2}^{T/2} dt \left[\tfrac{1}{2} \dot{\phi}^2 + V \right] \qquad \text{Euclidean action}$$

In the limit $T \to \infty$:

$$S = \int_{-\infty}^{+\infty} dt \left[\tfrac{1}{2} \left(\dot{\phi} - \sqrt{2V} \right)^2 + \sqrt{2V} \, \dot{\phi} \right]$$

$$= \int_{-\infty}^{+\infty} dt \; \tfrac{1}{2} \left(\dot{\phi} - \sqrt{2V} \right)^2 + \int_{-v}^{v} \sqrt{2V(\phi)} \, d\phi$$

The extremum of the action satisfies $\dot{\phi} = \sqrt{2V}$, $S_0 = \int_{-v}^{v} \sqrt{2V(\phi)} \, d\phi$

1-param family: $\phi = v \tanh\left[\tfrac{1}{2} v \sqrt{\lambda} (t - t_0) \right]$ 'instanton'

$$S_0 = \frac{2}{3} \lambda^{1/2} v^3$$

Quadratic fluctuations around this solution lead to the 'one-loop determinant'

$$\sqrt{\frac{S_0}{2\pi\hbar}} \, T \cdot \mathcal{N} \left(\det{}' \left[-\partial_t^2 + V''(\phi) \right] \right)^{-1/2} e^{-S_0/\hbar} \qquad (\det{}': \text{remove 0 eigenvalue})$$

(recall that for a quadratic potential, $\mathcal{N} \left(\det(-\partial_t^2 + \omega^2) \right)^{-1/2} \xrightarrow[T \to \infty]{} \left(\frac{\omega}{\pi\hbar} \right)^{1/2} e^{-\omega T/2}$

consistently with $E_0 = \tfrac{1}{2} \hbar \omega$

$$|\langle x = 0 | n = 0 \rangle|^2 = \sqrt{\frac{\omega}{\pi\hbar}}$$

The factor $\sqrt{\frac{S_0}{2\pi\hbar}}$ originates from trading 0-eigenvalue against time translation

Thus $\langle v | e^{-HT/\hbar} | v \rangle = T \cdot \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\omega T/2} \, K \, e^{-S_0/\hbar} + \ldots$

where $K = \sqrt{\frac{S_0}{2\pi\hbar}} \left(\frac{\det -\partial_t^2 + \omega^2}{\det' (-\partial_t^2 + V'')} \right)^{\frac{1}{2}}$

More generally, there are multi-instanton configurations which jump between $\varphi = -v$ and $\varphi = v$ at times $t_1 \leq t_2 \leq \ldots \leq t_n$

Their contribution is

$$\sum_{n \text{ odd}} \int_{-T/2,\ t_1 \leq t_2 \leq \ldots \leq t_n}^{T/2} dt_1 \, dt_2 \ldots dt_n \cdot \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\omega T/2} \cdot K^n \, e^{-nS_0/\hbar}$$

$$= \frac{T^n}{n!} K^n \left(\frac{\omega}{\pi\hbar}\right) e^{-\omega T/2} \, e^{-nS_0/\hbar}$$

[Rk: Dominant contrib to $\sum_n \frac{x^n}{n!}$ are from $n \lesssim x$
here $x = TK e^{-S_0/\hbar}$ so $\frac{n}{T} \lesssim K e^{-S_0/\hbar} \ll 1$]

Summing them up:

$$\langle v | e^{-HT/\hbar} | -v \rangle = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\omega T/2} \sinh\left(K \cdot T \cdot e^{-S_0/\hbar}\right)$$

so the ground state degeneracy is lifted to

$$E_\pm^0 = \frac{1}{2}\hbar\omega \pm \hbar K T e^{-S_0/\hbar}$$

(only the difference can be trusted here)

<u>Exercise</u> Consider a periodic potential $V = \frac{\alpha}{\beta^2}(1 - \cos\beta\varphi)$

Identify the instanton interpolating between 2 subsequent classical vacua, compute their action S_0

Show that the lowest energy states consist of a continuum of states labelled by $\theta \in [0, 2\pi]$, with energy

$$E(\theta) = \frac{1}{2}\hbar\sqrt{\alpha} + 2\pi\hbar K \cos\theta \, e^{-S_0/\hbar}$$

$\boxed{D=2}$

- Up to now, we considered euclidean solutions describing tunnelling process. In higher dimension, it is possible to find similar solutions to the lorentzian eq. of motion, describing some lumps of energy which propagate <u>without dissipation</u>, much as ordinary quantum particles.

<u>Ex</u> consider the same double well potential, now in $D=1+1$,

$$H = \frac{\dot\phi^2}{2} + \frac{\phi'^2}{2} + V(\phi) \qquad , \qquad \mathcal{L} = -\frac{\dot\phi^2}{2} + \frac{\phi'^2}{2} + V(\phi)$$

$$V(\phi) = \frac{\lambda}{8}(\phi^2 - v^2)^2$$

For static solutions, we are back to same pb as before, with x playing the rôle of Euclidean time

The same Bogomolny trick shows that in the sector with $\begin{cases}\phi(x\to-\infty) = -v \\ \phi(x\to+\infty) = +v\end{cases}$, the lowest energy classical solution

is $\phi' = \sqrt{2\phi}$, with energy $E = \frac{2}{3}\lambda^{1/2}v^3 = \frac{2}{3}\left(\frac{m^2}{\lambda}\right)\cdot m$

This is much bigger than the energy of perturbative quanta

$$m = \lambda^{1/2} v$$

at weak coupling $\lambda \ll 1$

Due to Lorentz invariance, these solitons exist for any velocity and satisfy relativistic kinematics.

These solitons are classically stable, since they carry a conserved quantum number $Q = \int_{-\infty}^{+\infty} \partial_x\phi = \phi(+\infty) - \phi(-\infty)$, from the conserved current $j^\mu = \varepsilon_{\mu\nu}\partial^\nu\phi$.

Two solitons of opposite charge can annihilate.

- Unlike in $D=1$, there exist no finite action saddle point of the classical action (Derrick's theorem).

Indeed, if there was a saddle point $\phi(x)$, with action

$$S = \int d^Dx\, V + \int d^Dx\, (D\phi)^2 \quad \left(+ \int d^Dx\, (D\phi)^4 + \dots \right)$$
$$= S_0 + S_1 \quad (+ S_2 + \dots)$$

the rescaled configuration $\phi_\lambda(x) = \phi(\lambda x)$ would have action

$$S(\lambda) = \lambda^{D} S_0 + \lambda^{2-D} S_1 \quad (+ \lambda^{4-D} S_2 + \dots)$$

At $\lambda = 1$: $\quad D S_0 + (D-2) S_1 + \quad (+ (D-4) S_2 + \dots) = 0$

If S_0, S_1 are positive definite, this is impossible for $D \geq 2$ (though higher derivative may allow it)

- In the presence of gauge fields however, instanton solutions may exist.

Eg: $V = \frac{\lambda}{8} \left[|\phi|^2 - v^2 \right]^2$ $\qquad$ Abelian Higgs model in 2D

$$S = |D_\mu \phi|^2 - \frac{1}{4} F_{\mu\nu}^2 - V$$

with $A_\mu = 0$, solutions with $\varphi \xrightarrow{r \to \infty} v\, f(r)\, e^{in\theta}$, $f(\infty) = 1$

have log divergent action

$$S = \int r\, dr \,.\, v^2 \left[f'(r)^2 + \frac{n^2}{r^2} f(r)^2 \right]$$
$$\sim 2\pi n^2 v^2 \int^{\infty} \frac{dr}{r}$$

(vortex solutions in 2+1 dimensions)

Allowing for $A = \frac{n\, a(r)}{e}\, d\theta$ $\quad a(\infty) = 1$, $dA = \frac{n\, a'(r)}{e}\, dr \wedge d\theta$

one can arrange for the action to be finite :

$$S = 2\pi v^2 \int_0^\infty dr\, r \left[f'^2 + \underbrace{\frac{n^2}{r^2} (a-1)^2 f^2}_{\to 0} + V + \frac{n^2}{e^2 v^2 r^2} a'^2 \right]$$

(Abrikosov Nielsen Olsen vortices in 2+1 dimensions)

Such a configuration carries a quantized flux

$$\Phi = \oint_{S_1(\infty)} A = \frac{2\pi n}{e} = \iint F$$

A is pure gauge at ∞ : $\quad A \to U^{-1} dU$

$\qquad U = e^{in\theta} \in U(1)$

n is the winding number of the map $S_1(\infty) \xrightarrow{V} U(1)$

In general, one should sum over all winding sectors with some weight $A(n)$:

$$\langle \mathcal{O} \rangle = \frac{\sum_n A(n) \int_{\nu(\phi)=n} [d\phi]\, \mathcal{O}(\varphi)\, e^{-S_E[\varphi]/\hbar}}{\sum_n A(n) \int_{\nu(\phi)=n} [d\phi]\, e^{-S_E[\varphi]/\hbar}}$$

Cluster decomposition requires that $A(n_1 + n_2) = A(n_1) A(n_2)$

ie $\qquad A(n) = e^{i\theta n} \qquad \theta \in [0, 2\pi[$

This amounts to adding a term $\quad \frac{\theta . e}{2\pi} \int F \quad$ in the action.

This is a total derivative, so does not affect equations of motion.

Due to instanton corrections, the vacuum energy depends on θ:

$$\int_{\text{Box } T\times L} [d\varphi]\; e^{i\theta\nu(\varphi) - \frac{S(\varphi)}{\hbar}}$$

$$\sim \sum_{n,\bar n} \frac{e^{i\theta(n-\bar n)}\cdot\left(K L T e^{-S_0}\right)^{n+\bar n}}{n!\,\bar n!}$$

$$\sim \exp\left[\, KLT\left(e^{-S_0+i\theta} + e^{-S_0-i\theta}\right)\right]$$

So $\quad \dfrac{E_0(\theta)}{L} \sim -2K\cos\theta\; e^{-S_0}$

Similarly

$$\langle F\rangle \sim \frac{\int [d\varphi]\;\frac{2\pi}{eLT}\,\nu(\varphi)\; e^{i\theta\nu(\varphi)-S(\varphi)/\hbar}}{\int [d\varphi]\; e^{i\theta\nu(\varphi)-S(\varphi)/\hbar}}$$

$$\sim \frac{2\pi}{eLTi}\,\frac{d}{d\theta}\log\left(\int [d\varphi]\; e^{i\theta\nu(\varphi)-S(\varphi)}\right)$$

$$\sim \frac{2\pi}{eLTi}\; 2KLT\sin\theta\, e^{-S_0} \sim \frac{4\pi}{ie}\; K e^{-S_0}\sin\theta$$

Thus, θ-vacua are physically distinct.

In fact, we may show that instantons in the 2D Abelian Higgs model drastically change the physics: the Higgs phenomenon no longer takes place, and electric charges are confined. (just as they would be in the unbroken phase $v^2<0$, due to linear Coulomb potential in D=2)

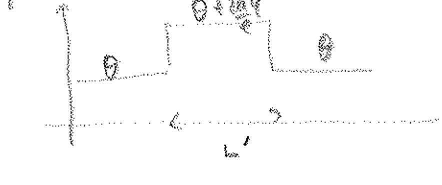
<u>Ex</u> compute the rectangular Wilson line in the dilute gas approximation:

$$W = \exp\left(-\frac{q}{e}\oint A\,dx\right) \sim \exp\left(2\pi i\frac{q}{e}\, n^{\text{inside}}\right)$$

$$\langle W\rangle \sim \exp\left[\, 2K e^{-S_0}\left\{(LT - L'T')\cos\theta + L'T'\cos\left(\theta + \frac{2\pi q}{e}\right) - LT\cos\theta\right\}\right]$$

$$E(L') = -\lim_{T'\to\infty} \ln\langle W\rangle \sim 2L'Ke^{-S_0}\left(\cos\theta - \cos\left(\theta+\frac{2\pi q}{e}\right)\right)$$

linear potential $\Rightarrow$ confinement



$\boxed{D=3}$

. As before, instanton configurations in D Euclidean configurations lift to solitonic configurations in D+1

instantons of Abelian Higgs model $\rightsquigarrow$ Nielsen-Olsen strings.
More generally, <u>vortices</u> in D=2+1 gauge theory with classical space of vacua G/H are classified by the homotopy group $\pi_1(G/H)$

Similarly, instanton configurations in D=3 Euclidean dimensions are classified by maps $S_2 \to G/H$, ie $\pi_2(G/H)$

[A thm in topology states that this is the same as $\pi_1(H)$ if G is simply connected]

<u>e.g:</u> Georgi-Glashow model $\qquad S = -\frac{1}{2}(D^\mu\phi)^2 - V(\varphi) - \frac{1}{4}F_{\mu\nu}^2$
SU(2) gauge theory in 3D (or 3+1 D)
with a triplet of scalar fields

$$V = \frac{1}{8}\lambda\left(\phi^a\phi^a - v^2\right)^2 \qquad m_W = e.v$$

SU(2) is spontaneously broken to U(1) by $\langle\varphi^a\rangle = v.\underbrace{\hat{\phi}^a}_{\text{unit vector}}$

Maps from $S_2 \to \frac{SU(2)}{U(1)} = S_2$ are classified by

$$n = \frac{1}{8\pi}\int d^2\theta\,.\ \varepsilon^{abc}\,\varepsilon^{ij}\,\hat{\varphi}^a\,\partial_i\hat{\varphi}^b\,\partial_j\hat{\varphi}^c$$

$$= \frac{1}{8\pi}\int \vec{\hat{\varphi}}\cdot d\vec{\hat{\varphi}}\wedge d\vec{\hat{\varphi}} \in \mathbb{Z}$$

which is invariant under deformations

According to Derrick's theorem, there are no saddle points unless one also excites the gauge field.

Let's assume $\varphi^a = v\, x^a\, f(r)/r$ $f(\infty)=1,\ f(0)=0$

$A^a = a(r)\, \varepsilon^{aij} x_j / e r^2$ $a(\infty)=1,\ a(0)=0$

$D_i \varphi^a = O(1/r^2)$ at $r\to\infty$

so $\int (D_i\varphi^a)^2 r^2 dr$ is convergent

$$S_E = \int \tfrac{1}{2}(D_i\varphi^a)^2 - \tfrac{1}{4}F_{ij}^2 + V$$

$$= \int 4\pi r^2 dr \left\{ \frac{v^2}{2r^2}\left[2(1-a)^2 f^2 + r^2 f'^2\right] + \frac{1}{2e^2 r^4}\left[2r^2 a'^2 + (2a-a^2)^2\right] + \frac{1}{8}\lambda v^4 (f^2-1)^2 \right\}$$

Using the Bogomolnyi trick, one has the lower bound ($B_i = \varepsilon_{ijk}\partial_j A_k$)

$$S_E = \int \tfrac{1}{2}(B_i \pm D_i\varphi)^2 + V(\varphi) \mp B_i D_i \varphi$$

$$= \frac{4\pi |n| v}{e} + \int \tfrac{1}{2}(B_i + \mathrm{sign}(n)\, D_i\varphi)^2 + V(\varphi)$$

In the formal limit, $\lambda\to 0$, one arrives at the BPS monopole

$a' = (1-a) f$ $\rho = e v r$

$f' = (2a-a^2)/\rho^2$ $' = d/d\rho$

$$a(\rho) = 1 - \frac{\rho}{\sinh\rho}$$

$$f(\rho) = \coth\rho - \frac{1}{\rho}$$

$\boxed{D=4}$

recall $F = dA + A\wedge A$
$dF + A\wedge F - F\wedge A = 0$

- Consider pure Yang Mills theory in D (Euclidean) dimensions, group G

Stationary points of the Euclidean action only occur for $D=4$:

$$A_\lambda(x) = \lambda A(\lambda x)$$

$$S(\lambda) = \lambda^{4-D} S_0 \quad \longrightarrow \text{stationary only if } D=4$$

For the action be finite, A_μ must be pure gauge at $r\to\infty$:

$$A_\mu(x) \to \frac{1}{i}\, g^{-1}(\hat{x})\, \partial_\mu g(\hat{x})$$

with $\quad g: S^3 \to G$

Such maps are classified by $\pi_3(G) = \mathbb{Z}$ for any compact connected Lie group

This 'winding number' can be computed as the integral of a local density:

$$n = \frac{1}{8\pi^2}\int \mathrm{Tr}(F\wedge F) = \frac{1}{16\pi^2}\,\mathrm{Tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu}$$

It may also be measured at ∞ using the Chern-Simons density

$$\mathrm{Tr}(F\wedge F) = d\,\omega_{CS}$$

$$\omega_{CS} = \mathrm{Tr}\left[\, A\,dA + \tfrac{2}{3} A\wedge A\wedge A \,\right]$$

$$= \mathrm{Tr}\left[\, A\,\underbrace{(dA + A\wedge A)}_{F} - \tfrac{1}{3} A\wedge A\wedge A \,\right]$$

At large radius, F vanishes, so we're left with

$$n = \frac{1}{24\pi^2}\int_{S_3(\infty)} \mathrm{Tr}\left(g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg\right)$$

As usual, sectors with different values of n must be weighted by

$$e^{in\theta} = e^{\frac{i\theta}{8\pi^2}\int \mathrm{Tr}\,F\wedge F}$$

[constraint on electric neutron dipole $\Rightarrow |\theta| < 10^{-9}$. 'strong CP problem']

which does not affect equations of motion but in general breaks CP.

Why is this a magnetic monopole?

For $\hat{\varphi}^a = \delta^{a,3}$, the electromagnetic field of the unbroken $U(1)$ is $f = \partial_i A_j^3 - \partial_j A_i^3 = F_{ij}^3$

For arbitrary $\hat{\varphi}$, a gauge invariant definition of $\mathcal{F}$ is

$$\mathcal{F}_{\mu\nu} = \hat{\varphi}^a F_{\mu\nu}^a - \frac{\varepsilon^{abc}}{e} \hat{\varphi}^a D_\mu \hat{\varphi}^b D_\nu \hat{\varphi}^c \qquad \text{with } \hat{\varphi} = \frac{\varphi}{|\varphi|}$$

$$= \partial_\mu (\hat{\varphi}^a A_\nu^a) - \partial_\nu (\hat{\varphi}^a A_\mu^a) - \frac{\varepsilon^{abc}}{2e} \varepsilon^{abc} \hat{\varphi}^a \partial_\mu \hat{\varphi}^b \partial_\nu \hat{\varphi}^c$$

so it satisfies Bianchi identity $d\mathcal{F} = 0$

The flux carried by the 'monopole' is then

$$\int_{S_2(\infty)} \mathcal{F} = \frac{-4\pi n}{e} \qquad n = \frac{1}{8\pi} \int d^2\theta \, \varepsilon^{abc} \hat{\varphi}^a \partial \hat{\varphi}^b \partial \hat{\varphi}^c$$

One could choose a singular gauge where $\hat{\varphi}^a = \delta^{a,3}$ everywhere.

In this case one recovers the Dirac monopole in $U(1)$ gauge theory:

(at large distance only; the action is finite, unlike for Dirac mon.)

$$A_N = g(1-\cos\theta)\, d\varphi$$

$$A_S = g(1+\cos\theta)\, d\varphi \qquad dA = g \sin\theta \, d\theta \, d\varphi$$

$$A_N - A_S = 2g\, d\varphi$$

[in Cartesian coordinates: $\vec{A} = g \frac{\vec{u} \wedge \vec{r}}{r(r + \vec{r}.\vec{u})}$ $|\vec{u}| = 1$; direction of the Dirac string]

$$\int A_N - A_S = 4\pi g$$

$$4\pi g . \frac{e}{\hbar c} \in \mathbb{Z} \qquad \frac{2eg}{\hbar c} \in \mathbb{Z} \quad \text{: Dirac quantization condition}$$

Going back to 3D Euclidean theories, Polyakov has shown that multi-instanton effects are responsible for confinement in the Georgi-Glashow model, or in Abelian gauge theory with compact $U(1)$.

To construct such non-trivial saddle points, we use again the Bogomolny's trick

$$S_E = \int \frac{1}{8g^2} \mathrm{Tr}\left(F_{\mu\nu} \mp \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}\right)^2 \pm \frac{1}{8g^2} \varepsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

Saddle points occur for self dual field configurations

$$F_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}$$

and have an action $S_0 = \frac{8\pi^2}{g^2} |n|$ where $n \in \mathbb{Z}$

For $n = 1$, $G = SU(2)$, one can find an explicit solution

$$A_\mu = \frac{i}{g} \frac{r^2}{r^2 + a^2} U^{-1} \partial_\mu U \qquad r = |\vec{x}|^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2$$

$$U(\hat{x}) = \frac{x_0 + i \vec{x} \cdot \vec{\sigma}}{r}$$

This solution has 8 collective coordinates:

$\left.\begin{array}{l} \text{– center in } \mathbb{R}^4 \\ \text{– size } a \end{array}\right\} AdS_5$

– orientation in $SU(2)$

Multi-instantons contribute to the vacuum energy density schematically as

by dimensional analysis, $E/V \sim 1/L^4$ (volume element in AdS_5)

$$\frac{E(\theta)}{V} = -\cos\theta \, e^{-\frac{8\pi^2}{g^2}} g^{-8} \int_0^\infty \frac{d\rho}{\rho^5} f(\rho\mu)$$

↑ mass scale

Here μ and g both run, but $\frac{1}{g^2} - \frac{\beta_1}{8\pi^2} \ln\mu = -\frac{\beta_1}{8\pi^2} \ln \Lambda_{QCD}$ is RG-invariant.

where $\beta_1 = 11 (\mathrm{Tr}\, R_A - \mathrm{Tr}\, R_\psi)$ is the 1-loop beta function

Now $\left(i\not{D}_x - \not{k}\right)^2 = -k^2 + 2i\, k\cdot D - \not{D}^2$

$$= -k^2 + 2i\, k\cdot D_x + D^2 + g\, S^{\mu\nu} F_{\mu\nu}(x)$$

Then, rescaling $k \to Mk$

$$\mathcal{A}(x) = -2M^4 \int \frac{d^4k}{(2\pi)^4} \operatorname{Tr}\left\{ \gamma_5 t \exp\left(-k^2 + 2i\frac{k\cdot D_x}{M} + \frac{D^2}{M^2} + g\frac{S^{\mu\nu}F_{\mu\nu}}{M^2}\right)\right\}$$

As $M \to 0$, the only term contributing is $\frac{1}{2} g^2 \left(S^{\mu\nu}F_{\mu\nu}\right)^2 / M^4$ in the exponent

$$\operatorname{Tr} S^{\mu\nu} S^{\rho\sigma} \gamma_5 = \operatorname{Tr} \frac{i}{2}\gamma^\mu\gamma^\nu \frac{i}{2}\gamma^\rho\gamma^\sigma \gamma_5$$

$$= i\, \varepsilon^{\mu\nu\rho\sigma}$$

hence

$$\mathcal{A}(x) = -\frac{2}{2} \cdot g^2\, i \operatorname{Tr} F_{\mu\nu} F_{\rho\sigma}\, \varepsilon^{\mu\nu\rho\sigma} \int \frac{d^4k}{(2\pi)^4} \underbrace{e^{-k^2}}_{(\sqrt{\pi})^2}$$

$$= -\frac{1}{16\pi^2} \varepsilon_{\mu\nu\rho\sigma} F_\alpha^{\mu\nu} F_\beta^{\rho\sigma} \underbrace{\operatorname{Tr}\left[t_\alpha t_\beta t\right]}_{\frac{1}{2}\delta_{\alpha\beta} \operatorname{Tr} t}$$

for a global symmetry.

$\rightarrow$ the $U(1)_R$ symmetry is broken by the UV regulator.

Nevertheless, it only depends on the low energy spectrum, and must be realized in the low energy field theory.

Eg QCD with 1 massless flavor:

$\delta u = i\alpha \gamma_5 u$ $\quad\quad$ $\pi_0 \sim u\bar{u}$ transforms as $\delta \pi^0 = \alpha \underbrace{F_\pi}_{184\,\text{MeV}}$

$\delta d = -i\alpha \gamma_5 d$

hence there must exist a term $\dfrac{\pi^0 e^2 N_c}{48\pi^2 F_\pi} \varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu}(x) F^{\rho\sigma}(x)$ π^0

contributing (and explaining, for $N_c = 3$) to $\pi^0 \longrightarrow 2\gamma$

Thus we must have $e^{-\frac{8\pi^2}{g^2}} f(\rho\mu) = e^{-\frac{8\pi^2}{g^2} + \beta_1 \log(\rho\mu)}$

$$= (\rho \Lambda_{QCD})^{\beta_1}$$

The integral over ρ is UV finite but IR divergent.

Eventually, on dimensional ground one must have

$$\frac{E(\theta)}{V} \sim f(\theta) \Lambda_{QCD}^4$$

but it is hard to get a quantitative result...

One qualitative consequence however is a resolution of the 'U(1) problem'!

Recall that in a theory with massless quarks, there is a global symmetry

$$U(N_f)_L \times U(N_f)_R \qquad N_f = 2 \text{ for } \{u, d\} \text{ quarks}$$

chiral symmetry, which rotates independently the left/right Weyl spinors.

The diagonal $U(1)_V$ is the conservation of (Baryon - lepton) number.

$SU(2)_V$ isospin : $\begin{bmatrix} p \\ n \end{bmatrix}$: 938 MeV, 940 MeV

$\begin{bmatrix} \pi^0 \\ \pi^\pm \end{bmatrix}$: 135 MeV, 140 MeV

The off-diagonal $SU(2)_A$ is spontaneously broken, with $\pi^0, \pi_\pm$ identified as Goldstone bosons.

This is a non-perturbative process involving condensation of fermion bilinears

(just like confinement, χSB has never been demonstrated rigorously)

The off-diagonal $U(1)_A$ on the other hand is anomalous, ie

$$\partial_\mu j^A_\mu = \frac{2 g^2}{16\pi^2} \mathrm{Tr}(F^a_{\mu\nu} \tilde{F}^a_{\mu\nu}) = 2N_f n \qquad \text{if } n \text{ is the instanton number}$$

This results in non conservation of the $U(1)_A$ charge, e.g in decays

$$p + n \longrightarrow e^+ + \bar{\nu}_\mu \qquad \text{violating baryon and lepton number}$$

(though not their difference)

Chiral anomalies from Fujikawa's method

[Weinberg p 362]

Consider the fermion integration measure in fixed electromagnetic background A

Under $\psi(x) \to U(x)\,\psi(x)$

with $U(x) = \exp\left[\, i\gamma_5\, \alpha(x)\, t \,\right]$

$\uparrow$ hermitian matrix in flavor space

$$[D\psi][D\bar\psi] \to \frac{1}{\mathrm{Det}\,U \cdot \mathrm{Det}\,\bar U}\,[D\psi][D\bar\psi]$$

where $U = \bar U = \delta^4(x-y)\, e^{i\gamma_5 \alpha t}$

Using $\ln \mathrm{Det} = \mathrm{Tr}\,\mathrm{Log}$, for $\alpha(x)$ infinitesimal,

$$\frac{1}{(\mathrm{Det}\,U)^2} = \exp\left[-2i \int d^4x\, \alpha(x)\, \mathrm{Tr}\{\gamma_5 t\}\, \delta^4(x-x)\right]$$

$$= \exp\left[\, i \int d^4x\, \alpha(x)\, \mathcal{A}(x)\,\right]$$

$\mathcal{A}(x)$ is identified as $\langle \partial_\mu J^\mu_5 \rangle_A$ so the symmetry is anomalous if $\mathcal{A} \neq 0$.

$\mathcal{A}(x)$ needs to be regulated in a gauge invariant way, e.g.

$$\mathcal{A}(x) = -2\, \mathrm{Tr}\left[\left\{ \gamma_5 t \exp\left(-\not{D}_x^2/M^2\right)\right\} \delta^4(x-y)\right]_{y\to x}$$

to be computed at finite M, and then $M \to \infty$.

Using Fourier representation of $\delta^4(x-y)$: , and $f(\partial)e^{ikx} = e^{ikx} f(\partial + ik)$

$$\mathcal{A}(x) = -2 \int \frac{d^4k}{(2\pi)^4}\, \mathrm{Tr}\left\{ \gamma_5 t \exp\left[-\not{D}_x^2/M^2\right] e^{ik(x-y)} \right\}$$

$$= \quad '' \quad \mathrm{Tr}\left\{ \gamma_5 t\, e^{ik(x-y)} \exp\left[-(i\not{k} + \not{D}_x)^2/M^2\right]\right\}$$

$$= \quad '' \quad \mathrm{Tr}\left\{ \gamma_5 t \exp -(i\not{k} + \not{D}_x)^2/M^2 \right\}$$

Let us now introduce a Higgs field in $D=3+1$ Yang Mills:

In general gauge symmetry is broken to H,

with the gauge fields in $G-H$ becoming massive,

'eating the G/H goldstone bosons'; electric charge $m \in$ weight lattice of H

Still, there can be non trivial topological excitations classified

by $\pi_2(G/H) = \pi_1(H)$, corresponding to magnetic monopoles,

with charge $n \in$ coweight lattice of H [H simply laced $\Rightarrow$ coroot = weight]

More generally, there exist excitations carrying both electric

and magnetic charges, called 'dyons'. Such states are

obtained by allowing for $A_0^a = \hat{x}_a\, J(r)/er\, dt + \ldots$

The electric field is now $F_{0i} = \hat{x}_i \frac{d}{dr}\left(\frac{J(r)}{er}\right)$

and leads to $Q = \int_{S_2(\infty)} dS_i\, F_{0i}$

The quantization condition is that $N \in \mathbb{Z}$ where

N is the generator of gauge transformations in H.

$$N = \int \frac{\partial \mathcal{L}}{\partial\, \partial_0 A_\mu} \cdot \frac{D_\mu \phi^a}{ev} = \int \left(E_i^a + \theta B_i^a\right) D_i \phi^a = Q + \theta P$$

(Witten effect)

The Dirac condition also generalizes to $PQ' - P'Q \in \mathbb{Z}$.

Electric magnetic duality in 4D gauge theories

_ Maxwell' equations

$$\partial_\mu F_{\mu\nu} = j^e_\nu$$
$$\partial_\mu \tilde{F}_{\mu\nu} = j^m_\nu$$

are symmetric under exchange $F_{\mu\nu} \to \tilde{F}_{\mu\nu}$, provided one introduces a conserved magnetic current j^m_ν.

_ At the path integral level,

$$\int DA \ \exp\left[\frac{1}{g^2}\int F\wedge * F\right]$$
$$= \int DF\, D\tilde{A} \ \exp\left[\int \frac{1}{g^2}(F\wedge * F) + \int A_D \wedge dF\right.$$

$\hookrightarrow$ Lagrange multiplier for Bianchi identity $dF=0$

$$= \int D\tilde{A} \ \exp\left[\ g^2 (F_D \wedge * F_D)\right] \quad : \quad Z(g) = Z(1/g) \ !$$

More generally, in the presence of a coupling $\frac{\theta}{8\pi^2} F\wedge F$

$$Z\left(\frac{a\tau+b}{c\tau+d}\right) = Z(\tau) \qquad \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \qquad \text{'S-duality'}$$

_ This manipulation in general fails in the presence of interacting charged matter, or for non Abelian gauge fields.

However, for suitable particle contents such that the gauge theory is superconformal, it is conjectured that S duality is exact.

<u>Ex</u> $N=4$ SYM in Coulomb phase.

$$e^{2\pi i \tau} = e^{-\frac{8\pi^2}{g^2}}$$

Summary of lecture 1

* In general, n-pt functions of quantum fields may receive contributions from non-trivial stationary configurations of the Euclidean path integral: instantons

* Such non-trivial saddle points typically exist only if there exists a topological obstruction for deforming them into the trivial saddle point, measured by some topological charge n

ex $D = 0+1$: , $n = \frac{1}{2v}\int_{-\infty}^{+\infty} \mathring{\phi}\, dt = \frac{\phi(+\infty)-\phi(-\infty)}{2v} = \begin{cases} 0 \\ 1 \end{cases}$

for double well potential

; $\mathring{\phi} = \sqrt{2V(\phi)}$

. $n = \frac{2\pi}{\beta}\int_{-\infty}^{+\infty} \mathring{\phi}\, dt \in \mathbb{Z}$

for periodic potential

proceed to S2

ex $D = 1+1$ Abelian Higgs model $\quad V = \frac{\lambda}{8}\left[|\phi|^2 - v^2 \right]^2$

$$n = \oint_{S_1(\infty)} \frac{d\phi}{2\pi i\, \phi} \in \mathbb{Z} \qquad D\phi = d\phi + ieA \to 0$$

$$= \frac{e}{2\pi}\oint_{S_1(\infty)} A = \frac{e}{2\pi}\int d^2x\, F$$

; n can be written as a local functional of ϕ: $n = \int \nu(\phi)$

* Unlike the trivial vacuum, these saddle points are not isolated but (in infinite volume) are labelled by some collective coordinates, $\mathcal{C}_n$ in particular the location of the center; These are in 1-1 correspondence with zero-modes of the linear operator $Q = \delta S / \delta\phi(x)\delta\phi(x')$ around the instanton background.

eg $D = 0+1$:
$$Q = \delta(t-t')\left(-\partial_t^2 + V(\phi)\right)$$
$$\mathring{\phi} = \sqrt{2V(\phi)}$$
$\delta\phi = \mathring{\phi}$ satisfies $\left(-\partial_t^2 + V(\phi)\right)\cdot \delta\phi = 0$

— proceed to S3

* If the space of classical vacua is $\mathcal{M}$, (D space-time dimensions) the finiteness of the action requires $\phi \in \mathcal{M}$ on $S_{D-1}(\infty)$

thus topological sectors are classified by

$$\pi_{D-1}(\mathcal{M})$$

ex $D = 1 \Rightarrow$ connected components of $\mathcal{M}$

$D = 2 \Rightarrow$ fundamental group of $\mathcal{M}$

* Unless $D = 1$, Derrick's thm shows that finite action solutions cannot be obtained only by exciting the scalar field ϕ

$$S = \int (D\phi)^2 + V(\phi) + \frac{1}{4} F_{\mu\nu} \; d^D x = S_0 + S_1 + S_2$$

$$\phi_\lambda(x) = \phi(\lambda x)$$

$$A_\mu^\lambda(x) = \lambda A_\mu(\lambda x)$$

$$S(\lambda) \propto \lambda^{2-D} S_0 + \lambda^{-D} S_1 + \lambda^{4-D} S_2$$

$$(2-D) S_0 - D S_1 + (4-D) S_2 = 0$$

The problem originates from the angular contribution to S_0.

It can be fixed by requiring that at ∞, (ϕ, A_μ) is locally gauge equivalent to the vacuum:

$$\phi(x^\mu) \longrightarrow g(\hat{x}) \cdot \phi_0 \qquad \mathcal{M} = \mathcal{G}/H$$

$$A_\mu \longrightarrow g^{-1} \partial_\mu g(\hat{x})$$

e.g. $D=1$ Abelian Higgs model

$$\varphi = v \, f(r) \, e^{in\theta}$$

$$A_\theta = \frac{n \, a(r)}{e} \, d\theta \qquad A_r = 0 \text{ (gauge choice)}$$

—— back to S_1

* Instantons in D $\to$ Solitons in D+1 $\to$ string in D+2

$\downarrow$

...

* In the limit $\hbar \to 0$ (which is the same as $\lambda \to 0$, since for a ϕ^n interaction, the correlation functions depend only on $\hbar \lambda^{\frac{2}{n-2}}$),

$$\langle 0 | T\phi(x_1) \cdots \phi(x_n) | 0 \rangle$$

$$= \mathcal{N} \sum_n a(n) \int_{\mathcal{C}_n} d\mu \; \underbrace{\left[\det{}'(Q)\right]^{-1/2}}_{\hbar^{-\#\text{ fermionic zero modes}/2}} \; e^{-S(\phi_{cl})/\hbar} \cdot \phi_{cl}(x_1) \cdots \Big(1 + \underbrace{\sum}_{\text{Wick contractions using } Q}\Big)$$

* Configurations with $n > 1$ can be approximated by superpositions of one-instanton configurations which are far separated, (dilute gas approx) with collective coordinates $\mathcal{C}^{n+\bar{n}}/S_{n+\bar{n}}$ (indistinguishable), with action $(n+\bar{n})S$ but charge $n-\bar{n}$.

$$\hookrightarrow \quad \sum_{n,\bar{n}} a(n,\bar{n}) \frac{(TVK)^{n+\bar{n}}}{n!\;\bar{n}!} e^{-(n+\bar{n})S_1}$$

where $K \propto \left[\det{}'(Q_1)\right]^{-1/2}$

* In quantum mechanics, transition amplitudes from $|m\rangle$ to $|m'\rangle$ involve

$$a(n,\bar{n}) = \delta_{m-m'+n-\bar{n}}$$

$$= \int \frac{d\theta}{2\pi} e^{i\theta(m-m'+n-\bar{n})}$$

$\rightsquigarrow$ one param of low energy states, with

$$\delta E(\theta) \propto \cos\theta \; e^{-S_0/\hbar}$$

* In QFT, cluster decomposition requires that a should be a local functional of ϕ, satisfying $a(n_1,\bar{n}_1)\,a(n_2,\bar{n}_2) = a(n_1+n_2,\bar{n}_1+\bar{n}_2)$

$\hookrightarrow \quad a(n,\bar{n}) = e^{i\theta(n-\bar{n})} = e^{i\theta\,\nu(\phi)}$ 'θ vacua'

TD 1: Instantons and solitons

• Consider the integral $A(g,I)=\int_{-\infty}^{\infty} e^{-\frac{1}{2}\phi^2-g\,I(\phi)}$ where I is a positive function of ϕ such that the integral converges. Let $A_k(I)$ be the k-th Taylor coefficient of $A(g,I)$ around $g=0$. Using the uniform bound $0\le\sum_{k=0}^{\infty}(-x)^k/(k+N+1)!\le 1/(N+1)!$, show that for any $I(\phi)$,

$$|A(g,I)-\sum_{k=0}^{N}A_k(I)g^k|\le A_{N+1}(I)\,g^{N+1}\ . \tag{1}$$

For $I(\phi)=\phi^n/n$, show that $A_k(I)$ grows like $\alpha^k\,(k!)^{\frac{n}{2}-1}$ as $k\to\infty$, for some α. Find the optimum truncation N for given value of g, and the corresponding error in the perturbative computation of $A(g,I)$.

• Consider a one-dimensional model with Lagrangian

$$\mathcal{L}_{1D}=\frac{1}{2}\dot\phi^2-V(\phi)\ ,\qquad V(\phi)=\frac{\alpha}{\beta^2}(1-\cos\beta\phi)\ . \tag{2}$$

Find the instanton interpolating between two subsequent classical vacua, and its action S_0. Show that the lowest energy spectrum in the classical limit $\hbar\to0$ consists of a continuum of states labelled by $0\le\theta<2\pi$, with energy

$$E(\theta)=E_0+2\pi\hbar\,K\,\cos\theta\,e^{-S_0/\hbar} \tag{3}$$

where K is a certain determinantal factor, and $E_0=\frac{1}{2}\hbar\sqrt{\alpha}+\mathcal{O}(\hbar^2)$. What is the interpretation of this instanton solution in the context of a two-dimensional model with Lagrangian $\mathcal{L}_{2D}=\mathcal{L}_{1D}-\frac{1}{2}(\phi')^2$.

• In the context of the 1+1-dimensional Abelian Higgs model

$$\mathcal{L}_{2D}=|D_\mu\phi|^2-\frac{1}{4}F_{\mu\nu}^2-\tfrac{1}{8}\lambda(|\phi|^2-v^2)^2\ , \tag{4}$$

compute the instanton corrections to the expectation value of a rectangular Wilson line of size $L'\times T'$ in a much larger volume $L\times T$, in the dilute gas approximation. Show that this implies a linear potential between external charged particles, hence confinement.

TD 2: Instantons and anomalies

- For $W(t)$ a bounded function, we wish to compute the determinant of the operator $\det(-\partial_t^2 + W)$ in the space of functions vanishing at both $t = -T/2$ and at $t = T/2$. For this purpose, let $\phi_{W,\lambda}(t)$ be the solution of

$$(-\partial_t^2 + W)\phi_{W,\lambda} = \lambda\,\phi_{W,\lambda} \quad \phi_{W,\lambda}(-T/2) = 0\ , \quad \dot{\phi}_{W,\lambda}(-T/2) = 1 \tag{1}$$

Show that for two functions $W_1(t)$, $W_2(t)$,

$$\frac{\det(-\partial_t^2 + W_1 - \lambda)}{\det(-\partial_t^2 + W_2 - \lambda)} = \frac{\phi_{W_1,\lambda}(T/2)}{\phi_{W_2,\lambda}(T/2)} \tag{2}$$

This implies that the factor N defined by $\pi\hbar N^2 = \det(-\partial_t^2 + W)/\phi_{W,0}(T/2)$ is independent of W. We choose this factor N as the normalization factor in the functional integral $\langle\phi_f|e^{-HT/\hbar}|\phi_i\rangle = N\int_{\phi(-T/2)=\phi_i,\phi(T/2),\phi(T/2)=\phi_f} D\phi\, e^{-S_E/\hbar}$. Show that for the harmonic oscillator,

$$N\left[\det(-\partial_t^2 + \omega^2)\right]^{1/2} = \left[\frac{\pi}{\hbar\omega}\sinh(\omega T)\right]^{-1/2} \tag{3}$$

in agreement with the expected late time behavior of $\langle 0|e^{-HT/\hbar}|0\rangle$ (Coleman, 'The uses of instantons', Appendix 1).

- Compute the angular momentum $\vec{L} = \frac{1}{4\pi c}\int d^3r\,\vec{r}\wedge\vec{E}\wedge\vec{B}$ carried by the electromagnetic field sourced by a magnetic monopole of charge p at $\vec{r} = \vec{r}_1$ and an electric source of charge q at $\vec{r} = \vec{r}_2$. Show that $\vec{L}$ is independent of the distance between the two sources, and aligned with $\vec{r}_1 - \vec{r}_2$. Show that the quantization of $\vec{L}$ reproduces the Dirac quantization condition $2pq/c\hbar \in \mathbb{Z}$. (Jackson, sec 6.13).

- Compute the chiral anomaly à la Fujikawa using an arbitrary regulator

$$\partial_\mu j_A^\mu = -2\lim_{M\to\infty}\left[\mathrm{Tr}\,\gamma_5\, f(\not{D}^2/M^2)\,\delta^4(x-y)\right]_{y\to x} \tag{4}$$

where $f(x)$ is an arbitrary function of x such that $f(0) = 1, f(\infty) = 0$ and $xf'(x) = 0$ at $x = 0$ and $x = \infty$. Check that the anomaly is independent of $f(x)$. (Weinberg, sec 22.2).

- Consider the Euclidean action for the electromagnetic field in the absence of sources

$$S(F,\tau) = \frac{1}{g^2}\int F\wedge\star F - \frac{\theta}{8\pi^2}\int F\wedge F\ , \qquad \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}\ . \tag{5}$$

By adding a Lagrange multiplier $\int A_D\wedge dF$ for the Bianchi identity $dF = 0$ and integrating out F, show that the dynamics of $F_D = dA_D$ is governed by the action $S(F_D,\tau_D)$ where $\tau_D = -1/\tau$. Identify the group generated by the discrete symmetries $\tau\mapsto -1/\tau$ and $\tau\mapsto\tau+1$.