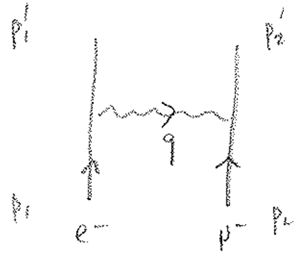
Infrared divergences in QED

Consider Coulomb scattering $e^- \mu^- \to e^- \mu^-$

At tree level:

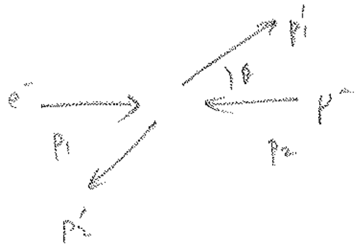


$$= \frac{ie^2}{q^2} \; \bar{u}(p_1')\gamma^\mu u(p_1) \cdot \bar{u}(p_2')\gamma_\mu u(p_2)$$

Average over initial state, sum over final states:

$$\frac{1}{4}\sum |\mathcal{M}|^2 = \frac{e^4}{4q^4} \underbrace{\mathrm{Tr}\left[(\not{p}_1' + m_e)\gamma^\mu(\not{p}_1 + m_e)\gamma^\nu\right]}_{4\left[p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}(p\cdot p' + m^2)\right]} \cdot \mathrm{Tr}\left[(\not{p}_2' + m_\mu)\gamma_\mu(\not{p}_2 + m_\mu)\gamma_\nu\right]$$

$$= \frac{8e^4}{t^2}\left[(p_1 p_2')(p_2 p_1') + (p_1 p_2)(p_1' p_2') - m_\mu^2\, p_1 p_1' - m_e^2\, p_2 p_2' + \frac{1}{2} m_e^2 m_\mu^2\right]$$



$p_1 = (E_e, k\hat{z})$ $\quad p_1' = (E_e, \vec{k})$

$p_2 = (E_\mu, -k\hat{z})$ $\quad p_2' = (E_\mu, -\vec{k})$

$E_e^2 = k^2 + m_e^2$ $\quad E_e + E_\mu = E_{cm}$

$E_\mu^2 = k^2 + m_\mu^2$ $\quad q^2 = -2k^2(1-\cos\theta)$

$m_e \equiv 0$

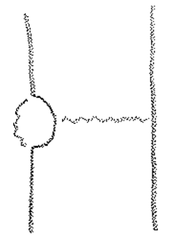
$$\frac{1}{4}\sum|\mathcal{M}|^2 = \frac{2e^4}{k^2(1-\cos\theta)^2}\left[(E+k)^2 + (E + k\cos\theta)^2 - m_\mu^2(1-\cos\theta)\right]$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{|\mathcal{M}|^2}{64\pi^2(E+k)^2}$$

$\left(\text{recall: } \frac{1}{2E_A \cdot 2E_B\, |v_A - v_B|} \frac{|p_1|}{(2\pi)^2\, 4E_{cm}} |\mathcal{M}|^2\right.$

$$\left(\frac{d\sigma}{d\Omega}\right) \sim \frac{1}{\theta^4} \quad \text{as } \theta \to 0 \quad \text{due to nearly on-shell virtual photon}$$

At 1-loop, one needs to include



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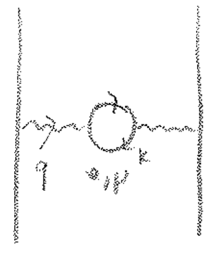

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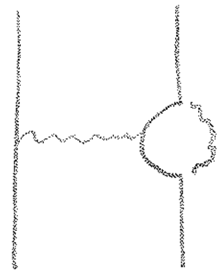


vertex correction | vacuum polarization | vertex correction

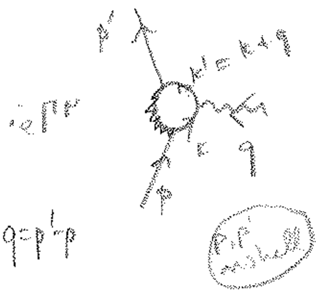
$$i\Pi_2^{\mu\nu}(q) = -(-ie)^2 \int \frac{d^4k}{(2\pi)^4} \, \mathrm{Tr}\left[\gamma^\mu \frac{i}{\not k - m} \gamma^\nu \frac{i}{\not k + \not q - m}\right]$$

$$\sim -4e^2 \int \frac{d^4k}{(2\pi)^4} \frac{k^\mu (k+q)^\nu + k^\nu (k+q)^\mu - g^{\mu\nu}(k(k+q) - m^2)}{(k^2 - m^2)((k+q)^2 - m^2)}$$

UV divergent, but IR finite

use $q_\mu \Gamma^\mu = 0$
$\not p u(p) = m\, u(p)$
$\bar u(p') \not p' = \bar u(p') m$

In contrast, the vertex corrections are IR divergent.



$$\Gamma^\mu(p', p) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}}{2m} q_\nu F_2(q^2)$$

$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$
$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

recall Gordon's identity
$\bar u(p')\gamma^\mu u(p) = \bar u(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu} q_\nu}{2m}\right] u(p)$

$F_1(0) = 1$: charge of electron

$F_2(0) = \frac{1}{2}(g-2)$: anomalous magnetic moment

$$\bar u\, \delta\Gamma^\mu u = \int \frac{d^4k}{(2\pi)^4} \frac{-i g_{\nu\rho}}{(k-p)^2 + i\varepsilon} \, \bar u(p') (-ie\gamma^\nu) \frac{i(\not k' + m)}{k'^2 - m^2 + i\varepsilon} (\gamma^\mu) \frac{i(\not k + m)}{k^2 - m^2 + i\varepsilon} (-ie\gamma^\rho)\, u(p)$$

$$\delta\Gamma^\mu(q) = 2ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{\not k \gamma^\mu \not k' + m^2 \gamma^\mu - 2m(k+k')^\mu \mathbb{1}}{((k-p)^2 + i\varepsilon)((k+q)^2 - m^2 + i\varepsilon)(k^2 - m^2 + i\varepsilon)}$$

This is manifestly UV divergent $\left(\int d^4k / k^4\right)$

but also infrared divergent due to the pole at $k \simeq p$. For now we put a small photon mass: $(k-p)^2 - \mu^2 + i\varepsilon$

intermediate result:

$$\bar{u}(p')\,\delta\Gamma^{\mu}(p',p)\,u(p) = 2ie^2 \int \frac{d^4\ell}{(2\pi)^4} \int_0^1 dx\,dy\,dz\ \delta(x+y+z-1) \cdot \frac{2}{(\ell^2-\Delta+i\varepsilon)^3}$$

$$\times\ \bar{u}(p')\Big[\ \gamma^{\mu}\left(-\tfrac{1}{2}\ell^2 + (1-x)(1-y)q^2 + (1-4z+z^2)m^2\right) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\cdot 2m^2 z(1-z)\ \Big]\,u(p)$$

$$\int \frac{d^D\ell}{(2\pi)^D}\ \frac{1}{(\ell^2-\Delta)^n} = \frac{(-1)^n\, i}{(4\pi)^{D/2}} \cdot \frac{\Gamma(n-D/2)}{\Gamma(n)} \cdot \left(\frac{1}{\Delta}\right)^{n-\frac{D}{2}}$$

$D=4,\ n=3$: $\quad \dfrac{-i}{(4\pi)^2}\cdot\dfrac{1}{\Delta}$.

$$\int \frac{d^D\ell}{(2\pi)^D}\ \frac{\ell^2}{(\ell^2-\Delta)^n} = \frac{(-1)^{n-1}\, i}{(4\pi)^{D/2}} \cdot \frac{D}{2} \cdot \frac{\Gamma(n-\frac{D}{2}-1)}{\Gamma(n)} \quad \left(\frac{1}{\Delta}\right)^{n-\frac{D}{2}-1}$$

$D=4,\ n=3$: diverges due to pole in $\Gamma(0)$

but the difference $\displaystyle\int \frac{d^D\ell}{(2\pi)^D}\left[\frac{\ell^2}{(\ell^2-\Delta)^n} - \frac{\ell^2}{(\ell^2-\Delta')^n}\right]$ is finite, equal to $(D=4+\varepsilon,\ \varepsilon\to 0)$

$$\frac{i}{(4\pi)^{2+\frac{\varepsilon}{2}}}\left(2+\frac{\varepsilon}{2}\right)\cdot\frac{\Gamma(-\varepsilon/2)}{2}\cdot\Big[\underbrace{\Delta^{\varepsilon/2} - \Delta'^{\,\varepsilon/2}}_{\frac{\varepsilon}{2}\log\left(\frac{\Delta}{\Delta'}\right)}\Big]$$

$$= \frac{-i}{(4\pi)^2}\log\left(\frac{\Delta}{\Delta'}\right)$$

here $\Delta' = z\Lambda^2$

$\Delta = (1-z)^2 m^2 - xy\,q^2 + z\mu^2$

$$\hookrightarrow\ \frac{\alpha}{\pi}\int_0^1 dx\,dy\,dz\ \delta(x+y+z-1)$$

$$\times\ \bar{u}(p')\Big[\ \gamma^{\mu}\left(\log\left(\frac{z\Lambda^2}{\Delta}\right) + \frac{1}{\Delta}\Big[(1-x)(1-y)q^2 + (1-4z+z^2)m^2\Big]\right) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\ \frac{1}{\Delta}\ 2m^2 z(1-z)\ \Big]\,u(p)$$

Using the Feynman trick

$$\frac{1}{A_1 A_2 \dots A_n} = \int_0^1 dx_1 \dots dx_n \,\delta(\Sigma x_i - 1) \frac{(n-1)!}{[x_1 A_1 + x_2 A_2 + \dots + x_n A_n]^n}$$

we can combine denominators

$$\frac{1}{(k-p)^2 + i\varepsilon} \frac{1}{k'^2 - m^2 + i\varepsilon} \frac{1}{k^2 - m^2 + i\varepsilon} = \int_0^1 dx\, dy\, dz\, \delta(x+y+z-1) \frac{2}{D^3}$$

$$D = x\,(k^2 - m^2) + y\,((k+q)^2 - m^2) + z\,((k-p)^2 \cancel{-\mu^2}) + (x+y+z)\, i\varepsilon$$

$$= k^2 + 2k\,(yq - zp) + \cancel{yq^2} + zp^2 - (x+y)m^2 + i\varepsilon$$

$$= \ell^2 - \Delta + i\varepsilon \qquad \ell = k + yq - zp$$

$$\Delta = x(x-1)\,q^2 + m^2(1-z) + (p+q)z\,\big(p(z-1) + q(z+2x-1)\big)$$

For on-shell electrons: $p^2 = p'^2 = m^2$, $q^2 = -2pq$

$$\boxed{\Delta = -xy\,q^2 + (1-z)^2\, m^2 + z\mu^2}$$

Moreover, using $\not{p}\, u(p) = m\, u(p)$

and the Gordon identity $\bar{u}(p')\,\gamma^\mu\, u(p) = \bar{u}(p')\left[\left(\frac{p'^\mu + p^\mu}{2m}\right) + \frac{i\sigma^{\mu\nu} q_\nu}{2m}\right] u(p)$

as well as

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\mu}{D^3} = 0,$$

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\mu \ell^\nu}{D^3} = \frac{1}{4} g^{\mu\nu} \int \frac{d^4\ell\, \ell^2}{D^3}$$

give intermediate step

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2 - \Delta + i\varepsilon)^m} = \frac{i\,(-1)^m}{(4\pi)^2} \frac{1}{(m-1)(m-2)\,\Delta^{m-2}}$$

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^2}{(\ell^2 - \Delta + i\varepsilon)^m} = \frac{i\,(-1)^m}{(4\pi)^2} \frac{2}{(m-1)(m-2)(m-3)\,\Delta^{m-3}}$$

For $m=3$, the second integral is log divergent in the UV. We cure it by replacing the photon propagator $\frac{1}{(k-p)^2 - \mu^2 + i\varepsilon} \longrightarrow \frac{1}{(k-p)^2 - \mu^2 + i\varepsilon} - \frac{1}{(k-p)^2 - \Lambda^2 + i\varepsilon}$

and take $\Lambda \to \infty$ at the end.

This replaces $\int \frac{d^4\ell}{(2\pi)^4}\left[\frac{\ell^2}{(\ell^2-\Delta+i\varepsilon)^3} - \frac{\ell^2}{(\ell^2-\Delta-z\Lambda^2)^3}\right] \sim \int \frac{d^4\ell}{\ell^6}$ UV convergent

$\sim \frac{i}{(4\pi)^2}\log\left(\frac{z\Lambda^2}{\Delta}\right)$

Both the UV and IR divergence only affect F_1.

We also need to subtract a term proportional to tree-level vertex, to ensure $F_1(q^2)=0$

$$F_1(q^2) = 1 + \frac{\alpha}{2\pi}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)$$
$$\left[\log\frac{m^2(1-z)^2}{m^2(1-z)^2-q^2xy} + \left(\frac{m^2(1-4z+z^2)+q^2(1-x)(1-y)}{m^2(1-z)^2-q^2xy+\mu^2 z} - \left(\begin{matrix}\text{same at}\\ q^2=0\end{matrix}\right)\right)\right]$$

$$F_2(q^2) = \frac{\alpha}{2\pi}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\left[\frac{2m^2z(1-z)}{m^2(1-z)^2-q^2xy}\right] + O(\alpha^2)$$

In particular $F_2(0) = \frac{\alpha}{2\pi}\int_0^1 dz\int_0^{1-z}dx\,\frac{2z}{1-z} = \frac{\alpha}{2\pi}\int_0^1 2z\,dz = \frac{\alpha}{2\pi}$

Thus $\boxed{g-2 = \frac{\alpha}{\pi} \sim \frac{1}{430}}$ Schwinger 1948

On the other hand

$$F_1(q^2) \underset{q^2\to 0}{\sim} 1 + \frac{e^2}{24\pi^2}\left[\ln\frac{\mu^2}{m^2} + \frac{23}{60}\right] \quad ; \qquad \text{(Weinberg 11.3.31)}$$

$\frac{\alpha}{6\pi}$

more to later

$$= 1 - \frac{q^2a^2}{6} + \dots$$

where a is the charge radius. $a^2 \sim \frac{-e^2}{4\pi^2m^2}\ln\left(\frac{\mu^2}{m^2}\right)$

$a^2>0$ for electrons in atoms...

The logarithmic divergence in F_2 can be traced to the region $z \to 1$, $(x,y) \to 0$. Eg at $q^2 = 0$:

$$\frac{\alpha}{2\pi} \int dx\,dy\,dz\,\delta(x+y+z-1) \cdot \frac{(1-4z+z^2)\,m^2}{m^2(1-z)^2+\mu^2 z}$$

$$\int^{1-\mu/m} \frac{dz}{1-z} \sim \Big[-\log(1-z)\Big]^{1-\mu/m} \sim -\log\left(\frac{\mu}{m}\right)$$ ↳ effective cut-off at $1-z = \frac{\mu}{m}$

The log term gives $\frac{\alpha}{2\pi} \int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\, \frac{x\,y}{(1-z)^2}\,\frac{q^2}{m^2}$.

$$\int_0^{1-z} dx \int_0^{1-x-z} dy\; xy = \frac{(1-z)^4}{24}$$ so this is regular.

More generally, in the region $x, y \to 0$, $z \to 1$:

$$F_1(q^2) = \frac{\alpha}{2\pi} \int^1 dz \int_0^{1-z} dy \quad \frac{-2m^2+q^2}{m^2(1-z)^2 - q^2 y(1-z-y) + \mu^2} \quad - \text{(same at } q^2=0)$$

Change variable to
$w = 1-z$
$y = (1-z)\,\xi \qquad 0 \le \xi \le 1$
$dz\,dy = -dw\,.\,w\,.\,d\xi$

$\int \frac{dx}{x+\mu}$

$$= \frac{\alpha}{2\pi} \int_0^1 d\xi \int_0^{\delta} w\,dw \left[\frac{q^2-2m^2}{[m^2-q^2\xi(1-\xi)]\,w^2+\mu^2} - (q^2=0) \right]$$

$$= \frac{\alpha}{2\pi} \int_0^1 d\xi \quad \frac{q^2-2m^2}{m^2-q^2\xi(1-\xi)} \log\left(\frac{m^2-q^2\xi(1-\xi)}{\mu^2}\right) \quad - (q^2=0)$$

In the limit $\mu \to 0$: $\quad -\frac{\alpha}{2\pi} \int_0^1 d\xi\, \frac{q^2-2m^2}{m^2-q^2\xi(1-\xi)} \log \mu^2$

So $\quad F_1(q^2) \sim 1 - \frac{\alpha}{2\pi} f_{IR}(q^2) \log\left(\frac{-q^2 \text{ or } m^2}{\mu^2}\right)$, $\quad \boxed{f_{IR} \equiv \int_0^1 \frac{m^2-q^2/2}{m^2-q^2\xi(1-\xi)}\,d\xi - 1}$

In the limit $q^2 \to 0$:

$$f_{IR} \sim \frac{-q^2}{3m^2} \qquad \text{so} \qquad F_1 \sim 1 + \frac{\alpha q^2}{6\pi m^2} \log\left(\frac{m^2}{\mu^2}\right)$$

in agreement with charge radius

In the limit $q^2 \to \infty$:

$$f_{IR} \sim \int_0^1 \frac{-q^2/2 \; d\xi}{-q^2\xi(1-\xi) + m^2}$$

$$\sim -\frac{1}{2}\int_0 \frac{q^2}{m^2 - q^2\xi} \quad -\frac{1}{2}\int^1 \frac{q^2}{m^2 - q^2(1-\xi)}$$

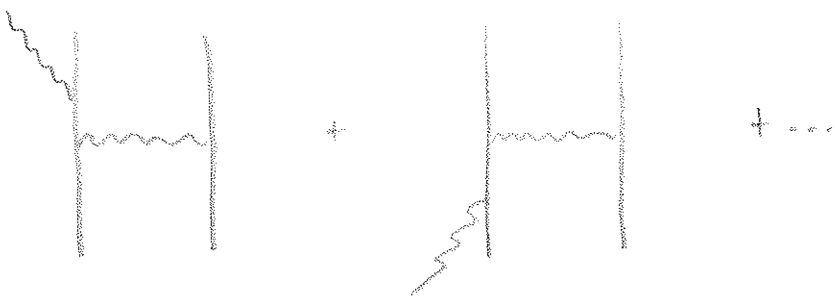
$$= \log\left(\frac{-q^2}{m^2}\right)$$

const.

so $\quad F_1\,(-q^2 \to \infty) \sim 1 - \dfrac{\alpha}{2\pi} \log\left(\dfrac{-q^2}{m^2}\right) \log\left(\dfrac{-q^2}{\mu^2}\right) + O(\alpha^2)$

'double logarithm'

At the end of the day, the dependence on μ must drop from physical observables. As we shall see, the $\log \mu$ divergence of $F_1(q^2)$ will be cancelled by 'Bremsstrahlung diagrams'



which are also IR divergent due to soft photon emission ...

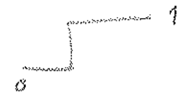
Classical Bremsstrahlung (PSchröder 6.1 p 177)

Consider an electron receiving a sudden kick at $t=0$, changing momentum from p to p':

Current density $\quad j^\mu(x) = e\int d\tau \, \frac{dy^\mu}{d\tau} \, \delta^{(4)}(x-y(\tau))$

(from minimal coupling $\int A_\mu \frac{dy^\mu}{d\tau} d\tau$ along worldline)

$$y^\mu(\tau) = p^\mu \tau \quad \theta(-\tau) + p'^\mu \tau \quad \theta(\tau)$$

$\theta =$ 

In Fourier space. $\quad j^\mu(k) = \int d^4x \, e^{ikx} \, j^\mu(x)$

$$= e\int d\tau \, \frac{dy^\mu}{d\tau} \, e^{iky(\tau)}$$

$$= e\Big[\int_{-\infty}^{0} p^\mu e^{ikp\tau} d\tau + \int_0^\infty p'^\mu e^{ikp'\tau} d\tau \Big]$$

$\uparrow \times e^{\varepsilon\tau} \qquad \uparrow \times e^{-\varepsilon\tau}$

Note current conservation: $k_\mu j^\mu = 0$

$$= \frac{e\, p^\mu}{ikp+\varepsilon} - \frac{e\, p'^\mu}{ikp'-\varepsilon}$$

$$= ie\left(\frac{p'^\mu}{kp'+i\varepsilon} - \frac{p^\mu}{kp-i\varepsilon} \right)$$

The electromagnetic field radiated by the source, in Lorentz gauge $\partial_\mu A^\mu = 0$, is

$$A^\mu(k) = -\frac{1}{k^2} j^\mu(k)$$

$e^{-i(k_0 t - \vec{k}\cdot\vec{r})}$

$$A^\mu(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \cdot \frac{-ie}{k^2} \cdot \left(\frac{p'^\mu}{k.p'+i\varepsilon} - \frac{p^\mu}{kp-i\varepsilon} \right)$$

The integral over k^0 can be performed as a contour integral.

To implement the retarded propagator, the poles at $k^0 = \pm|k|$ must be displaced below the real axis.

Thus $\vec{\mathcal{E}}\cdot\vec{\mathcal{E}}^* = \vec{k}^2 (A^0)^2 + (k^0)^2 \vec{A}^2 - 2(k^0)^2 (A^0)^2$

$$= (k^0)^2 \left(\vec{A}^2 - (A^0)^2 \right)$$

$$= |\vec{k}|^2 \, (A_\mu)^2$$

$$E = \int \frac{d^3k}{(2\pi)^3} \, \frac{e^2}{2} \cdot (-g_{\mu\nu}) \left[\left(\frac{p'^\mu}{k\cdot p'} - \frac{p^\mu}{k\cdot p} \right)\left(\frac{p'^\nu}{k\cdot p'} - \frac{p^\nu}{k\cdot p} \right) \right] \qquad \hat{k} = \begin{pmatrix} 1 \\ \vec{k}/k \end{pmatrix}$$

$$= \frac{e^2}{(2\pi)^2} \int dk \; I(p,p') \qquad d^3k = k\,dk\,d\Omega \qquad k = |\vec{k}|$$

where $I(p,p') = \int \frac{d^2\Omega_k}{4\pi} \left[\frac{2\, p\cdot p'}{(\hat{k}\cdot p)(\hat{k}\cdot p')} - \frac{m^2}{(\hat{k}\cdot p)^2} - \frac{m^2}{(\hat{k}\cdot p')^2} \right]$

$I(p,p')$ is independent of k, so the total energy is UV divergent.
But this comes from assuming instantaneous change of momentum,
in practice,
$$E = \left(\int_0^{k_{max}} \frac{e^2}{(2\pi)^2} \, dk \right) I(p,p')$$

Note also that the integrand in $I(p,p')$ is peaked where $\hat{k}\cdot p$ or $\hat{k}\cdot p'$ are maximal, ie in the direction of the incoming and outgoing particles.

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let $p = E\begin{pmatrix} 1 \\ \vec{v} \end{pmatrix} \quad p' = E\begin{pmatrix} 1 \\ \vec{v}' \end{pmatrix}$

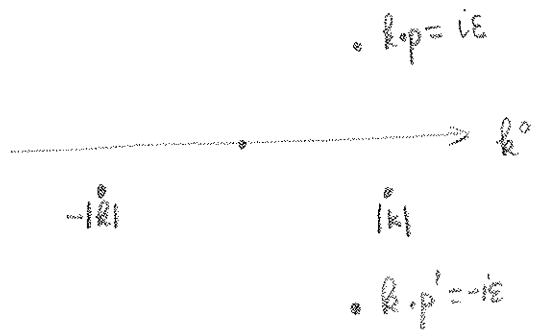
$$q^2 = (p'-p)^2 = -E^2 \left(\vec{v}^2 + \vec{v}'^2 - 2\vec{v}\vec{v}' \right) \xrightarrow{\text{relativistic limit}} -2E^2 (1 - \vec{v}\cdot\vec{v}')$$

$p\cdot p' = E^2 (1 + \vec{v}\cdot\vec{v}')$ so in relativistic limit, $q^2 \approx -2p\cdot p'$

$$I(p,p') \sim \int \frac{\sin\theta \, d\theta \, d\varphi}{4\pi} \; \frac{2(1-\vec{v}\vec{v}')}{(1-\hat{k}\cdot\vec{v})(1-\hat{k}\cdot\vec{v}')}$$

$$\sim \int_{v\cdot v'}^1 d\cos\theta \, \frac{1}{1-v'\cos\theta} + \int^1 d\cos\theta \, \frac{1}{1-v\cos\theta}$$

$$\sim \log\frac{1-vv'}{1-v'} + \log\frac{1-vv'}{1-v} \sim 2\log\left(\frac{-q^2}{m^2} \right)$$



In reference frame where

$$p = E\begin{pmatrix} 1 \\ \vec{v} \end{pmatrix} \qquad k = \begin{pmatrix} k_0 \\ \vec{k} \end{pmatrix}$$

$$p' = E\begin{pmatrix} 1 \\ \vec{v}' \end{pmatrix}$$

$$k \cdot p = E\left(k_0 + \vec{k}\cdot\vec{v} \right)$$

$$k \cdot p' = E\left(k_0 + \vec{k}\cdot\vec{v}' \right)$$

For $t<0$, the contour can be closed upward ↶ picking pole at $k.p = i\varepsilon$

$$A_\mu = \int \frac{d^3x}{(2\pi)^3} \cdot e^{i\vec{k}\vec{x}} \cdot e^{-i\vec{k}\cdot\vec{p}\frac{t}{p^0}} \cdot \frac{2\pi i}{2\pi} \cdot \frac{ie}{k^2} \frac{p^\mu}{p^0}$$

In frame where $\vec{v}=0$:
$$+e\int \frac{d^3x}{(2\pi)^3}\, \frac{e^{i\vec{k}\cdot\vec{x}}}{\vec{k}^2} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}^\mu$$

→ field of particle $\vec{p}$

For $t>0$: the contour closes downward.

The pole at $kp' = -i\varepsilon$ similarly produces the Coulomb field of other particle $\vec{p}'$

The field radiated is then the contribution of the poles at $k_0 = \pm|k|$:

$$A^\mu_{rad} = \int \frac{d^3k}{(2\pi)^3} \cdot \frac{-e}{2|k|} \left\{ e^{-ikx} \left(\frac{p'^\mu}{k.p'} - \frac{p^\mu}{k.p} \right) + cc \right\}_{k^0 = |k|}$$

$$\mathcal{A}_\mu(\vec{k}) = \frac{-e}{|k|} \left(\frac{p'^\mu}{k.p'} - \frac{p^\mu}{k.p} \right) \qquad (k^0 = |k| \text{ implicit})$$

After some work, one finds the radiated energy

$$E = \frac{1}{2}\int d^3x \left(\vec{E}(x)^2 + \vec{B}(x)^2 \right)$$

$$= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\, \vec{\mathcal{E}}(k) \cdot \vec{\mathcal{E}}^*(k)$$

$$\vec{\mathcal{E}}(k) = -i\vec{k}\,\mathcal{A}^0 + ik^0 \vec{\mathcal{A}}$$

$$\vec{\mathcal{E}}^*(k) = i\vec{k}\,\mathcal{A}^0 - ik^0 \vec{\mathcal{A}}$$

$$\mathcal{E}.\mathcal{E}^* = \vec{k}^2(\mathcal{A}^0)^2 + (k^0)^2\vec{\mathcal{A}}^2 - 2k^0\mathcal{A}^0\vec{k}\cdot\vec{\mathcal{A}}$$

Transversality → $-k^0\mathcal{A}^0 + \vec{k}\cdot\vec{\mathcal{A}} = 0$

$$\boxed{\text{Computing} \quad I(p,p') = \int \frac{d^2\Omega_{\hat{k}}}{4\pi} \left[\frac{2\, p\cdot p'}{(\hat{k}\cdot p)(\hat{k}\cdot p')} - \frac{m^2}{(\hat{k}\cdot p)^2} - \frac{m'^2}{(\hat{k}\cdot p')^2} \right]}$$

where $p^2 = m^2$ (not necessarily equal)

$p'^2 = m'^2$

$\hat{k} = \begin{pmatrix} 1 \\ \vec{\hat{k}} \end{pmatrix}$, $|\vec{\hat{k}}| = 1$

- $$\int \frac{d\Omega_{\hat{k}}}{4\pi} \frac{m^2}{(\hat{k}\cdot p)^2} = \int \frac{d\theta \sin\theta \, d\varphi}{4\pi} \frac{m^2}{(p^0 - p\cos\theta)^2}$$

$$= \frac{1}{2}\int_{-1}^{1} d\cos\theta \frac{m^2}{(p^0 - p\cos\theta)^2} = \frac{1}{2p}\left[\frac{m^2}{p^0 - p} - \frac{m^2}{p^0+p}\right] = \frac{m^2}{(p^0)^2 - p^2} = 1$$

- $$\int \frac{d\Omega_{\hat{k}}}{4\pi} \frac{1}{(\hat{k}\cdot p)(\hat{k}\cdot p')} = \int_0^1 d\xi \frac{d\Omega_{\hat{k}}}{4\pi} \frac{1}{\left(\xi(\hat{k}\cdot p) + (1-\xi)\,\hat{k}\cdot p'\right)^2}$$

$$= \int_0^1 d\xi \frac{1}{[\xi p' + (1-\xi) p]^2}$$ by same computation as above

$$[\xi p' + (1-\xi)p]^2 = \xi^2 (m')^2 + (1-\xi)^2 m^2 + 2\xi(1-\xi)\, p\cdot p'$$

$$= \xi^2 [m'^2 + m^2 - 2\, p\cdot p'] - 2\xi [m^2 + 2\, p\cdot p'] + m^2$$

In terms of $\beta = \sqrt{1 - \frac{m^2 m'^2}{(p\cdot p')^2}}$, ie the relative velocity of each particle in the rest frame of the other

$$\left(\begin{array}{l} p = \begin{pmatrix} E \\ \vec{p} \end{pmatrix} \quad p' = \begin{pmatrix} E' \\ 0 \end{pmatrix} \qquad \begin{array}{l} E = \gamma m \\ E' = m' \end{array} \\ p\cdot p' = E E' = \gamma m m' \\ \text{So } \gamma = \frac{p\cdot p'}{m m'}, \quad \beta = \sqrt{1 - \frac{1}{\gamma^2}} \end{array} \right)$$

one arrives at $$\int \frac{d^2\Omega_{\hat{k}}}{4\pi} \frac{p\cdot p'}{(\hat{k}\cdot p)(\hat{k}\cdot p')} = +\frac{1}{2\beta} \ln \frac{1+\beta}{1-\beta}$$ Weinberg 13.2.9

Altogether, $$\boxed{I(p,p') = \frac{1}{\beta} \ln \frac{1+\beta}{1-\beta} - 2}$$

$$\downarrow$$

$$A = \frac{e^2}{4\pi^2} I(p,p')$$

In the ultrarelativistic limit, taking $m = m'$,

$$q^2 = (p-p')^2 \approx -2\,p\cdot p' \rightarrow -\infty$$

$$\beta = \sqrt{1 - \frac{4m^4}{(q^2)^2}} \sim 1 - \frac{2\,m^4}{(q^2)^2} \longrightarrow 1$$

$$\frac{1}{\beta}\ln\frac{1+\beta}{1-\beta} - 2 \rightarrow \ln\frac{2\,(q^2)^2}{2\,m^4} \approx 2\ln\frac{-q^2}{m^2}$$

so $I(p,p')$ grows logarithmically in the UR limit

In the nonrelativistic limit, $\beta \rightarrow 0 = \sqrt{1 - \frac{m^4}{(m^2 - \frac{q^2}{2})^2}} \sim \sqrt{\frac{-q^2}{m^2}}$

$$I(p,p') \sim \frac{2\,\beta^2}{3} \sim \frac{-2\,q^2}{3m^2}$$

<u>Rk</u> Going back to the Feynman representation above, and setting $m = m'$, $2pp' = 2m^2 - q^2$,

we have $$\int \frac{d\Omega_k}{4\pi}\,\frac{1}{(\hat{k}\cdot p)(\hat{k}\cdot p')} = \int_0^1 d\xi\,\frac{1}{[m^2 - \xi(1-\xi)q^2]}$$

and therefore $$I(p,p') = \int_0^1 \frac{2m^2 - q^2}{m^2 - \xi(1-\xi)q^2} - 2$$

This is recognized as $2\,f_{IR}(q^2)$, the coef of the IR log divergence in the vertex

<u>Thm</u> $$\boxed{I(p,p') = 2\,f_{IR}(q^2)}$$

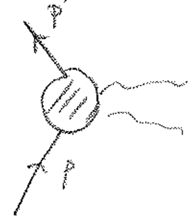
Thus $\quad \text{Energy} \longrightarrow \frac{2\alpha}{\pi}\int_0^{k_{max}} dk \cdot \log\left(\frac{-q^2}{m^2}\right)$

The number of photon is then $\quad \frac{2\alpha}{\pi}\int_0^{k_{max}} \frac{dk}{k} \cdot I(p,p')$

Which is now <u>IR divergent</u>: this will be cancelled by IR divergence of vertex correction.

For now this may be regulated by putting a small photon mass

<u>Quantum computation</u>:

Let $\quad \mathcal{M}_0(p';p) =$ 

hard A $\quad$ amplitude for $p \to p' + A$

The amplitude for $\quad p \to p' + k + A \quad$ is then

$$i\mathcal{M} = -ie\,\bar{u}(p')\left(\mathcal{M}_0(p', p-k)\; i\frac{\not p - \not k + m}{(p-k)^2 - m^2}\,\gamma^\mu\,\varepsilon^*_\mu(k)\right.$$

$$\left. + \;\gamma^\mu \varepsilon^*_\mu \; i\frac{\not p' + \not k + m}{(p'+k)^2 - m^2}\,\mathcal{M}_0(p'+k, p)\right)\cdot u(p)$$

In the soft limit $\quad |\vec{k}| \ll |\vec{p}' - \vec{p}|$:

$$\mathcal{M}_0(p', p-k) \sim \mathcal{M}_0(p'+k, p) \sim \mathcal{M}_0(p', p)$$

$$(\not p + m)\,\gamma^\mu u(p) = \left(2p^\mu + \gamma^\mu(m - \not p)\right)u(p) = 2p^\mu u(p)$$

$$\bar{u}(p')\,\gamma^\mu(\not p' + m) = \cdots = 2p'^\mu\,\bar{u}(p')$$

$$(p-k)^2 - m^2 \sim -2\,p\cdot k$$

$$(p'+k)^2 - m^2 \sim 2\,p'\cdot k$$

hence $$i\mathcal{M} \sim \bar{u}(p')\,\mathcal{M}_0\, u(p) \cdot \underbrace{e\left(\frac{p'\cdot\varepsilon^*}{p'\cdot k} - \frac{p\cdot\varepsilon^*}{p\cdot k}\right)}$$

classical electromag field radiated by

in polarization ε^*.

The cross section is similarly

$$d\sigma(p\to p'+\gamma+A) = d\sigma(p\to p'+A)\cdot \frac{d^3k}{(2\pi)^3}\,\frac{1}{2k}\sum_{\lambda=1,2} e^2\left|\frac{p'\cdot\varepsilon}{p'\cdot k} - \frac{p\cdot\varepsilon}{p\cdot k}\right|^2$$

Using $\sum_\lambda \varepsilon^*_\mu \varepsilon_\nu = -g_{\mu\nu}$ (when contracted with conserved current)

we arrive at

$$\frac{d\sigma(p\to p'+\gamma+A)}{d\sigma(p\to p'+A)} = \frac{e^2}{(2\pi)^2}\,\frac{dk}{k}\,I(p,p')$$

↳ same as # of radiated photons in classical Bremsstrahlung.

Putting things together:

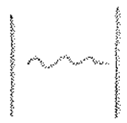
Let us compute the scattering correction

for $\quad p + A \longrightarrow p' + A' \quad$ + no photon of energy $> E_\ell$

$\uparrow$ detector resolution

$$d\sigma\Big|_{\text{measured}} = d\sigma(p+A \to p'+A') + \int_\mu^{E_\ell} d\sigma(p+A,\, p'+A'+\gamma)$$
$$+ \int_\mu^{E_\ell} d\sigma(p+A,\, p'+A'+2\gamma)$$
$$+ \ \ldots$$

At leading order: $\quad d\sigma_0 \quad \propto \quad |\mathcal{M}_0|^2$

e.g. 

At first subleading order:

$$d\sigma_{\text{measured}} = d\sigma_0 \Big[1 - \frac{\alpha}{\pi} f_{IR}(q^2) \log\Big(\frac{-q^2 \text{ or } m^2}{\mu^2}\Big)$$
$$+ \frac{\alpha}{2\pi} I(p,p') \log \frac{E_\ell^2}{\mu^2} + \mathcal{O}(\alpha^2) \Big]$$

Now recall $\quad \boxed{I(p,p') = 2 f_{IR}(q^2)}$

Sudakov double log

Thus $\quad d\sigma_{\text{measured}} = d\sigma_0 \Big[1 - \frac{\alpha}{\pi} f_{IR}(q^2) \log\Big(\frac{-q^2 \text{ or } m^2}{E_\ell^2}\Big)\Big]$

has a smooth limit $\mu \to 0$; for $q^2 \to -\infty$: $\quad d\sigma_0 \Big[1 - \frac{\alpha}{\pi} \log\Big(\frac{-q^2}{m^2}\Big) \log\Big(\frac{-q^2}{E_\ell^2}\Big)\Big]$

However, it can be made arbitrarily negative by sending $E_\ell \to 0$!

All order cancellation of IR divergences

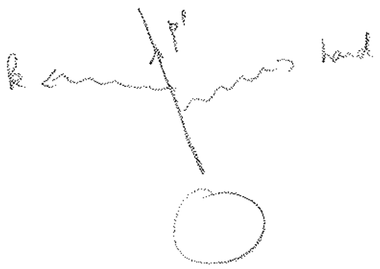
We want to show the cancellation of the leading

$$\mathcal{O}\left[\left(\alpha \log\left(\frac{-q^2}{\mu^2}\right)\log\left(\frac{-q^2}{m^2}\right)\right)^n\right]$$

divergence at order n in perturbation ~~theory~~ in QED

(the cancellation holds also for subleading divergences but is harder to demonstrate, cf Yennie Frautschi Suura (1961))

IR Divergences arise only from photons attached to external electron legs

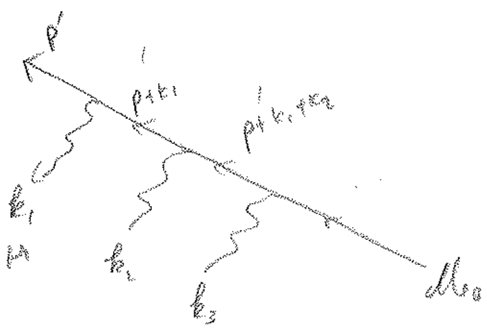


p' is on shell

the propagator before photon emission

is $\dfrac{1}{(p'-k)^2-m^2} \sim \dfrac{1}{-2p'\cdot k}$ as $k\to 0$

Consider outgoing electron line:



$$\bar{u}(p')\cdot(-ie\gamma^{\mu_1})\frac{i(\not p'+\not k_1+m)}{2p'\cdot k_1}(-ie\gamma^{\mu_2})\frac{i(\not p'+\not k_1+\not k_2+m)}{2p'(k_1+k_2)}$$

$\dots \quad i\mathcal{M}_0$

omit $\not k$ in numerator

$$\bar{u}(p')\gamma^{\mu_1}(\not p'+m)\gamma^{\mu_2}(\not p'+m)\dots = \bar{u}(p')\,2p'^{\mu_1}\,2p'^{\mu_2}\dots$$

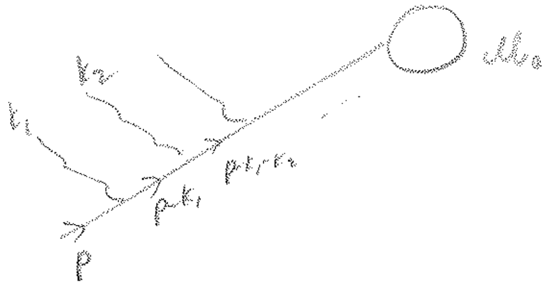
hence

$$\bar{u}(p')\frac{e\,p'^{\mu_1}}{p'\cdot k_1}\frac{e\,p'^{\mu_2}}{p'(k_1+k_2)}\dots\frac{e\,p'^{\mu_n}}{p'(k_1+\dots+k_n)}$$

Summing over permutations of $\{k_i\}$:

$$\bar{u}(p')\cdot\frac{e\,p'^{\mu_1}}{p'k_1}\cdot\frac{e\,p'^{\mu_2}}{p'k_2}\cdot\frac{e\,p'^{\mu_n}}{p'k_n}$$

Similarly, for incoming electron line,



$$\left(\frac{-e\,p^{\nu_1}}{(p\cdot k_1)}\right)\cdots\left(\frac{-e\,p^{\nu_n}}{p\cdot k_n}\right)$$

Thus, including photons attached to both incoming and outgoing lines,

$$\bar{u}(p')\, i\mathcal{M}_0\, u(p) \quad \times \quad e\left(\frac{p'_{\mu_1}}{p'k_1}-\frac{p_{\mu_1}}{p'k_1}\right)\cdots\left(\frac{p'_{\mu_n}}{p'\cdot k_n}-\frac{p_{\mu_n}}{p\cdot k_n}\right)$$

Virtual photons are obtained by picking two momenta k_i, k_j, setting $k_j = -k_i = k$, multiplying by photon propagator, and integrating:

$$X = \frac{e^2}{2}\int\frac{d^4k}{(2\pi)^4}\,\frac{-i}{k^2+i\epsilon}\left(\frac{p'}{p'\cdot k}-\frac{p}{p\cdot k}\right)\left(\frac{p'}{-p'\cdot k}-\frac{p}{-p\cdot k}\right)$$

Sym factor

$$= -\frac{\alpha}{2\pi}\, f_{IR}(q^2)\,\log\left(\frac{-q^2}{\mu^2}\right)$$

For n virtual photons, we get $\sum \frac{X^n}{n!} = \exp(X)$

Real photon emission: extra factor of

$$Y = \int\frac{d^3k}{(2\pi)^3}\,\frac{1}{2k}\,e^2(-g_{\mu\nu})\left(\frac{p'^{\mu}}{p'\cdot k}-\frac{p^{\mu}}{p\cdot k}\right)\left(\frac{p'^{\nu}}{p'\cdot k}-\frac{p^{\nu}}{p\cdot k}\right)$$

$$= \frac{\alpha}{\pi}\, I(v,v')\,\log\left(\frac{E_\ell}{\mu}\right)$$

E_ℓ = detection threshold

Observable cross section:

$$\sum_{n=0}^{\infty} d\sigma(p\to p'+n\gamma) = d\sigma_{\text{hard}}(p\to p')\cdot e^{2X}\cdot e^{Y}$$

← 'Sudakov form factor'

$$= d\sigma_{\text{hard}}\cdot \underbrace{\exp\left[-\frac{\alpha}{\pi}\, f_{IR}(q^2)\,\log\left(\frac{-q^2}{E_\ell^2}\right)\right]}_{\text{now always between 0 and 1}}$$

In addition to soft photon IR divergences, there can be extra IR divergences due to charged massless particles in initial/final state

e.g. . electrons in ultra relativistic regime

$$f_{IR}(q^2) \sim \ln \frac{-q^2}{m^2}$$

. gluons in QCD

There will be collinear divergences due to propagators $\frac{1}{(p_n \pm q)^2}$

when $p_n // q$:

$$\int d^2\hat{q} \, \frac{1}{(p \pm q)^2} \sim \mp \frac{1}{|\vec{p}|\,|\vec{q}|} \int_0^{\pi} \frac{\sin\theta \, d\theta}{1-\cos\theta} \quad \text{diverges from } \theta = 0$$

One can show that such divergences cancel provided one sums over suitable <u>initial</u> as well as final states

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but this goes beyond our scope ...