

Compactification

We have seen that closed bosonic strings are consistent only when the embedding coordinates $X^\mu(\tau, \sigma)$ are described by a conformal field theory of central charge $c=26$, so as to cancel the conformal anomaly of the b, c ghosts.

Choosing X^μ to be flat coordinates in $\mathbb{R}^{1,25}$ is one possibility but not the only one. That's fortunate, since we don't live in 26 dimensions!

More generally, one could choose $X^{\mu=0 \dots D-1}$ to be flat coordinates in $\mathbb{R}^{1, D-1}$, and the remaining coordinates $X^{i=D \dots 26}$ live in $(26-D)$ dimensional curved space db with metric $G_{ij}(x^m)$, two-form $B_{ij}(x^m)$, dilaton $\phi(x^m)$ satisfying the equations of motion. For energies less than $1/R$ where R is the characteristic scale of db , the physics would be effectively D -dimensional.

Even more generally, one could replace $X^{i=D \dots 26}$ by an abstract CFT of central charge $c=26-D$, with no geometric description, for example (Ising) ^{$\frac{26-D}{3}$} , where Ising is the 2D Ising model with $c=\frac{1}{2}$!!

Here we shall restrict ourselves to the case where M is locally flat, but not globally ∞ .

The simplest example is the circle $M=S^1 = \mathbb{R}/\mathbb{Z}$ with radius R .

A striking result is that closed strings cannot distinguish between $S^1(R)$ and $S^1(1/R)$!

1. Circle compactification

1.1. Field theory on $\mathbb{R}^{1, D-2} \times S_1(\mathbb{R})$

$S_1(\mathbb{R})$ can be constructed as the quotient

$$\mathbb{R}/\mathbb{Z} \text{ where } \mathbb{Z} \text{ acts by } x^{D-1} \rightarrow x^{D-1} + 2\pi R$$

Field theory on $\mathbb{R}^{1, D-2} \times S_1(\mathbb{R})$ is the same as field theory on $\mathbb{R}^{1, D-1}$, restricted to the sector

where fields are periodic under $x^{D-1} \rightarrow x^{D-1} + 2\pi R$.

In particular, plane waves $\psi = e^{i p_\mu x^\mu + i p_{D-1} x^{D-1}}$

must satisfy $p_{D-1} = \frac{m}{R}$ with $m \in \mathbb{Z}$

The mass-shell condition $p_\mu^2 + p_{D-1}^2 = -M^2$

for a field ϕ of mass M in dimension D becomes

$$p_\mu^2 = -M^2 - p_{D-1}^2 = -M_{D-1}^2, \quad M_{D-1} = \sqrt{M^2 + \left(\frac{m}{R}\right)^2}$$

Thus, ϕ gives rise to an infinite tower of fields ϕ_m ,

in dimension $D-1$, with mass $M_{D-1} = \sqrt{M^2 + \left(\frac{m}{R}\right)^2}$.

The lightest state is ϕ_0 , and the other states ϕ_m are its 'Kaluza-Klein excitations'.

If $M=0$, the masses of the KK states are evenly separated by $\frac{1}{R}$.

If ϕ carries spin, the spin of ϕ_m can be obtained by decomposing the $SO(1, D-1)$ rep under $SO(1, D-2)$

e.g. scalar \rightarrow scalar

vector $A_\mu \rightarrow$ vect: A_μ , scalar A_{D-1}

Spin-2 $g_{\mu\nu} \rightarrow$ Spin-2 $g_{\mu\nu}$, vector $g_{\mu, D-1}$, scalar $g_{D-1, D-1}$, etc

- The effective theory for the lightest modes is obtained by dimensional reduction, i.e. by assuming that ϕ is independent of x_{D-1} .

e.g.: gravity in dimension D

$$S_D = \int d^D x \frac{\sqrt{-G} R[G]}{E^2}$$

$$G_{mn} dx^m dx^n = g_{\mu\nu} dx^\mu dx^\nu + R^2 (dy + \omega_\mu dx^\mu)^2$$

'Kalusa-Klein Ansatz':

$$R[G] = R[g_{-1}] - 2 \frac{\square R}{R} - \frac{R^2}{4} F_{\mu\nu}^2$$

$$\text{where } F_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu$$

'Kalusa-Klein vector field'

$$S_D = \int d^{D-1} x \frac{R}{E^2} \sqrt{-g} \left(R[g_{-1}] - 2 \frac{\square R}{R} - \frac{R^2}{4} F_{\mu\nu}^2 \right)$$

This can be recast in the usual Einstein-Hilbert form by 'going to Einstein frame', i.e. rescale the metric

$$g = e^{2\alpha\phi} \hat{g} \quad \alpha^2 = \frac{1}{2(D-2)(D-3)}$$

$$R = e^{\beta\phi} \quad \beta = -(D-3)\alpha$$

so that

$$S_D = \int d^{D-1} x \sqrt{-g} \left(R - \frac{1}{2} (\partial\phi)^2 - \frac{1}{4} e^{-2(D-2)\alpha\phi} F^2 \right)$$

thus we get gravity in dim $D-1$, plus a massless scalar field ϕ (sometimes called 'dilaton', but it's not the same as the string theory dilaton!)

and a massless gauge field ω_μ .

The gauge symmetry $\omega_\mu \rightarrow \omega_\mu + \partial_\mu \lambda$ is the remnant of the diffeo symmetry in dim D $g_{mn} \rightarrow g_{mn} + \partial_m \lambda_n + \partial_n \lambda_m$ where $\lambda_{D-1} = 0$, λ_μ independent of y .

(4)

• Importantly, dimensional reduction is a consistent reduction, i.e. any solution of the $(D-1)$ -dimensional action gives a solution of the D -dim action, invariant under translations along S_1 .

• In the presence of fermionic fields, one must choose periodicity conditions for the fermions:

- periodic boundary conditions lead to $\mathcal{P}_{D-1} = \frac{m}{R}$, $m \in \mathbb{Z}$
'Rarmond bc'

- anti-periodic boundary conditions lead to $\mathcal{P}_{D-1} = \frac{m + \frac{1}{2}}{R}$, $m \in \mathbb{Z}$
'Neveu-Schwarz bc'

In addition, - spinors in odd dim D decompose into
left and right chiral spinors in dim $D-1$
- chiral spinors in even dim D reduce to
(non chiral) spinors in dim $D-1$

\Rightarrow the spectrum in $D-1$ dimensions is always non-chiral.

1.2. Circle compactification in bosonic closed string

The mode expansion of the coordinate $X \equiv X^{D-1}$ along S^1 must take into the fact that $X(\tau, \sigma + 2\pi)$ can differ from $X(\tau, \sigma)$ by a multiple of $2\pi R$.

$$X(\tau, \sigma) = X(\tau) + 2\pi n R \quad n \in \mathbb{Z}$$

'winding number'

The most general solution is

$$X = x + \frac{l_s^2}{2} p \tau + n R \sigma + \frac{i l_s}{\sqrt{2}} \sum_{k \neq 0} \left(\frac{\alpha_k}{k} e^{-ik(\tau-\sigma)} + \frac{\tilde{\alpha}_k}{k} e^{-ik(\tau+\sigma)} \right)$$

$$= X_L + X_R$$

with

$$X_L = \frac{x}{2} + \frac{l_s^2}{2} p_L(\tau + \sigma) + \dots$$

$$X_R = \frac{x}{2} + \frac{l_s^2}{2} p_R(\tau - \sigma) + \dots$$

with

$$p_L = \frac{m}{R} + \frac{nR}{l_s^2}$$

$$p_R = \frac{m}{R} - \frac{nR}{l_s^2}$$

} the only difference is in the zero-modes

$$L_0 = \frac{1}{4} \left(\frac{m l_s}{R} + \frac{n R}{l_s} \right)^2 + \sum_{k=1}^{\infty} \alpha_{-k} \alpha_k$$

$$\hat{L}_0 = \frac{1}{4} \left(\frac{m l_s}{R} - \frac{n R}{l_s} \right)^2 + \sum_{k=1}^{\infty} \alpha_{-k} \alpha_k$$

The spectrum can be constructed, e.g. in light-cone gauge:

$$p^+ p^- = \sum_{i=2}^{D-2} p_i^2 + p_L^2 + \frac{4}{l_s^2} \left(\sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i - \frac{D-2}{24} \right)$$

$$= \sum_{i=2}^{D-2} p_i^2 + p_R^2 + \frac{4}{l_s^2} \left(\sum_{n=1}^{\infty} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i - \frac{D-2}{24} \right)$$

↑
here $i = 2 \dots D-1$

Using $\frac{1}{2}(p_L^2 + p_R^2) = \left(\frac{m}{R}\right)^2 + \left(\frac{nR}{\alpha'}\right)^2$

$\frac{1}{2}(p_L^2 - p_R^2) = 2 \frac{mn}{\alpha'}$

we find the mass-shell condition

$$M_{D-1}^2 \equiv p^+ p^- - \sum_{i=2}^{D-2} p_i^2$$

$$= \left(\frac{m}{R}\right)^2 + \left(\frac{nR}{\alpha'}\right)^2 + \frac{2}{\alpha'^2} \left(\sum_{i=2}^{D-1} \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i + \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i - \frac{D-2}{12} \right)$$

$$= \left(\frac{m}{R}\right)^2 + \left(\frac{nR}{\alpha'}\right)^2 + M^2$$

while the matching condition is now

$mn = \sum_{-n}^i \alpha_{-n}^i \alpha_n^i - \sum_{-n}^i \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i = \tilde{N} - N$

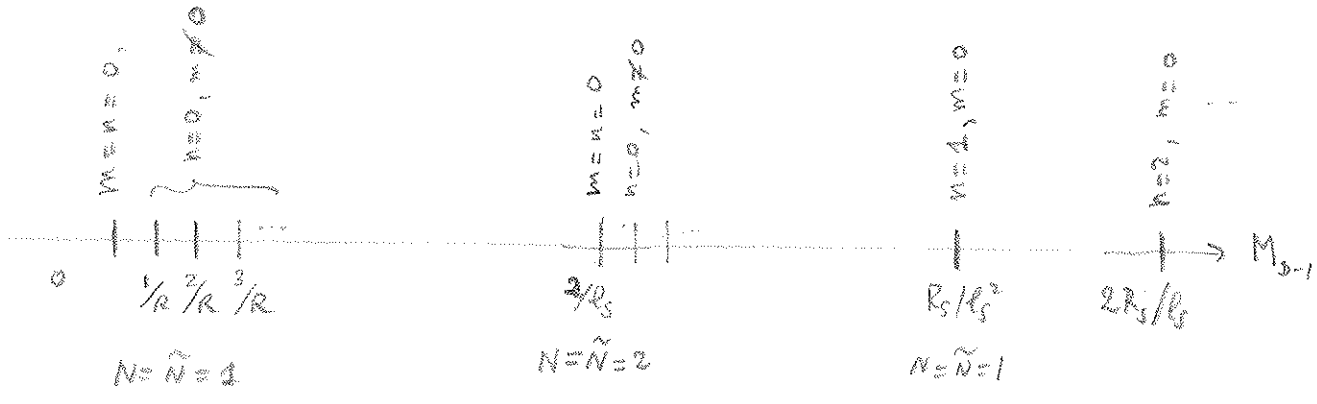
For $n=0$ (no winding), the result is the same as in field theory:

$M_{D-1}^2 = \left(\frac{m}{R}\right)^2 + M^2$: tower of Kaluza-Klein states

For $n \neq 0$ however, there is an additive contribution to the mass, corresponding to the tension of the string wound around the circle.

Moreover, if $mn \neq 0$, there are physical states in $(D-1)$ dimension which originate from 'unphysical states' in dimension D .

* If $R \gg \alpha'$, the energy spectrum looks like (ignoring the tachyon)



If $R \ll l_s$, the winding states are instead lighter than the Kaluza-Klein states

In fact, the spectrum is symmetric under

$$R \rightarrow \frac{l_s^2}{R}, \quad m \leftrightarrow n \quad ; \quad \text{'T-duality'} \quad T = \text{target space}$$

This is an exact symmetry of the conformal field theory of a compact boson, valid at each order in g_s (i.e. on surfaces of any genus)

It acts by

$$P_L \rightarrow P_L, \quad P_R \rightarrow -P_R$$

$$\alpha_{1c} \rightarrow \alpha_{1c}, \quad \tilde{\alpha}_{1c} \rightarrow -\tilde{\alpha}_{1c}$$

$$X = X_L + X_R \rightarrow \hat{X} = X_L - X_R$$

More generally, $\partial_\alpha X = \epsilon_{\alpha\beta} \partial^\beta \hat{X}$ on any worldsheet.

Thus, the bosonic closed string cannot distinguish between $S_1(R)$ and $S_1(1/R)$!

* The inclusion of these winding states is actually required for modular invariance; at one-loop, in Schwinger-time rep, the sum over momenta $P_{0-1} = \frac{m}{R}$ can be rewritten as a sum over windings in proper time, by Poisson resummation:

$$\sum_{m \in \mathbb{Z}} \exp \left[-\pi t \left(\frac{m}{R} \right)^2 \right] = \sum_{\tilde{m} \in \mathbb{Z}} \frac{R}{\sqrt{t}} \exp \left[-\pi R^2 \frac{\tilde{m}^2}{t} \right]$$

$\frac{R^2 \tilde{m}^2}{t}$ is the action for the configuration $X(\tau) = 2\pi \tilde{m} R \tau + x$

$$\int_0^1 \frac{1}{2} (\partial_\sigma X)^2 d\sigma, \quad e(\sigma) = t \quad X(1) = X(0) + 2\pi \tilde{m} R$$

but worldsheet diffs can exchange τ and σ , so one must also include winding in σ .

Indeed, the partition function of the compact boson

$$Z(R) = \text{Tr} \left(q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{c}{24}} \right)$$

$$= \frac{\sum_{m, n \in \mathbb{Z}^2} q^{\frac{1}{2} k^2} \bar{q}^{\frac{1}{2} \tilde{k}^2}}{\eta \bar{\eta}} = \sum_{m, n} e^{-\pi \tau_2 \left(\frac{m^2 R^2}{R^2} + \frac{n^2 R^2}{\ell_s^2} \right) + 2\pi i m n \tau_1}$$

is modular invariant. This can be made manifest by Poisson resummation $m \rightarrow \tilde{m}$:

(recall $\sum_{m \in \mathbb{Z}} e^{-\pi a m^2 + 2\pi i m b} = \frac{1}{\sqrt{a}} \sum_{\tilde{m} \in \mathbb{Z}} e^{-\frac{\pi}{a} (\tilde{m} - b)^2}$)

$$Z(R) = \frac{1}{\eta \bar{\eta}} \cdot \frac{R}{\ell_s \sqrt{\tau_2}} \cdot \sum_{\tilde{m}, n} \exp \left[-\frac{\pi R^2}{\ell_s^2} \frac{(\tilde{m} - n \tau_1)^2 + n^2 \tau_2^2}{\tau_2} \right]$$

$$= \underbrace{\frac{R}{\ell_s} \frac{1}{\sqrt{\tau_2} \eta \bar{\eta}}}_{\text{modular invariant}} \cdot \underbrace{\sum_{\tilde{m}, n} \exp \left[-\frac{\pi R^2}{\ell_s^2} \frac{|\tilde{m} - n \tau|^2}{\tau_2} \right]}_{\text{invariant under } \tau \rightarrow \frac{a\tau + b}{c\tau + d}}$$

$$\begin{pmatrix} \tilde{m} \\ n \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tilde{m} \\ n \end{pmatrix}$$

This can be viewed as a sum over classical configurations

$$X = 2\pi R \left(\tilde{m} \sigma_2 + n \sigma_2 \right) + X_{\text{fluct.}}$$

\uparrow worldsheet time \uparrow worldsheet space.

* For $R > l_s$ the only massless states are those with $m=n=0$
 \rightarrow just the dimensional reduction of the D -dimensional spectrum

$\alpha_{-1}^\mu \tilde{\alpha}_{-1}^\nu |0\rangle$: graviton, $B_{\mu\nu}$, dilaton in $D-1$ dimension

$\alpha_{-1}^{D-1} \tilde{\alpha}_{-1}^\mu |0\rangle, \alpha_{-1}^\mu \tilde{\alpha}_{-1}^{D-1} |0\rangle$: two gauge fields $G_{\mu y}, B_{\mu y}$

$\alpha_{-1}^{D-1} \tilde{\alpha}_{-1}^{D-1} |0\rangle$: a scalar field $R \rightarrow$ radius of S^1

The conserved charge associated to $G_{\mu y} =$ momentum m
 $B_{\mu y} =$ winding w

* For $R = l_s$, new massless states appear:

$$\alpha_{-1}^\mu |m = \pm 1, n = \mp 1\rangle = W_\mu^\pm$$

$$\tilde{\alpha}_{-1}^\mu |m = \pm 1, n = \pm 1\rangle = \tilde{W}_\mu^\pm$$

have mass $M_{0-1}^2 = \left(\frac{1}{R} - \frac{R}{l_s^2} \right)^2 \rightarrow 0$ at $R = l_s$
 'self dual radius'

At this point, the Abelian symmetry $U(1) \times U(1)$

is enhanced to $SU(2) \times SU(2)$

$$\begin{aligned} &\hookrightarrow W_\mu^\pm \text{ and } \omega_\mu - \tilde{\omega}_\mu \\ &\hookrightarrow \tilde{W}_\mu^\pm \text{ and } \omega_\mu + \tilde{\omega}_\mu \end{aligned}$$

T-duality $R \rightarrow \frac{l_s^2}{R}$ can be viewed as a gauge transformation

$\mathbb{Z}_2 \subset SU(2)$!

1.3 Circle compactification for open strings

Open strings with Neumann boundary conditions cannot wind, so the spectrum of open bosonic strings on $S_1(\mathbb{R})$ is just

$$M_{D-1}^2 = M_D^2 + \left(\frac{m}{R}\right)^2 \quad m \in \mathbb{Z}$$

momentum

like in QFT.

Thus, the D -dim massless gauge boson A_M reduces to a $(D-1)$ -dim massless gauge boson A_μ

↓ scalar $\chi = \int A_y dy$: holonomy of A_M around the circle
 + their Kaluza-Klein towers.

If $\chi \neq 0$, states which carry charge q under A_M must have momentum $p_{D-1} = \frac{m}{R} - \frac{\chi q}{2\pi R}$

such that the wavefunction $e^{i p_{D-1} x^{D-1}}$ transforms with a phase $e^{-iq\chi}$ under $x^{D-1} \rightarrow x^{D-1} + 2\pi R$

Yet we can ask what is the effect of replacing

$X_L + X_R$ by $X_L - X_R$;

The mode expansion for Neumann bc

$$X = x + 2\alpha' p z + i\sqrt{2} \alpha' \sum_{n \neq 0} \frac{\alpha_n}{n} \cos k\sigma e^{-ikz}$$

becomes the one for Dirichlet conditions:

$$\tilde{X} = \tilde{x} + 2\alpha' p \sigma + i\sqrt{2} \alpha' \sum_{n \neq 0} \frac{\alpha_n}{n} \sin k\sigma e^{-ikz}$$

↑ vanishes at $\sigma = 0, \pi$

hence $\tilde{X}(\sigma=0, \tau) = \tilde{x}$

$$\tilde{X}(\sigma=\pi, \tau) = 2\alpha l_s^2 p = 2\pi l_s^2 \frac{m}{R} = 2\pi m \tilde{R}$$

Thus the open string winds m times around the dual circle, and its ends are attached on a D-brane located at $X = \tilde{x} \text{ mod } 2\pi \tilde{R}!$

\Rightarrow D-branes are required by T-duality

D_pbrane wrapped around $S^1 \xrightarrow{T} \text{D}(p-1)\text{brane at fixed pos on } \tilde{S}^1$

D_p-branes can be viewed as solitons of string theory. They carry a worldvolume gauge field $A_{p=0 \dots p}$, and $25-p$ scalar fields corresponding to their transverse fluctuations.

Their dynamics is induced by fluctuations of open strings, and described at low energies by the Born-Infeld action

$$S = -T_p \int d^{p+1}x e^{-\phi} \sqrt{-\det(G_{\mu\nu} + 2\alpha l_s^2 F_{\mu\nu})}$$

\nearrow follow from disk diagram \uparrow target space metric.

 closed string mode

They can also be described by semi-classical solutions of the low energy effective theory of closed strings

Many applications to model building (Universe as a brane)

- strongly coupled gauge theories (AdS/CFT)

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