Quantum Attractor Flows and Black Hole Partition Functions

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References

- Summary: Gunaydin, Neitzke, BP and Waldron [hep-th/0512296]
- Preview: Lecture notes, BP [hep-th/0607227]
- Geometry: Neitzke, BP and Vandoren [hep-th/0701214]
- Physics: Gunaydin, Neitzke, BP and Waldron [arXiv:0707.0267]
- Repres. Theory: Gunaydin, Neitzke, BP [arXiv:0707.1669]

Introduction

- As shown by Strominger, Vafa and many others, string theory provides a good microscopic understanding of the Bekenstein-Hawking entropy of a large class of extremal and near-extremal black holes in D=4 and D=5 supergravity.
- More recently, much progress has been made in extending this agreement beyond the thermodynamical (large charge) limit, where higher-derivative corrections in the low energy effective action and subleading corrections to Cardy's formula become important.
- In some cases, exact formulae for the degeneracies of black hole micro-states have been proposed, and tested with some success:

Exact black hole degeneracies

- For 1/4 BPS dyonic black holes in $\mathcal{N} = 4$, $D = 4$ supergravity, DVV have conjectured

$$\Omega(q_e, q_m) = (-1)^{q_e \cdot q_m} \oint \frac{e^{i(q_e^2 \rho + q_m^2 \sigma + 2 q_e \cdot q_m \nu)}}{\Phi_{10}(\rho, \sigma, \nu)} \, d\rho \, d\sigma \, d\nu$$

where Φ_{10} is a Siegel cusp form of weight 10.

- For 1/2 BPS- black holes in $\mathcal{N} = 2$, $D = 4$ supergravity, OSV have proposed

$$\Omega(p^I, q_I) \sim \int d\phi^I \, |\Psi_{top}(p^I + i\phi^I)|^2 \, e^{\phi^I q_I}$$

where $\Psi_{top}(p^I)$ is the topological string amplitude in the real polarization.

Automorphic Black Hole Partition Functions I

- Both of these proposals have been "proven" several times over. Yet they still raise questions: Why should $\Omega(q_e, q_m)$ depend on $q_e^2, q_m^2, q_e \cdot q_m$ only ? What is the physical origin of the $Sp(4, \mathbb{Z})$ symmetry ? How to incorporate multi-centered configurations and lines of marginal stability in OSV formula ? etc.
- The goal of this talk will be to propose a general framework for constructing black hole partition functions, inspired by both of these proposals, which can potentially resolve these difficulties.

Automorphic Black Hole Partition Functions I

- In particular, the three-dimensional duality group $G_3(\mathbb{Z})$ is proposed to play the rôle of a spectrum generating symmetry for black holes in 4 dimensions. This is closely related to the fact that black holes in D=4 correspond to instantons in D=3.
- More specifically, we propose that the partition function of black hole micro-states is an automorphic form of $G_3(\mathbb{Z})$, "attached" to a particular representation of $G_3(\mathbb{R})$ obtained by performing the radial quantization of stationary, spherically symmetric BH: hence we'll study *quantum attractors*.
- For $\mathcal{N} = 4$ SUGRA, this suggests that the Siegel modular form should be replaced by an automorphic form of $SO(8, 24, \mathbb{Z})$. For $\mathcal{N} = 2$ SUGRA, this suggests the existence of a one-parameter generalization of the topological string amplitude, and an automorphic form attached to any Calabi-Yau 3-fold.

Outline

Outline

Stationary solutions and KK* reduction I

- Stationary solutions in $D = 3 + 1$ gravity can be parameterized as

$$ds_4^2 = -e^{2U}(dt + \omega)^2 + e^{-2U} ds_3^2 \ , \quad A_4^I = \zeta^I dt + A_3^I$$

  where $ds_3, U, \omega, A_3^I, \zeta^I$ and the 4D scalars $z^i \in \mathcal{M}_4$ are independent of time. The $D = 3 + 1$ theory reduces to a field theory in three Euclidean dimensions.

- In contrast to the usual KK ansatz,

$$ds_4^2 = e^{2U}(dy + \omega)^2 + e^{-2U} ds_{2,1}^2 \ , \quad A_4^I = \zeta^I dy + A_3^I$$

  we reduce along a time-like direction.

- For the usual KK reduction to $D = 2 + 1$, the one-forms (A_3^I, ω) can be dualized into pseudo-scalars $(\tilde{\zeta}_I, \sigma)$, where σ is the twist (or NUT) potential. The 4D Einstein-Maxwell equations reduce to 3D gravity + scalars living in a Riemannian space

$$\mathcal{M}_3 = R^+|_U \times \mathcal{M}_4 \times |_{z^i}\mathbb{R}^{2n_v+3}|_{\zeta^I,\tilde{\zeta}_I,\sigma}$$

with positive-definite metric

$$ds^2 = 2(dU)^2 + g_{ij}dz^i dz^j + \frac{1}{2}e^{-4U}\left(d\sigma + \zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I\right)^2$$
$$+-e^{-2U}\left[t_{IJ}d\zeta^I d\zeta^J + t^{IJ}\left(d\tilde{\zeta}_I + \theta_{IK}d\zeta^K\right)\left(d\tilde{\zeta}_J + \theta_{JL}d\zeta^L\right)\right]$$

Stationary solutions and KK* reduction III

- The KK* reduction is simply related to the KK reduction by letting $(\zeta^I, \tilde{\zeta}_I) \to i(\zeta^I, \tilde{\zeta}_I)$. As a result, the scalar fields live in a pseudo–Riemannian space $\mathcal{M}_3^*$, with non-positive definite signature.
- $\mathcal{M}_3^*$ always has $2n_V + 4$ isometries corresponding to the shifts of $\zeta, \tilde{\zeta}_I, \sigma, U$, satisfying the graded Heisenberg algebra

$$\begin{aligned} [p^I, q_J] &= 2\delta^I_J\, k \\ \left[m, p^I\right] = p^I\ , \quad [m, q_I] &= q_I\ , \quad [m, k] = 2k \end{aligned}$$

- The notation anticipates the identification of the corresponding conserved charges with the electric and magnetic charges q_I and p_I, NUT charge k and ADM mass m.

G_3 as a solution generating symmetry I

- Some times, $\mathcal{M}_3^*$ has more isometries or structure:

4D	Sigma model in 3D
Pure Einstein-Gravity	$Sl(2)/U(1)$
Einstein-Maxwell	$SU(2,1)/Sl(2) \times U(1)$
N=2 Supergravity	Quaternionic-Kähler manifold
N=4 supergravity	$SO(8, n_v + 2)/SO(8) \times SO(n_v + 2)$
N=8 supergravity	$E_8/SO^*(16)$

Ehlers; Kinnersley; Mazur; Breitenlohner Gibbons Maison

- When $\mathcal{M}_3^* = G_3/K_3$ is a symmetric space, the group G_3 is a solution generating symmetry for stationary solutions in 4D !
- 5D Black holes with $U(1)$ isometry can also be described that way.

Spherically symmetric BH and geodesics I

- Now, restrict to spherically symmetric solutions, with spatial slices

$$ds_3^2 = N^2(\rho) d\rho^2 + r^2(\rho) d\Omega_2^2$$

- The sigma-model action becomes, up to a total derivative (G_{ab} is the metric on $\mathcal{M}_3^*$):

$$S = \int d\rho \left[\frac{N}{2} + \frac{1}{2N} \left(\dot{r}^2 - r^2 G_{ab} \dot{\phi}^a \dot{\phi}^b \right) \right]$$

- This is the Lagrangian for the geodesic motion of a fiducial particle with unit mass on the (hyperbolic) cone $\mathbb{R}^+ \times \mathcal{M}_3^*$. The einbein $\sqrt{N}$ enforces invariance under reparameterizations of ρ.

Spherically symmetric BH and geodesics II

- The equation of motion of N imposes the Hamiltonian constraint, or Wheeler-De Witt equation

$$H_{WDW} = (p_r)^2 - \frac{1}{r^2} G^{ab} p_a p_b - 1 \equiv 0$$

- The gauge choice $N = r^2$ allows to separate the problem into radial motion along r, and geodesic motion on $\mathcal{M}_3^*$:

$$G^{ab} p_a p_b = C^2 \,, \quad (p_r)^2 - \frac{C^2}{r^2} - 1 \equiv 0 \quad \Rightarrow \quad r = \frac{C}{\sinh C\rho} \,,$$

Thus, the problem reduces to affinely parameterized geodesic motion on the three-dimensional moduli space $\mathcal{M}_3^*$.

Spherically symmetric BH and geodesics III

- It turns out that $C = 2T_H S_{BH}$ is the extremality parameter: extremal (in particular BPS) black holes correspond to light-like geodesics on $\mathcal{M}_3^*$. Since $r = 1/\rho$, the 3D spatial slices are flat.
- Other gauges are also possible: e.g. $N = e^U$ identifies ρ with the radial geodesic distance to the horizon.
- For the purpose of defining observables such as the horizon area, $A_H = 4\pi e^{-2U} r^2|_{U\to-\infty}$ and ADM mass $M_{ADM} = r(e^{2U} - 1)|_{U\to 0}$, it may convenient to leave the gauge unfixed.

Isometries and conserved charges

- The isometries of $\mathcal{M}_3$ imply conserved Noether charges, whose Poisson bracket reflect the Lie algebra of the isometries:

$$\begin{aligned} [p^I, q_J] &= 2\delta^I_J\, k \\ \left[m, p^I\right] = p^I, \quad [m, q_I] &= q_I\ , \quad [m, k] = 2k \end{aligned}$$

- If $k \neq 0$, the off-diagonal term in the 4D metric

$$ds_4^2 = -e^{2U}(dt + k\cos\theta d\phi)^2 + e^{-2U}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]$$

implies the existence of closed time-like curves around ϕ direction, near $\theta = 0$.

- Bona fide 4D black holes arise in the “classical limit” $k \to 0$. Keeping $k \neq 0$ will allow us to greatly extend the symmetry.

Conserved charges and black hole potential

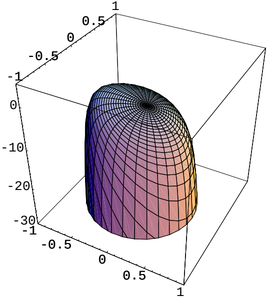
- Setting $k = 0$ for simplicity, one arrives at the Hamiltonian,

$$H = \frac{1}{2}\left[p_U^2 + p_i g^{ij} p_j - e^{2U} V_{BH}\right] \equiv C^2$$

where V_{BH} is the "black hole potential",

$$V_{BH}(z^i, p^I, q_I) = \frac{1}{2}(q_I - \mathcal{N}_{IJ}p^J)t^{IK}(q_K - \bar{\mathcal{N}}_{KL}p^L) + \frac{1}{2}p^I t_{IJ} p^J$$

- The potential $V = -e^{2U} V_{BH}$ is unbounded from below.



Quantizing geodesic motion I

- The classical phase space is the cotangent bundle $T^*(\mathcal{M}_3^*)$, specifying the initial position and velocity.
- Quantization proceeds by replacing functions on phase space by operators acting on wave functions in $L_2(\mathcal{M}_3^*)$, subject to

$$\Delta_3 \Psi(U, z^i, \zeta^I, \tilde{\zeta}_I, \sigma) = C^2 \Psi$$

  where Δ_3 is the Laplace-Beltrami operator on $\mathcal{M}_3^*$.
- The electric, magnetic and NUT charges may be diagonalized as

$$\Psi(U, z^i, \zeta^I, \tilde{\zeta}_I, \sigma) = \Psi_{p,q}(U, z)\ e^{i(q_I \zeta^I + p^I \tilde{\zeta}_I)}$$

$$\left[-\partial_U^2 - \Delta_4 - e^{2U} V_{BH} - C^2 \right] \Psi_{p,q}(U, z) = 0$$

Quantizing geodesic motion II

- The black hole wave function $\Psi_{p,q}(U,z)$ describes quantum fluctuations of the 4D moduli as one reaches the horizon at $U \to -\infty$. Naively, it should be peaked at the attractor point.
- Restoring the variable r, one could also describe the quantum fluctuations of the horizon area $4\pi r^2 e^{-2U}$, around the classical value $4 S_{BH}$.
- The natural inner product is the Klein-Gordon inner product at fixed U, famously NOT positive definite. A standard remedy in quantum cosmology is "third quantization", possibly relevant for black hole fragmentation / multi-centered solutions.

Attractor flow in $N = 2$ supergravity

- Consider $N = 2$ SUGRA coupled to n_V abelian vector multiplets [*hypers decouple at tree-level*]: the vector multiplet scalars z^i take values in a special Kähler manifold $\mathcal{M}_4$. For type IIA on $X = CY_3$, z^i parameterize the complexified Kähler structure of X.
- After reduction to 3 dimensions, the vector multiplet scalars take value in a quaternionic-Kähler space $\mathcal{M}_3$, known as the $c - map$ of the special Kähler space $\mathcal{M}_4$.
- Under T-duality along the 4th direction, this becomes the hypermultiplet space for type IIB compactified on X at tree-level.
- The manifold $\mathcal{M}_3^*$ obtained by analytic continuation is sometimes called "para-quaternionic-Kahler manifold"; it has split signature $(2n_V + 2, 2n_V + 2)$

Cortes Mayer Mohaupt Saueressig

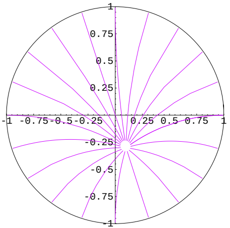
Attractor flow and semi-classical BPS wave function

- The black hole potential splits into two pieces,

$$H = \frac{1}{2}\left[p_U^2 + p_i g^{i\bar{j}} p_{\bar{j}} - e^{2U}\left(|Z|^2 + \partial_i|Z|\; g^{i\bar{j}}\; \partial_{\bar{j}}|Z|\right)\right]$$

where Z is the central charge $Z = e^{K/2}(q_I X^I - p^I F_I)$.

- Supersymmetric solutions are obtained by cancelling each term separately, leading back to the attractor flow equations,



$$\begin{aligned} r^2 \frac{dU}{dr} &= e^U |Z| \\ r^2 \frac{dz^i}{dr} &= 2 e^U g_{i\bar{j}}\, \partial_{\bar{j}} |Z| \end{aligned}$$

Attractor flow and semi-classical BPS wave function

- At this stage, one could already quantize the attractor flow equations and guess the BPS wave function:

$$\begin{cases} p_U & = & -e^U|Z| \\ p_{\bar{z}^{\bar{i}}} & = & -e^U\partial_{\bar{i}}|Z| \end{cases} \quad \Rightarrow \Psi_{p,q}(U, z^i, \bar{z}^{\bar{i}}) \sim \exp\left[2ie^U|Z|\right]$$

  The effective Planck constant $\hbar = e^{-U}$ blows up towards the horizon at $U \to -\infty$. The phase is stationary at the classical attractor points in the opposite limit $U \to +\infty$.
- Using twistor techniques, we shall be able to resolve ordering ambiguities, and compute the BPS wave function exactly.

Supersymmetric quantum mechanics

- More rigorously, the full $D = 4, N = 2$ SUGRA including fermions, reduces to $D = 1, N = 4$ supergravity:

$$S = \int d\rho \; G_{ab}\dot{\phi}^a\dot{\phi}^b + \psi^A \frac{D}{D\rho}\psi_A + (\psi^A\psi_A)(\psi^A\psi_A) + \dots$$

- The supersymmetry variations are $\delta\psi^A = V^{AA'}\epsilon_{A'}$, where $V^{AA'}$ $(A = 1,..2n_V + 2, A' = 1, 2)$ is the quaternionic vielbein afforded by the restricted holonomy $Sp(2) \times Sp(2n_V + 2)$.
- Thus, SUSY trajectories are characterized by

$$\exists \epsilon_\alpha \;/\; V_\mu^{AA'}\;\dot{\phi}^\mu\;\epsilon_{A'} = 0 \quad \Leftrightarrow \quad V^{A[A'}V^{B']B} = 0$$

This reproduces the attractor flow equations (generalized to $k \neq 0$)

Gutperle Spalinski

Outline

Improved SUSY mechanics - HKC and twistors I

- This SUSY mechanics is rather unusual, insofar as the SUSY comes from a triplet of non-integrable almost complex structures.
- It is possible to remedy this problem by combining the Killing spinor $\epsilon_{A'} \in \mathbb{C}^2$ with the coordinates $\phi^a \in QK$, i.e. extend the QK space into its Hyperkähler cone (HKC), or Swann bundle,

$$\mathbb{R}^4 \to HKC \to QK$$

  By cancelling the $Sp(2)$ holonomy on QK against the $SU(2)$ holonomy on S^3, the three almost complex structures on QK become genuine (integrable) complex structures on HKC.
- Geodesic motion on HKC is equivalent to geodesic motion on QK after gauging the $SU(2)$ and dilation symmetries.

The twistor space

- For many purposes it is sufficient to work with the twistor space Z, a two-sphere bundle over QK with a Kähler-Einstein metric. The sphere coordinate z keeps track of the Killing spinor, $z = \epsilon_1/\epsilon_2$.
- In the presence of triholomorphic isometries, the geometry of HKC is given by the "Legendre transform construction" $G(\eta^L)$,

$$\langle K(v^L, \bar{v}^L, w_L + \bar{w}_L) + x^L(w_L + \bar{w}_L)\rangle_{w+\bar{w}} = \mathrm{Im} \oint \frac{d\zeta}{2\pi i \zeta} G[\eta^L(\zeta), \zeta]$$

where η^L is the projective "O(2) multiplet"

$$\eta^L = v^L/\zeta + x^L - \bar{v}^L \zeta$$

and $G[\eta^L]$ is a holomorphic function of η^L, homogeneous of degree 1, known as the generalized prepotential.

Hitchin Lindstrom Rocek; De Wit Rocek Vandoren

Twistor space for the c-map

- When HKC is the Swann bundle of the c-map of a SK manifold, the generalized prepotential G is related to the prepotential F via

$$G(\eta^L, \zeta) = F(\eta^I)/\eta^\flat$$

Berkovits; Rocek Vafa Vandoren

- The inhomogeneous coordinates $\xi^I = v^I/v^\flat, \tilde{\xi}_I = -2iw_I$, $\alpha = 4iw_\flat - \xi^I \tilde{\xi}_I$ are complex coordinates on Z, adapted to the Heisenberg symmetries, given by the twistor lines:

$$\begin{aligned} \xi^I &= \zeta^I + i\, e^{U+\mathcal{K}(X)/2} \left(z\, \bar{X}^I + z^{-1} X^I \right) \\ \tilde{\xi}_I &= \tilde{\zeta}_I + i\, e^{U+\mathcal{K}(X)/2} \left(z\, \bar{F}_I + z^{-1}\, F_I \right) \\ \alpha &= \sigma + \zeta^I \tilde{\xi}_I - \tilde{\zeta}_I \xi^I \end{aligned}$$

- Conversely, the coordinates on the base $\mathcal{M}_3$ are $SU(2)$ invariant combinations of $\xi^I, \tilde{\xi}_I, \alpha$.

Neitkze BP Vandoren 07

BPS black holes and holomorphic curves

- Upon lifting the geodesic motion to Z, SUSY is preserved iff the momentum is holomorphic in the canonical complex structure on Z, at any point along the trajectory: 1st class constraints !
- BPS solutions correspond to holomorphic curves $\xi^I(\rho), \tilde{\xi}_I(\rho), \alpha(\rho)$ at constant $\bar{\xi}^I, \bar{\tilde{\xi}}_I, \bar{\alpha}$, and are algebraically determined by the conserved charges: integrable system !
- The SUSY phase space is the twistor space Z, equipped with its Kähler symplectic form. Its dimension is $4n_V + 6$, almost half that of the generic phase space $T^*(\mathcal{M}_3^*)$.

The Penrose Transform

- At fixed values of $U, z^i, \zeta^I, \tilde{\zeta}_I, \sigma$, the complex coordinates $\xi^I, \tilde{\xi}_I, \alpha$ on Z are holomorphic functions of the twistor coordinate z: the fiber over each point is a rational curve in Z.
- Starting from a holomorphic function Φ on Z (more precisely a class in $H^1(Z, O(-2))$), we can produce a function Ψ on QK

$$\Psi(U, z^i, \bar{z}^{\bar{i}}, \zeta^I, \tilde{\zeta}_I, \sigma) = e^{2U} \oint \frac{dz}{2\pi i z} \Phi\left[\xi^I(z), \tilde{\xi}^I(z), \alpha(z)\right]$$

which then satisfies some generalized harmonicity condition:

$$\left(\epsilon^{A'B'} \nabla_{AA'} \nabla_{BB'} - R_{AB}\right) \Psi = 0$$

- This generalizes the usual Penrose transform between holomorphic functions on CP^3 and conformally harmonic functions on S^4 to the quaternionic setting.

Salamon; Baston Eastwood

Outline

The BPS Hilbert space I

- In terms of geodesic motion on the QK base, the classical BPS conditions $V^{A[A'}V^{B']B} = 0$ become a set of 2nd order differential operators which have to annihilate the wave function Ψ:

$$\left(\epsilon_{A'B'}\nabla^{AA'}\nabla^{BB'} - R^{AB}\right)\Psi = 0$$

- In terms of the twistor space, the BPS condition $p_{\bar{L}} = 0$ requires that Ψ should be a holomorphic function on Z. More precisely, depending on the fermionic state, it should be a class in the sheaf cohomology group $H^1(Z, \mathcal{O}(-\ell))$. Take $\ell = 2$ for simplicity.
- The equivalence between the two approaches is a consequence of the Penrose transform discussed previously.

The BPS Black Hole Wave-Function I

- Thus, the BPS black hole wave function on $\mathcal{M}_3$ is given by

$$\Psi(U, z^i, \bar{z}^I, \zeta^I, \tilde{\zeta}_I, \sigma) = e^{2U} \oint \frac{dz}{2\pi i z} \, \Phi\left[\xi^I(z), \tilde{\xi}^I(z), \alpha(z)\right]$$

where Ψ is entirely determined (up to normalization) by the black hole charges. For zero NUT charge k,

$$\begin{aligned} \Phi &= \exp\left[i(p^I\tilde{\xi}_I - q_I\xi^I)\right] \\ &= \exp\left[i(p^I\tilde{\zeta}_I - q_I\zeta^I) + ie^{U+K(X)/2}(z\bar{W}_{p,q}(\bar{X}) + z^{-1}W_{p,q}(X))\right] \end{aligned}$$

$$\Rightarrow \quad \Psi = e^{2U} \, J_0\left(2 \, e^U \, |Z_{p,q}|\right) \, e^{i(p^I\tilde{\zeta}_I - q_I\zeta^I)}$$

- The exact result is in qualitative agreement with our naive guess $\exp(2ie^U\,|Z_{p,q}|)$. This is peaked around the classical attractor points in the “far horizon” limit $U \to \infty$, but quantum fluctuations become infinite near the horizon.
- This is probably due to the large fine-tuning needed to produce a BPS solution. Can this be taken as support for the “fuzzball proposal” ?
- Ooguri, Vafa and Verlinde had proposed to interpret the topological string amplitude, which lives in a Hilbert space of dimension $n_v + 1$, as a wave function for the (near horizon) radial quantization of BPS black holes. Instead, the BPS radial quantization produces a much bigger Hilbert space $H_l(Z, O(-l))$, of functional dimension $2n_v + 3$.

Generalized topological amplitude

- On the other hand, the holomorphic quantization of the HKC leads to a Hilbert space of dimension $(4n_v + 8)/4 = n_v + 2$, morally the space of "triholomorphic functions" on HKC.
- In the symmetric cases, this still carries a unitary rep of G_3 known as the minimal representation. Upon fixing the value of k, this yields the Schödinger-Weil representation of G_4, the usual habitat of the topological string amplitude !
- This suggests a one-parameter generalization of the topological string amplitude, controlling higher-derivative corrections on hypermultiplet spaces.

Outline

Black Hole Partition Functions

- The moduli space $\mathcal{M}_3 = R^+|_U \times \mathcal{M}_4 \times |_{z^i}\mathbb{R}^{2n_v+3}|_{\zeta^I,\tilde{\zeta}_I,\sigma}$ appears to provide all the desirable parameters for a partition function for black hole micro-states: the inverse temperature $\beta = e^{2U}$, asymptotic moduli z^i, chemical potentials $\zeta^I, \tilde{\zeta}_I$.
- Upon compactification to D=3, the effective action will receive instanton contributions from black holes winding the Euclidean time direction, and will have to be invariant under $G_3(\mathbb{Z})$.
- This suggests that the exact degeneracies of black hole micro-states should be given by Fourier coefficients of an automorphic form of $G_3(\mathbb{Z})$.

Which automorphic form ?

- Automorphic forms of G generally require three ingredients: (i) a unitary representation ρ of G, (ii) a K-invariant (or spherical vector) f_K, and (iii) a $G(\mathbb{Z})$-invariant vector $f_{\mathbb{Z}}$:

$$\Psi(g) = \langle f_{\mathbb{Z}} | \rho(g) | f_K \rangle$$

  For example the Jacobi theta series is obtained from the metaplectic representation of $Sl(2,\mathbb{Z})$, using $f_K = e^{-x^2}$ and $f_{\mathbb{Z}} = \sum_{m\in\mathbb{Z}} \delta_{x-m}$.
- The radial quantization of spherically symmetric black holes provides a unitary representation of G_3. In particular, the BPS Hilbert space $H_1(Z, O(-\ell))$ furnishes a family of quaternionic discrete| (QD) of G_3.

Abelian and non-Abelian Fourier coefficients

- Upon Fourier expanding in $\zeta^I, \tilde{\zeta}_I, \sigma$, one gets Abelian ($k=0$) and non-Abelian ($k \neq 0$) Fourier coefficients:

$$\Psi = \sum_{p,q} \Omega(p,q) \Psi_{p,q}(U, z^i, \bar{z}^i)\, e^{i(q_I \zeta^I - p^I \tilde{\zeta}_I)} + \sum_k \sum_{r^I \in \mathbb{Z}/k\mathbb{Z}}$$

$$\Omega(k, r^I) \sum_{p^I} f_k(U, z^i, \bar{z}^i; \zeta^I + k p^I) \exp\left[i(kp^I + r^I)\tilde{\zeta}_I + ik(\sigma - \zeta^I \tilde{\zeta}_I) \right]$$

When Ψ is in the quaternionic discrete series, $\Psi_{p,q}(U, z^i, \bar{z}^i)$ is the black hole wave function that we have computed !

Black hole degeneracies as Fourier coefficients

- We propose that $\Omega(p, q)$ are the exact black hole degeneracies, for a suitable choice of ℓ and Ψ. To see that we may be on the right track, note Wallach's theorem: $\Omega(p, q) = 0$ unless $I_4(p, q) \geq 0$.
- For 1/4-BPS BH in $\mathcal{N} = 4$ SUGRA, this suggests that the DVV-type formula, based on a Siegel modular form, should be subsumed into an automorphic form of $SO(8, n_v + 2, \mathbb{Z})$ in the QD series.
- For 1/8-BPS BH in $\mathcal{N} = 8$ SUGRA, we expect an automorphic form of $E_{8(8)}$ in the QD series, of Kirillov dimension 57. For 1/2 BPS, in the minimal representation, of Kirillov dimension 29.
- In order to reproduce the growth $\Omega(p, q) \sim \exp[\pi\sqrt{I_4(p, q)}]$, we need to allow for singularities worse than the poles appearing in DVV's formula. No concrete candidate yet.

Outlook

- Multi-centered configurations can be described by certain harmonic maps from $\mathbb{R}^3$ to QK: does that correspond to “second quantization”, i.e. including vertices ?
- Could one compute the radial wave function for extremal non-BPS black holes ? need to implement the fine-tuning of the boundary conditions at infinity.
- Can one construct automorphic forms of G in the quaternionic discrete series, with suitable exponential growth of Fourier coefficients ? Eg. via theta-lifts, or residues of Eisenstein series.
- Can one make progress on understanding instanton corrections to hypermultiplets using these techniques ?