Conformal quantum mechanics and quantum cosmology

Boris Pioline

LPTHE, Paris

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w/ A. Waldron, hep-th/0209044
w/ D. Kazhdan, A. Waldron, hep-th/0107222

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Conformal symmetry in various dimensions

- In the extreme infrared or ultraviolet regime, physics is described by a fixed point of the renormalization group, most often trivial, sometimes not: invariance under the Lorentz (super)group $SO(1,d-1)$ is then promoted to an invariance under the (super)conformal group $SO(2,d)$, including dilation and special conformal generators.

- This is especially powerful in study of critical phenomena in 2 dimensions, where the infinite dimensional Virasoro extension of $SO(2,2)$ often allows to determine critical exponents valid within universality classes of theories.

- Conformal invariance also occurs in a number of 4-dimensional gauge theories such as $N=4$ SYM or softly broken versions thereof, which remain conformal quantum mechanically at high energy; as embodied in the AdS/CFT correspondence.

- Non-trivial conformal fixed points can also surprisingly be found in gauge theories in higher dimensions: the UV behaviour of $D=5$ SYM is described by *(2,0) theory in* $D=5+1$, etc

Conformal symmetry in one dimension

- The conformal group in $D = 0 + 1$, $SO(2,1) = Sl(2,R) = Sp(1)$, corresponds to invariance under fractional transformations of time, $t \to (at+b)/(ct+d)$, with $ad - bc = 1$.

- The simplest example was introduced by de Alfaro Fubini Furlan (DFF, 1976) : a 1-dimensional non-relativistic particle in a $1/x^2$ potential.

- The near-horizon geometry of charged (Reissner-Nordström) black holes, $AdS_2 \times S_2$, has isometry group $Sl(2) \times SU(2)$. The $Sl(2)$ symmetry is realized in the dynamics of test particles through conformal transformations of time.

- More generally, string theory on AdS_2 should have a dual description through a conformal quantum mechanical "gauge" theory in 1D: *AdS fragmentation,. . .*

- In this talk, we will consider a different setting where $D = 0{+}1$ conformal invariance arises: gravity near a space-like (cosmological) singularity.

Outline

1. Conformal quantum mechanics and black-holes

2. Quantum cosmology at a spacelike singularity

3. Coadjoint orbits and generalized CQM

4. Outlook

Conformal quantum mechanics

- Conformal quantum mechanics was first introduced in 1976 by de Alfaro, Fubini and Furlan (DFF) as an attempt to understand soft breaking of conformal invariance:

$$L = \frac{1}{2}\left(\frac{dq}{dt}\right)^2 - \frac{g}{q^2}\ , \qquad g > 0$$

- The Lagrangian is invariant under conformal transformations of the time axis (up to total der.),

$$t \to \frac{at+b}{ct+d}\ , \quad q(t) \to \frac{q(t)}{ct+d}\ , \quad ad - bc = 1$$

- The Noether charges generating these transformations at $t = 0$ read

$$E_+ = \frac{1}{2}\left(p^2 + \frac{g}{q^2}\right) = H\ , \quad D = -\frac{1}{2}pq\ , \quad E_- = \frac{1}{2}q^2$$

- They represent the conformal group $SO(2,1) = Sl(2)$ in 0+1 dimensions,

$$\{E_+, E_-\} = 2D, \quad \{D, E_\pm\} = \pm E_\pm$$

- Upon quantization, $p \to i\partial_q$, conformal invariance fixes ordering ambiguities:

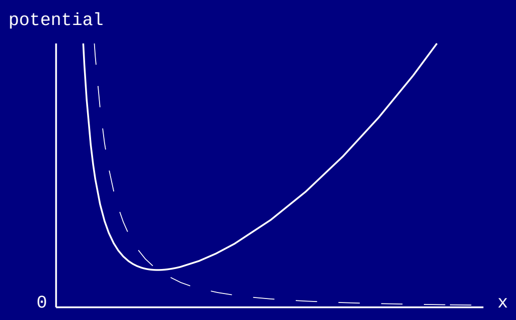
$$E_+ = \frac{1}{2}\left(p^2 + \frac{g}{q^2}\right) = H\ , \quad D = -\frac{1}{4}(pq+qp)\ , \quad E_- = \frac{1}{2}q^2$$

Mass can preserves conformal invariance !

- The Hamiltonian $H = E_+$ is a parabolic element of $SO(2,1)$. It has a delta-normalizable continuous positive spectrum starting at 0, with eigenfunctions

$$\psi_E(q) = q^{1/2} J_{2r-1}(x\sqrt{2E}) \to q^{2r-1/2} \text{ as } E \to 0$$

- The spectrum may be rendered discrete by deforming the Hamiltonian into $H = E_+ + \Lambda^2 E_-$ where $1/\Lambda$ is a new length scale, which can however be changed by acting with D.



- The Hamiltonian is now a compact (elliptic) element of $SO(2,1)$, with discrete normalizable spectrum, generated by the rising and lowering operators,

$$L_\pm = E_+ - E_- \pm iD \ , \quad [H, L_\pm] = \pm L_\pm$$

acting on the vacuum,

$$L_- \psi_0 = 0 \quad \Rightarrow \quad \psi_0(q) = q^{2r-\frac{1}{2}} e^{-q^2\Lambda^2/2}$$

- We thus have an (even) integer spaced spectrum, with eigenmodes

$$\psi_n(x) = (q\Lambda)^{2r-\frac{1}{2}} \, e^{-q^2\Lambda^2/2} L_n^{2r-1}(q^2\Lambda^2)$$

where L_n are Laguerre polynomials.

CQM and RN black holes

- Reissner-Nordström black holes have a near-horizon geometry given by $AdS_2 \times S_2$,

$$\begin{aligned} ds^2 &= -(2M/r)^4 dt^2 + (2M/r)^2 dr^2 + M^2 d\Omega^2 \; , \\ A &= (2M/r)^2 dt \end{aligned}$$

- The Hamiltonian of a free particle of mass m and charge q in static gauge is, in the limit $M \to \infty$ with $M^2(m-q)$ fixed,

$$H = \frac{p_r^2}{2m} + \frac{g}{2r^2} \, ,$$

with

$$g = 8M^2(m-q) + 4\ell(\ell+1)/m \, .$$

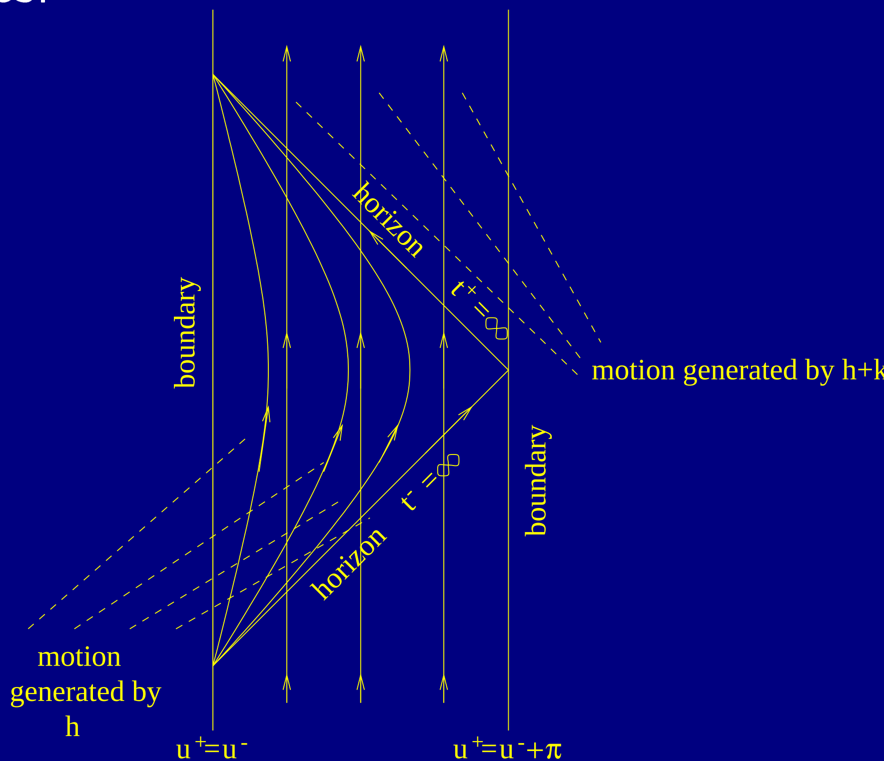
Claus Derix Kallosh Kumar Townsend Van Proeyen

- At finite M, one has in fact a relativistic generalization of DFF. A superconformal version of this model can also be found by considering a superparticle on the near-horizon geometry.

Claus Derix Kallosh Kumar Townsend Van Proeyen
de Azcarraga Izquierdo Perez Bueno Townsend

From parabolic to elliptic, now justified

- Instead of working with asymptotic time ∂_t with has a degenerate Killing horizon, one may choose instead a global time, e.g. $(u+v)$ in "Kruskal" coordinates:



- The Hamiltonian wrt to the new Killing vector is just the combination introduced by DFF,

$$\partial_u + \partial_v = E_+ + E_- = \frac{p_r^2}{2} + \frac{1}{2r^2} + \frac{r^2}{2}$$

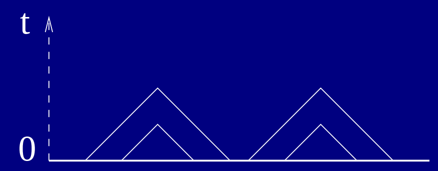
yielding a discrete spectrum of normalizable states.

Claus Derix Kallosh Kumar Townsend Van Proeyen; Kallosh

2. Spacelike singularity and CQM

- As one approaches a cosmological (spacelike) singularity, the dynamics of points separated by more than a cosmological horizon $\sim cT$ decouple.

  As $T \rightarrow 0$, this reduces to a set of decoupled 0+1-dimensional (quantum) mechanical systems at each point on the spacelike slice !

  Belinskii Khalatnikov Lifschitz; Misner

- In this limit, a minisuperspace ansatz is legitimate,

$$ds^2 = -\alpha^2 dt^2 + g_{ij}(t) dx^i dx^j$$

  with analogous ansatz for gauge fields. Evaluating Einstein's action one obtains the motion of a fictitious particle on the moduli space of (spatially constant) metrics and gauge fields = an constant negative curvature homogeneous space.

- "Integrating out" off-diagonal dof yields potential terms for diagonal radii. They become reflection walls towards the singularity: this leads to an hyperbolic billiard picture, with chaotic motion consisting of Kasner flights separated by bounces.

  Damour Henneaux + Nicolai Julia de Buyl Schomblond . . .

2+1-gravity at a spacelike singularity

- For simplicity, we consider 2+1 dimensional Einstein gravity, dimensionally reduced to 0+1 at a spacelike singularity:

$$ds^2 = -\left[\frac{\eta}{V}\right]^2 dt^2 + V \frac{\left[(dx_1 + U_1 dx_2)^2 + U_2^2 dx_2^2\right]}{U_2}$$

  where V is the volume and $U = U_1 + iU_2 \in Sl(2)/U(1)$ the "complex structure" of the spatial slice. We refrain from integrating U_1 out.

- The Einstein-Hilbert action becomes, after integrating by part,

$$S = \int dt \left[\frac{1}{2\eta}\left(-\dot{V}^2 + V^2 \frac{\dot{U}_1^2 + \dot{U}_2^2}{U_2^2}\right) - 2\eta\Lambda\right]$$

  This action is invariant by under general time reparameterization, keeping ηdt fixed.

- This is the Lagrangian of a free particle of mass $m^2 = 4\Lambda$ moving on the Lorentzian cone with metric

$$ds^2 = -dV^2 + V^2 \frac{dU_1^2 + dU_2^2}{U_2^2}$$

  Note this is flat $R^{2,1}$ in polar coordinates. For $\Lambda < 0$ the particle is tachyonic.

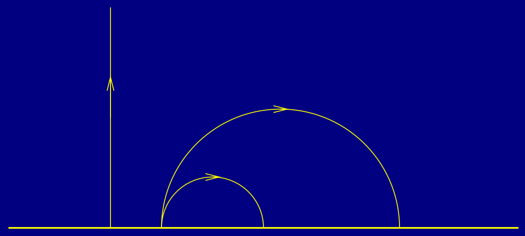
Moving on the cone

- The volume V appears with a negative signature: it can be chosen as a reference time, against which to measure other phenomena.

  DeWitt

- The motion is now easily integrated: in the gauge $\eta = V^2$, the motion of U decouples from V, hence U follows geodesics in the upper half plane.

- The charge p_1 associated to the isometry $U_1 \to U_1 + cste$ is conserved. The motion of U_2 effectively receives an harmonic potential $p_1^2 U_2^2$: for $p_1 \neq 0$, this prevents U_2 from reaching $+\infty$: trajectories are half circles centered on the boundary of the upper half plane.



- Now put $V = \rho^2$. Going to momentum variables $p = -4\rho\dot{\rho}/\eta, \quad p_1 = \rho^4 \dot{U}_1/(\eta U_2^2), \quad p_2 = \rho^4 \dot{U}_2/(\eta U_2^2),$ we get the Hamiltonian
$$H = \frac{\eta}{\rho^2}\left[-\frac{p^2}{8} - \frac{\Delta}{2\rho^2} + \frac{1}{8}\Lambda\rho^2\right]$$
- The Hamiltonian constraint $\delta H/\delta\eta \equiv 0$ reads
$$H_{WDW} = \frac{1}{2}p^2 + \frac{2\Delta}{\rho^2} - \frac{1}{2}\Lambda\rho^2 \equiv 0$$
- This is nothing but the Hamiltonian of conformal mechanics, upon identifying $g = 4\Delta$, where Δ is the angular momentum on $Sl(2)/U(1)$. The sign of g depends on boundary conditions on the upper half plane (square integrable modes have $\Delta < 0$)
- The quadratic potential is provided by the cosmological constant. For $\Lambda < 0$, we get an operator with discrete normalizable states.
- Even so, we are looking for a zero energy state, which will not be normalizable.
- For $\Lambda < 0$, we are looking for a state which is invariant under the compact generator $E_+ + E_-$: the wave function of the Universe is therefore the spherical vector of the representation.

DFF vs WDW

Despite formal identity between the two problems, there are some important differences:

- The WDW equation picks out zero-energy states only. So boundedness from below of H is no longer a requirement. Indeed, the sign of g depends on boundary conditions on S (square integrable wave functions have $g < 0$), and the sign of m^2 depends on Λ (discrete spectrum for $\Lambda < 0$)

- Usual quantum mechanics analysis requires wave functions to be square integrable. Here ρ should be thought as a time variable, square integrability along ρ should not be imposed. Instead perhaps, use a Klein-Gordon type norm on spacelike slices (and "third" quantize the system in order to get rid of negative norm states)

Those are problems in any quantum cosmology investigation, so we proceed anyway.

Reduction of $n+1$-dim gravity

- Let us know consider the reduction of $n+1$-dim Einstein gravity: The metric ansatz is

$$ds^2 = -\left[\frac{\eta(t)}{V(t)}\right]^2 dt^2 + V^{2/n}(t)\,\hat{g}_{ij}(t)\,dx^i dx^j\,,$$

where V is the spatial volume and $\det(\hat{g}) = 1$.

- The Einstein-Hilbert action reduces to

$$\int dt\left\{\frac{1}{2\eta}\left[-\frac{2(n-1)}{n}\dot{V}^2 + V^2\,\dot{U}^M G_{MN}\dot{U}^N\right] - 2\Lambda\eta\right\}$$

Here U^M coordinatize the negative curvature symmetric space $S = Sl(n)/SO(n)$ describing all spatially constant unit volume metrics $\hat{g}$.

- One recognizes the Lagrangian for a free particle propagating on the Lorentzian cone

$$d\sigma^2 = -\frac{2(n-1)}{n}\,dV^2 + V^2 dU^M G_{MN} dU^N\,.$$

- Change vars to $\rho = \sqrt{8(n-1)V/n}$ and go to canonical coordinates. The Hamiltonian now reads

$$H = \frac{\eta}{V}\left[\frac{1}{2}p^2 + \frac{4(n-1)}{n\rho^2}\Delta - \frac{n\Lambda}{4(n-1)}\rho^2\right]$$

The eom for η is again the DFF Hamiltonian, at zero energy, with $g = 8(n-1)\Delta/n$ related to the Laplacian Δ on S.

- The conformal symmetry is a direct consequence of the conical structure of moduli space, hence its having an closed homothetic Killing vector.

Dimensional reduction of supergravity

- In addition to the graviton, supergravity also contains scalar and gauge fields. Upon dimensional reduction, we still obtain the geodesic motion of a free particle on a Lorentzian cone with negatively curved sections G/K. E.g: gravity+dilaton+B yields a cone over $SO(n,n)/SO(n) \times SO(n)$.

- The positive roots in G correspond to off-diagonal metric and gauge fields; they can be eliminated by using the associated conserved Noether charges, producing a potential for the Cartan degrees of freedom, aka dilatonic scalars: as in the 2+1 case, these yield reflection walls keeping $R_1 \leq R_2 \leq \ldots$ in a fundamental chamber.

- In addition, there are potential terms originating from spatial gradients of the metric; these could be incorporated using a "dual" description of the graviton (e.g: in $11 \to 3$ reduction, $g_{\mu i}$ are 8 vectors fields dual to 8 scalars, hence yield a scalar field in $(8,1)$ Young tableau).

  Obers BP Rabinovici;Boulanger et al

- Duality implies an infinite set of such domain walls, corresponding to a roots of an hyperbolic Kac-Moody algebra E_{10}. Upon introducing these new degrees of freedom, one would expect to still have a conformal symmetry.

  Damour Henneaux Julia Nicolai

CQM and coadjoint orbits

Let us come back to classical conformal mechanics:

- Since D and E_- do not commute with the Hamiltonian, they evolve in time, but following the simple law,

$$\begin{aligned} dg/dt &= [h, g] \ , \\ g(t) &= \begin{pmatrix} D & E_- \\ -E_+ & -D \end{pmatrix}(t) \in sl(2)^* \ , \\ h &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in Sl(2) \end{aligned}$$

- The motion thus takes place on a coadjoint orbit of $Sl(2)$, flowing along the action of the nilpotent generator $h = E_+$,

$$G(t) = e^{th} G(0) e^{-th}$$

- Classically, the coupling constant is given by the invariant of the orbit, $\Delta \sim \det(G)$.

CQM and coadjoint orbits, in generality

- Let us consider the coadjoint orbit of a generic hyperbolic element of $sl(2)$:

$$\Omega = \left\{ g^{-1} J g \ , g \in Sl(2) \right\} \ , \quad J = \begin{pmatrix} \lambda & \\ & -\lambda \end{pmatrix}$$

- The orbit Ω can be viewed as $\Omega = Stab\backslash G$ where $Stab = \{g, g^{-1} J g = J\}$ is the stabilizer of J. A gauge slice can be chosen as

$$\Omega = \{ g = \begin{pmatrix} 1 & \\ \gamma & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \beta \\ & 1 \end{pmatrix} \}$$

$G = Sl(2)$ acts from the right on Ω, hence on (β, γ).

- A coadjoint orbit has a canonical invariant symplectic form, the Kirillov-Kostant symplectic form,

$$\omega = d\theta \ , \quad \theta = \mathrm{Tr}(J dg g^{-1}) = -2\lambda\beta d\gamma$$

- The right action of $h \in G$ preserves ω, hence can be represented by its moment map E_h such that $i_h \omega = dE_h$. h then acts by Poisson bracket with E_h on functions of (β, γ). Here:

$$E_+ = 2\lambda\gamma \ , \quad D = 2\lambda(1{+}2\beta\gamma) \quad E_- = -2\lambda\beta(1{+}\beta\gamma)$$

- This can be recast in the conformal quantum mechanics form through a canonical transformation,

$$E_+ = y^2 \ , \quad D = 2py \ , \quad E_- = \frac{1}{4}p^2 + \frac{\lambda}{2y^2}$$

- Note that this construction is purely classical: the non-trivial part is to quantize the coadjoint orbit. This can be done by induced representation methods.

- One could have started from a nilpotent element of $Sl(2)$ instead:

$$J = \begin{pmatrix} 0 & \\ 1 & 0 \end{pmatrix} \ , \qquad g = \begin{pmatrix} \sqrt{t} & \\ & 1/\sqrt{t} \end{pmatrix} \cdot \begin{pmatrix} 1 & \beta \\ & 1 \end{pmatrix}$$

$$\theta = t d\beta \ , \quad E_+ = t \ , \quad D = 2\beta t \ , \quad E_- = \beta^2 t$$

Redefining $t = y^2$ and $\beta = p/(2y)$ we get

$$E_+ = y^2 \ , \quad D = py \ , \quad E_- = \frac{1}{4}p^2$$

This is the usual harmonic oscillator. Its quantization gives the metaplectic representation of $Sl(2)$.

CQM from nilpotent orbits

- Quantization of coadjoint orbit of any group containing $Sl(2)$ will yield a conformal quantum mechanical model: simply need to find the right variables such that $D = pq$ etc.

- Generic coadjoint orbits have (even) dimension $n = \dim G - \mathrm{Rank} G$. Non-generic ones have a bigger stabilizer, hence correspond to a phase space of smaller dimension. They also have fewer parameters.

- The smallest coadjoint orbit is that of a minimal nilpotent element, ie the orbit of any root (for $Sl(n)$: only one 2×2 Jordan block). Its quantization leads to the minimal representation of G, analogous to the metaplectic representation of $Sl(2)$.

- Motivated by a conjecture about the BPS quantum supermembrane, we have constructed the minimal nilpotent orbit of ADE groups: this yields a family of q-mechanical systems where the conformal group is enhanced to any ADE group.

Kazhdan BP Waldron

- The simplest non-trivial model, based on D_4, turns out to be equivalent to the reduction of 2+1 gravity: this model thus exhibits hidden D_4 symmetry, which allows to get at the spherical vector.

Summary

- Conformal symmetry arises in many different problems where a universal regime is reached: non-trivial infrared dynamics of field theories, near horizon limit of black holes, and here: gravity near a spacelike singularity.

- The appearance of conformal symmetry here is perhaps not surprising, since we are expanding around a solution with power-like behavior, $g_{\mu\nu}(t,x) = t^\alpha g^0_{\mu\nu}(x)$: at least scaling symmetry is guaranteed.

- From a mathematical viewpoint, conformal quantum mechanics can be understood as free motion on a coadjoint orbit of $Sl(2)$. It can be generalized to any group G containing an $Sl(2)$. This allows for a general quantization of these models.

- Nilpotent orbits are particularly interesting, since they have the smallest phase space and parameter space: the minimal orbit has no parameter at all.

- We have identified the minimal orbit of D_4 with the dimensional reduction of 2+1 gravity. How about other ADE groups ? E_{10} ?

Poetry: gravity and fluid mechanics

- The dynamics of gravity at a spacelike singularity has a strong flavor of fully developped turbulency in fluid mechanics. Indeed, one may think of each of the fictitious particles as fluid elements moving on the moduli space, with the spatial position playing the role of the particle label in a Lagrangian description.

- Recall that Euler's perfect fluid equations can be thought of as a geodesic motion on the coadjoint orbit of volume preserving diffeomorphisms (V.I. Arnold) . Is there a similar picture for gravity ?

- The chaotic behavior is reminiscent of the energy cascade in turbulency. The conformal symmetry that we argued for may correspond to Kolmogorov's "inertial range". Do quantum fluctuations and particle production provide a dissipation cut-off ?