# Black hole entropy and topological string theory

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# Black hole thermodynamics and microscopic counting

- In general relativity, one associates to a macroscopic black hole with mass $M$, horizon area $A$ and surface gravity $\kappa$ an entropy $S_{BH} = A/4G_N$ and temperature $T = \kappa/2\pi$ such properties analogous to the standard laws of thermodynamics are obeyed

$$1)\ dS_{BH} = \frac{1}{T_H} dM + \omega dJ \ldots\ , \qquad 2)\ d(S_{BH} + S_{matter}) > 0$$

*Christodolou, Bekenstein, Hawking*

- String theory is famously known to provide a microscopic description of black hole microstates, reproducing the macroscopic Bekenstein-Hawking entropy. Eg, "4-charge" extremal black holes in 4D have a macroscopic entropy:

$$S_{BH} = 2\pi\sqrt{Q_1 Q_5 Q_{KK} P}\ ,$$

They can be represented as a D1-D5-P-KKM bound state, whose microstates are described by a 2D CFT. Their entropy can be counted by using the Cardy formula

$$S_{micro} = \ln\Omega \sim 2\pi\sqrt{cN/6} \sim S_{BH}$$

*Strominger Vafa; Maldacena Strominger; Johnson Khuri Myers*

# Black hole entropy beyond leading order

- This agreement relies on the "thermodynamical" limit where $A \gg G_N$, or $Q \gg 1$, and classical gravity can be trusted. Can we test this beyond leading order, and compare gravitational corrections to the Bekenstein-Hawking entropy to finite size effects on the microscopic side ?

# BH entropy beyond leading order (macroscopics)

- On the macroscopic side, the Bekenstein-Hawking "area law" receives corrections due to higher-derivative interactions in the low energy effective action. E.g, for 4D Einstein with polynomial interactions in $R_{\mu\nu\rho\sigma}$,

$$S_{BHW} = 2\pi \int_\Sigma \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} \epsilon^{\mu\nu} \epsilon^{\rho\sigma} \sqrt{h} d\Omega \sim \frac{1}{4} A + \dots$$

*Wald; Jacobson Kang Myers*

where $\epsilon^{\mu\nu}$ is the binormal on the horizon $\Sigma$. In addition, the geometry itself is deformed (sometimes in a drastic way).

- Recently, de Wit et al have been able to compute an infinite set of corrections to the BH entropy of extremal BH in type II string compactified on a Calabi-Yau 3-fold $Y$, due to a class of higher derivative interactions

$$F_h(X^A) R_+^2 F_+^{2h-2}$$

where $F_h$ is given by the genus-$h$ topological string amplitude.

# BH entropy beyond leading order (microscopics)

- On the microscopic side, the entropy is defined as the Legendre transform of the free energy, which depends on a choice of statistical ensemble. In the thermodynamical limit, the entropy is universal, but subleading corrections are not.
- Recently, Ooguri Strominger and Vafa (OSV) have proposed to identify the specific statistical ensemble implicit in the macroscopic computation as a “mixed” ensemble, where the magnetic charges are fixed micro-canonically but electric charges are allowed to fluctuate at a fixed electric potential:

$$Z(p^A, \phi^A) := \sum_{q_A \in \Lambda_{el}} \Omega(p^A, q_A) e^{-\phi^A q_A}$$

- In combination with the macroscopic computation, this gives a conjectural relation between microscopic degeneracies $\Omega(p^A, q_A)$ and the topological string amplitudes $F_h(X^A)$:

$$(OSV) \qquad Z(p^A, \phi^A) \stackrel{?}{=} |\exp\left(\frac{i\pi}{2}\Im(p^A + i\phi^A)\right)|^2$$

# Outline of the talk

- `Aim:` test the OSV proposal, in cases where the two sides of the equations can be computed to arbitrary accuracy.
- `Tools:` small black holes, heterotic/ type II duality, Rademacher formula
- `Report:` matched an infinite number of subleading corrections to the BHW entropy with a microscopic counting.

1. Review: BH entropy and D-brane counting
2. Attractor mechanism and the OSV conjecture
3. A benchmark case: $K_3 \times T^2$
4. $N = 4$ CHL strings
5. $N = 2$ orbifolds
6. Towards an exact OSV-type formula
7. Discussion

# Microscopic origin of the Bekenstein-Hawking entropy

*Strominger Vafa 1996*

Consider a 5D extremal Reissner-Nordström black hole in type II string theory on $K_3 \times S_1$:

- Solutions preserving 1/4 SUSY and carrying a minimum of 3 charges have a smooth event horizon, with an associated Bekenstein-Hawking entropy

$$S_{BH} = 2\pi\sqrt{N_1 N_5 P}$$

- Consider a configuration of $N_1$ D1-branes wrapping $S_1$, $N_5$ D5-branes wrapping $K_3 \times S_1$, with $P$ units of momentum along $S_1$. At strong string coupling, it becomes an extremal RN black hole, and carries the same quantum numbers as above.
- In the limit where $K_3$ is small the effective D=1+1 field theory on the brane is a supersymmetric (4,4) sigma model on the (deformed) permutation orbifold

$$(K_3)^{\otimes N_1 N_5} / S_{N_1 N_5}$$

since D1-branes can be seen as Yang-Mills instantons on the D5-brane world-volume.

- The central charge is therefore $c = 6N_1N_5$. BPS states with charge $P$ are ground states on the left, and carry level $P$ excitation on the right. By the Cardy formula,

$$S_{micro} = 2\pi\sqrt{cP/6} = 2\pi\sqrt{N_1N_5P}$$

which matches with the macroscopic entropy !

- Similar results hold in situations with a different amount of SUSY: 1/8-BPS black holes on $T^5$ have a BH entropy

$$S_{BH} = 2\pi\sqrt{I_3(Q)}$$

where $Q$ are the general 27 electric charges, and $I_3$ is the cubic invariant of $E_{6(6)}$, while 1/2 BPS black holes in M theory on CY can be described as bound states of M2-brane wrapped on 2-cycles, with BH entropy

$$S_{BH} = 2\pi\sqrt{D_{ABC}p^Ap^Bp^C}$$

were $D_{ABC}$ are the intersection numbers of 2-cycles. Microscopic counting remains ill understood in the latter case.

*Maldacena Moore Strominger*

*Katz Klemm Vafa*

- Microscopic degeneracies can also be computed by noting that the near-horizon geometry is $AdS_3 \times S_3 \times T^4$ factor. This admits an holographic description as a 2D CFT on the boundary, whose central charge is given by

$$c = \frac{3\Lambda}{2G_N}$$

reproducing the correct entropy via Cardy's formula.

*Brown Henneaux; Carlip; Strominger*

# Entropy of 4-dim black holes

- 4-dim 1/4 BPS black holes in type IIA / $K_3 \times T^2$ can be described by a D6-D2-NS5 bound state wrapped on $K_3$, with momentum along $S^1$:

$$S_{BH} = 2\pi\sqrt{Q_2 Q_5 Q_6 P}$$

By allowing the D2-branes to end on the NS5-branes, one can reproduce this entropy microscopically just as in the 5D case.

*Maldacena Strominger*

- Equivalently, the same system can be described by a bound state of D1-D5-P with $Q_K$ KK monopole: the same entropy arises by taking into account fractional D-branes in the ALE geometry.

*Johnson Khuri Myers, Constable Khuri Myers*

- More generally, in $N = 4$ backgrounds, the BH entropy is given by the $Sl(2) \times SO(6, n)$ invariant discriminant

$$S_{BH} = 2\pi\sqrt{(\vec{p}\cdot\vec{p})(\vec{q}\cdot\vec{q}) - (\vec{p}\cdot\vec{q})^2}$$

A formula for the exact degeneracies has been proposed, but remains to be tested.

*Dijkgraaf Verlinde Verlinde*

- In $N = 8$ backgrounds, the BH entropy is given by the $E_7$ quartic invariant,

$$S_{BH} = 2\pi\sqrt{I_4(Q)}$$

reproduced by a similar counting as above. Exact degeneracies are still unknown.

*Kallosh Kol*

- In $N = 2$ backgrounds, such as type II / CY, the tree-level BH entropy is

$$S_{BH} = 2\pi\sqrt{D_{ABC}p^A p^B p^C q_0}$$

but receives corrections from higher-derivative interactions. The first subleading correction can be obtained by considering an M5-brane wrapping $\gamma_4 \times S_1$, where $\gamma_4$ is a 4-cycle in CY. The reduced theory on $\gamma_4$ is a $(0, 4)$ sigma model, and the Cardy formula predicts

$$S_{micro} = 2\pi\sqrt{(D_{ABC}p^A p^B p^C + c_{2A}p^A/6)q_0}$$

in agreement with 1-loop $R^2$ corrections.

*Maldacena Strominger Witten; de Wit L.Cardoso Mohaupt*

- In general, the near-horizon geometry of these 4D extremal RN black holes is $AdS_2 \times S^2 \times M_6$. One expects a dual description by a conformal quantum mechanics leaving on the boundary, but no concrete example is known.

# The attractor mechanism

- Consider a general ansatz for a spherically symmetry RN BH in type IIA/CY:

$$ds^2 = -e^{2U(r)+2r}dt^2 + e^{-2U(r)}\left(dr^2 + r^2 d\Omega_2^2\right) + ds^2_{CY}$$

The shape of the CY is parameterized by Kähler moduli $X^A(r)$, and complex structure moduli. The latter decouple and can be taken to be constant.

- The tree-level lagrangian is controlled by the prepotential, an homogeneous holomorphic function $F(X^I)$ given by

$$F(X^A) = -\frac{1}{6}C_{ABC}\frac{X^A X^B X^C}{X^0} + \text{worldsheet instantons}$$

Notation: $F_A = \partial F/\partial X^A$.

- The SUSY equations can be rewritten as

$$\Re\left(X^A - \frac{d}{dr}X^A\right) = p^A\,, \quad \Re\left(F_A - \frac{d}{dr}F_A\right) = q_A$$

- At the horizon, the geometry becomes $AdS^2 \times S^2 \times CY$ where the Kähler moduli are fixed by the attractor equations,

$$Re(X^A) = p^A, \quad Re(F_A) = q_A$$

*Ferrara Kallosh Strominger*

- The Bekenstein-Hawking entropy is thus a function of the charges only,

$$S_{BH} = \frac{i\pi}{2}\left(q_A \bar{X}^A - p^A \bar{F}_A\right)$$

# The attractor mechanism, revisited

- This is usefully recast as follows: set $X^A = p^A + i\phi^A$ where $\phi^A$ is real. The second equation becomes

$$q_A = \frac{1}{2}\left(\partial F_0/\partial X^A + \partial \bar{F}_0/\partial \bar{X}^A\right) = \frac{1}{2i}\left(\partial F_0/\partial \phi^A - \partial \bar{F}_0/\partial \bar{\phi}^A\right)$$

hence

$$q_A = \pi\ \partial \mathcal{F}/\partial \bar{\phi}^A \quad \text{where} \quad \mathcal{F}_0(p^A, \phi^A) = \frac{1}{\pi}\text{Im} F_0(p^A + i\phi^A)$$

- In addition, the BH entropy may be rewritten as

$$S_{BH} = \mathcal{F}_0(p^A, \phi^A) + \pi\ q_A \phi^A$$

- The BH entropy $S_{BH}(p^A, q_A)$ is thus recognized as the Legendre transform of the free energy $\mathcal{F}_0(p^A, \phi^A)$ ! To compute the latter, no need to solve the attractor equations !

*Ooguri Vafa Strominger*

# Leading entropy of large black holes

- As an application, let us compute the tree-level entropy of a black hole with arbitrary charges, except for $p^0 = 0$: the tree-level superpotential is

$$F = -\frac{1}{6} C_{ABC} \frac{X^A X^B X^C}{X^0} \Rightarrow \mathcal{F}(p,\phi) = -\frac{\pi}{6} \frac{C(p)}{\phi^0} + \frac{\pi}{2} \frac{C_{AB}(p)\phi^A \phi^B}{\phi^0}$$

$$C(p) = C_{ABC} p^A p^B p^C \;, \quad C_{AB}(p) = C_{ABC} p^C \;, \quad A = 1, \dots n_V - 1$$

- The Legendre transform with respect to $\phi^A$ leads to

$$\phi^A_* = -C^{AB}(p) q_B \phi^0 \;, \qquad \phi^0_* = \pm\sqrt{-\hat{C}(p)/6\hat{q}_0}$$

$$\hat{q}_0 = q_0 + \frac{1}{2} q_A C^{AB}(p) q_B$$

- The tree-level Bekenstein-Hawking entropy is therefore the square-root of a quartic polynomial in the charges,

$$S_{BH} = 2\pi \sqrt{C(p)\hat{q}_0/6}$$

in agreement from the microscopic counting at leading order.

- When $C(p) = 0$, the tree-level BH entropy vanishes, indicating a singular solution. We shall be interested in such “small black holes”, which get a non-vanishing entropy from higher order contributions.

# Higher derivative interactions and the topological string

- Recall that the $(2,2)$ sigma-model on a CY threefold can be topologically twisted into the A-model topological string, which depends only on the Kähler moduli $X^A$. This defines a quantum field theory of Kähler structures, known as Kähler gravity.
- The topological A-model can be related to the physical type II superstring: the genus-h topological amplitude (without insertions) $F_h(X)$ is equal to the coefficient of the $R_+^2 F_+^{2h-2}$ amplitude in the low energy effective action

$$\int d^8\theta F(X;W^2) = \int d^8\theta \sum_{h=0}^{\infty} F_h(X) W^{2h} \sum_{h=0}^{\infty} F_h(X) R_+^2 F^{2h-2}$$

- The all-genus topological A-model thus resums an infinite number of higher-derivative F-term corrections. The topological coupling constant $\lambda$ is proportional to the graviphoton field-strength,

$$\lambda = \frac{\pi}{4}\frac{W}{X^0}$$

# The attractor mechanism, to all orders

- In the presence of $R_+^2 F_+^{2h-2}$ corrections, the attractor formalism goes through upon replacing the tree-level prepotential $F_0(X)$ by the generating function

$$F(X^A, W^2) = \sum_{h=0}^{\infty} F_h(X^A) W^2$$

and enforcing the additional attractor equation $W/X^0 = \pm 2^4$.

- The Bekenstein-Hawking-Wald entropy is thus the Legendre transform of the free energy

$$\mathcal{F}(p^A, \phi^A) = \frac{1}{\pi} \text{Im} \mathcal{F} \left( p^A + i\phi^A; (2^4 X^0)^2 \right)$$

- One may interpret $\mathcal{F}(p^A, \phi^A)$ as the free energy of a statistical ensemble of black holes with magnetic charge $p^A$ and electric potential $\phi_A$.

# The OSV conjecture for BH degeneracies

- It is thus natural to conjecture that the relevant microscopic statistical ensemble is a "mixed" ensemble, where magnetic charges are treated micro-canonically but electric charges are treated canonically:

$$Z(p^A,\phi^A) := \sum_{q_A\in\Lambda_{el}} \Omega(p^A,q_A) e^{-\phi^A q_A} \stackrel{?}{=} e^{\mathcal{F}(p^A,\phi^A)} = \left|\exp\left(\frac{i\pi}{2}F(p^A+i\phi^A)\right)\right|^2$$

*Ooguri Strominger Vafa*

- If correct, this provides a way to compute the microscopic degeneracies $\Omega(p^A,q_A)$ (or rather a suitable index) from the topological string amplitude $F(W,X)$, by inverse Laplace transform,

$$\Omega(p^A,q_A) \equiv \int d\phi^A \left|\exp\left(\frac{i\pi}{2}F(p^A+i\phi^A)\right)\right|^2 e^{\phi^A q_A}$$

- Conversely, one may hope to understand the non-perturbative completion of the topological string from the knowledge of black hole micro-states.

# More on the OSV conjecture

- There are several versions of the OSV conjecture: the weaker form is supposed to relate the topological string amplitude with some suitable index, and hold only asymptotically to all orders in inverse charges.
- The OSV proposal is somewhat formal: what is the precise integration measure and contour ? How about holomorphic anomalies, curves of marginal stability ? Should we count micro-states with arbitrary angular momentum or only $J = 0$ ? etc
- The proposal has been tested in the case of non-compact CY: $O(-m) \oplus O(m) \to T^2$: BPS states are counted by topologically twisted SYM on $N$ D4-brane wrapped on a 4-cycle $O(-m) \to T^2$, which is equivalent to 2D Yang Mills. Using the factorization properties in the large $N$ limit, one can show that OSV is correct to all orders in $1/N$.

  *Vafa; Aganagic Ooguri Saulina Vafa*
- A recent "proof" has been given by reinterpreting the BH partition function as (the inner product of) the wave function of the Universe in a minisuperspace formulation.

  *Ooguri Verlinde Vafa*

# Testing OSV: small black holes

- Our goal is to test the OSV conjecture in cases where black holes degeneracies are exactly known. For this, restrict to $K_3$-fibered CY, which admit a dual description as heterotic / $K^3 \times T^2$.
- The heterotic string admits a class of perturbative BPS states, known as Dabholkar-Harvey states:

$$|osc, N\rangle \otimes \overline{|osc, 0\rangle} \times |n_i, w^i\rangle$$

  satisfying the matching condition $N - 1 = n_i w^i$. They carry purely electric charge, in the natural heterotic polarization. They are counted by simple modular forms.
- At strong coupling, these states remain stable and become black holes, carrying both electric and magnetic charges, in the natural type II polarization. In contrast to the general “4-charge” black holes, they are singular at tree-level, but acquire a smooth horizon due to $R^2$ interactions.

*Sen 95; Dabholkar 04; Kallosh Maloney Dabholkar; Hubeny Maloney Rangamani; Bak Kim Rey*

# Large Black Hole degeneracies from OSV

- The A-model topological string amplitude on a CY $Y$ $F(X^A, W^2)$ is an homogeneous function of degree 2 in $(X^A, W)$:

$$F = -\frac{1}{6}C_{ABC}\frac{X^A X^B X^C}{X^0} - \frac{W^2}{64\cdot 24}\frac{c_A X^A}{X^0} - \frac{X_0^2}{(2\pi i)^3}\sum_{h=0}^{\infty}\sum_{\beta}\left(\frac{\pi W}{4X^0}\right)^{2h} N_{h,\beta}e^{2\pi i\beta_A X^A/X^0}$$

  where $A = 1..n_V - 1$ runs over 2-cycles of $Y$, $C_{ABC} = \int_Y J_A J_B J_C$ are triple intersection numbers, $X^A/X^0 = B^A + iV^A$ are the Kähler moduli, $c_A = \int_Y J_A c_2(T^{1,0}(X))$ and $N_{h,\beta}$ are rational numbers known as the Gromov-Witten invariants.

- From these we compute the free energy

$$\mathcal{F}(p,\phi) = -\frac{\pi}{6}\frac{\hat{C}(p)}{\phi^0} + \frac{\pi}{2}\frac{C_{AB}(p)\phi^A\phi^B}{\phi^0} + 2\mathrm{Re}(F_{GW})$$

  where

$$\hat{C}(p) = C(p) + c_A p^A\,, \quad C(p) = C_{ABC}p^A p^B p^C\,, \quad C_{AB}(p) = C_{ABC}p^C$$

- For now, let us drop $F_{GW}$ and compute the Laplace transform

$$\Omega_{OSV}(p^A, q_A) = \int d\phi^0 d\phi^A \exp\left(\mathcal{F}(p,\phi) + \pi\phi^A q_A\right)$$

can be computed exactly: the $\phi^A$ integral is Gaussian, with saddle at $\phi_*^A = -C^{AB}(p) q_B \phi^0$:

$$\Omega_{OSV}(p^A, q_A) = \int d\phi^0 \phi_0^{(n_V-1)/2} \det[C_{AB}(p)]^{-1/2} \exp\left(-\frac{\pi}{6}\frac{\hat{C}(p)}{\phi^0} + \pi\phi^0 \hat{q}_0\right)$$

with $q_0 = q_0 + \frac{1}{2} q_A C^{AB}(p) q_B$.

- The $\phi^0$ integral is now of Bessel type, with saddle at $\phi_*^0 = \pm\sqrt{-\hat{C}(p)/6\hat{q}_0}$. For an appropriate contour, we find

$$\Omega_{OSV}(p^A, q_A) = \det[C_{AB}(p)]^{-1/2} [\hat{C}(p)]^{(n_V+1)/2} \hat{I}_{(n_V+1)/2}\left[2\pi\sqrt{\hat{C}(p)\hat{q}_0/6}\right]$$

- Using the asymptotics

$$\hat{I}_\nu(z) \sim z^{-\nu-\frac{1}{2}} e^z \left(1 + a/z + b/z^2 + \dots\right)$$

we find the micro-canonical entropy predicted by OSV:

$$S_{OSV}(p^A, q_A) \sim 2\pi\sqrt{\hat{C}(p)\hat{q}_0/6} - \frac{n_V + 2}{2}\log[\hat{C}(p)\hat{q}_0] + \dots$$

- The leading square-root term reproduces the tree-level Bekenstein- Hawking entropy $S_{BH} = 2\pi\sqrt{C(p)\hat{q}_0}$ at large magnetic charge. The replacement $C(p) \to \hat{C}(p) = C(p) + C_A p^A$ is due to the one-loop $R^2$ interaction, and guarantees that the entropy is non-vanishing for “small black holes” which have $C(p)$.
- In general, our understanding of the microstates is too rough to allow us to test this prediction.

# Small black holes and $K3$-fibered CY

- Let us now restrict to type II on a K3-fibered CY 3-fold, dual to heterotic/ $K_3 \times T^2$. The Kähler moduli split into the modulus $X^1/X^0$ of the base, and the moduli $X^a/X^0$ of the fiber ($a = 2, \dots n_V - 1$). The intersection form factorizes into

$$C_{ABC} X^A X^B X^C = X^1 C_{ab} X^a X^b$$

- Further consider a state whose only non-vanishing magnetic charge is $p^1$:

$$C(p) = 0 \, , \quad \hat{C}(p) = 24 p^1 \, , \quad C_{AB}(p) = \begin{pmatrix} 0 & 0 \\ 0 & p^1 C_{ab} \end{pmatrix}$$

- The dependence on $\phi^1$ now disappears from the integrand. Since $F_{top}$ is invariant under monodromies $\phi_1 \to \phi_1 + \phi_0$, it is natural to restrict the integration range to $[0, \phi_0]$:

$$\Omega_{OSV}(p^1, q_A) = \int d\phi^0 \, \phi_0^{n_V/2} \exp\left( -\frac{4\pi p_1}{\phi^0} + \pi \phi^0 \hat{q}_0 \right) \sim \hat{I}_{(n_V+2)/2} \left[ 4\pi \sqrt{p^1 \hat{q}_0} \right]$$

where $\hat{q}_0 = q_0 + \frac{1}{2} C^{ab} q_a q_b / p_1$.

# Comments

- Integrals have been carried out somewhat formally. Since $C_{AB}(p)$ in general has signature $(1, n_V - 2)$, the gaussian integral needs to be computed by rotating the contour for $\phi^A$ to the imaginary axis.
- In addition to the Bessel $\hat{I}$ function, the OSV integration measure leads to extra $p$-dependent factors, which, if taken literally, contradict T-duality on the heterotic side. The ratio $\Omega_{OSV}(p, q)/\Omega_{OSV}(p', q)$ seems to be free of these ambiguities.
- In the derivation, we neglected GW instanton contributions. Non-degenerate instantons contributions are exponentially suppressed in the large charge limit, and can be consistently neglected if $(p^a)^2 q_0 \gg C(p)$. When $\chi \neq 0$, the series of point-like instantons appears to be strongly coupled but, after resummation to the Mac-Mahon representation, can be consistently neglected if $q_0 \gg p_1$.

# Pointlike instantons

- In particular, the point-like instantons with $\beta' = 0$ lead to $n_0^0 = -\chi/2$ ($\chi$=Euler number of CY). They contribute an infinite series of higher-genus contributions to the topological amplitude:

$$F_{point} = -\frac{\chi}{2}\left[\frac{\zeta(3)}{\lambda^2} + A + \sum_{h=2}^{\infty} \lambda^{2h-2} \frac{(2h-1)B_{2h}B_{2h-2}}{(2h-2)(2h)!}\right]$$

- The $\zeta(3)$ term follows from the tree-level $R^4$ amplitude in 10D, the term with $h \geq 2$ is proportional to the Euler number of the moduli space of genus-h Riemann surfaces without punctures, and $A$ is a naively divergent quantity, but, when properly regulated

$$A = \frac{1}{12}\log(2\pi/\lambda) + finite$$

- This asymptotic expansion is valid at $\lambda \ll 1$. If $\lambda$ is large, an alternative representation is provided by the Mac Mahon function,

$$F_{point} = -\chi/2 \sum_{n=0}^{\infty} n \log(1-q^n) \qquad q = e^{-\lambda}$$

leading to an infinite product representation for $e^F$.

# A benchmark case: $II/K3 \times T^2$ vs $Het/T^6$

- On the macroscopic side: thanks to $N = 4$, $F_{h>1} = 0$. $F_1$ can be extracted from $R^2$ coupling,

$$f_{R^2} \sim \log T_2 |\eta(T)|^4 \Rightarrow F_1 = \log \eta^{24}(T) \ , \quad T = X_1/X_0$$

- The gauge group is $U(1)^6 \times U(1)^{22}$, however upon decomposition into $N = 2$ multiplets 4 $U(1)$ are part of gravitino multiplets, and not covered by the attractor formalism. So $n_V = 24$.
- According to the above, the OSV prediction for small BH degeneracies is

$$\Omega_{OSV}(p^1, q_0) = \hat{I}_{13}\left[4\pi\sqrt{p^1 \hat{q}_0}\right]$$

- On the heterotic side, these small BPS BH are dual to Dabholkar Harvey states, enumerated by

$$\frac{1}{\eta^{24}} = \sum_{N=0}^{\infty} p_{24}(N) q^{N-1} \ , \quad N - 1 = p^1 q_0$$

- The leading exponential behavior is given by Cardy's formula $\log p_{24} = 2\pi\sqrt{N.24/6}$. Subleading corrections are given by the Rademacher formula...

# The Rademacher expansion

Consider a vector-valued modular form $f_{\mu=1..r}(\tau)$ of weight $w<0$,

$$f_\mu(\tau+1)=e^{2\pi i\Delta_\mu}f_\mu(\tau)\ ,\quad f_\mu(-1/\tau)=(-i\tau)^w S_{\mu\nu}f_\nu(\tau)$$

with Fourier expansion $f_\mu(\tau)=q^{\Delta_\mu}\sum_{m=0}^\infty\Omega_\mu(m)q^m$

- Claim: the Fourier coefs can be expressed as an infinite series

$$\Omega_\nu(n)=\sum_{c=1}^\infty\sum_{\mu=1}^r\sum_{m+\Delta_\mu<0}c^{w-2}Kl(n,\nu;m,\mu;c)|m+\Delta_\mu|^{1-w}$$

$$\times\Omega_\mu(m)\hat{I}_{1-w}\left[\frac{4\pi}{c}\sqrt{|m+\Delta_\mu|(n+\Delta_\nu)}\right]$$

where $Kl(n,\nu;m,\mu;c)$ are generalized Kloosterman sums, equal to $S^{-1}_{\nu\mu}$ for $c=1$ and $\hat{I}_\nu(z)$ is a modified, modified Bessel function of the 1st kind,

$$\hat{I}_\nu(z)=2\pi\left(\frac{z}{4\pi}\right)^{-\nu}I_\nu(z)\sim z^{-\nu-\frac{1}{2}}e^z(1+a/z+b/z^2+\dots)$$

- All $c > 1$ contributions are exponentially suppressed wrt to $c = 1$, yet they are exponentially large in an absolute sense.
- The Cardy-Hardy-Ramanujan formula emerges by keeping the leading term $c = 1, m = 0$, using $\Delta = c/24$:

$$\log \Omega_\nu(n) \sim 4\pi \sqrt{|\delta_\mu|(n + \Delta_\nu)} = 2\pi \sqrt{\frac{c(n + \Delta_\nu)}{6}}$$

- In addition to this leading term, there are log corrections, as well as an infinite series of power-suppressed terms.
- The Rademacher expansion depends only on the polar part $\sum_{m+\Delta_\mu<0} \Omega_\mu(m) q^{m+\Delta\mu}$ (and modular data). Indeed, one proof is to represent $f_\mu(\tau)$ (or rather its Farey transform $q\partial_q^{1-w} f$) as the Poincaré series (i.e. sum over $Sl(2, Z)$ images) of its polar part.

# Back to the bench

- In particular, for the inverse of the Dedekind function, $w = -12$, $\Delta = -1$,$\Omega(0) = 1$ hence

$$p_{24}(N) = \hat{I}_{13}\left[4\pi\sqrt{p^1\hat{q}_0}\right] + 2^{-14}\hat{I}_{13}\left[2\pi\sqrt{p^1\hat{q}_0}\right] + \dots$$

- Comparing to the OSV prediction, we find agreement to ALL orders in $1/(p^1 q_0)$ !
- However, OSV fails to reproduce subleading corrections which grow like $e^{2\pi\sqrt{p^1 q_0}}$.
- Note that for this to work, we had to drop non-holomorphic contributions from $f_{R^2}$, and consider the degeneracies of states with arbitrary angular momentum $j$.

# $N = 4$ CHL strings

- More general $N = 4$ models with $0 \leq k \leq 22$ vector multiplets of $N = 4$ can be constructed, either as orbifolds of type II/ $K3 \times T^2$ by an Enriques involution, or as freely acting asymmetric orbifolds of Het/$T^6$.
- In the untwisted sector of the orbifold, the BPS states are a projection of the DH states in the Het/$T^6$ model. Their degeneracies are now counted by a modular form of the form

$$Z_{untw} = \frac{1}{2}\left(\frac{\theta}{\eta^{24}} + \psi\right)$$

  where $\theta$ is a partition function for the lattice of electric charges under the $22 - k$ gauge fields which have been projected out, and $\psi$ enforces the projection. Modular weight:

$$w = \frac{1}{2}(22 - k) - 12 = -1 - k/2 \qquad \Rightarrow 1 - w = (k + 4)/2 = (n_V + 2)/2$$

  Degeneracies are dominated by $\theta/\eta^{24}$, and are in agreement with the OSV prediction.
- In addition, there are BPS states in the twisted sectors, which are counted by modular forms related to $\psi$ by modular transformation. Their asymptotics appears to be equal to that of the untwisted, unprojected sector, again vindicating OSV.

# $N = 4$ CHL strings (a case study)

- Consider the simplest case:

$$\Gamma_{6,22} = E_8(-1) \oplus E_8(-1) \oplus II^{1,1} \oplus II^{5,5}$$

orbifolded by $g|P_1, P_2, P_3, P_4\rangle = e^{2\pi i\delta \cdot P_3}|P_2, P_1, P_3, P_4\rangle$ This projects out the $U(1)$ associated to $P_1 - P_2$, leaving only the physical electric charges $Q = (P_1 + P_2, P_3, P_4)$.

- DH states arise in the untwisted sector by taking the ground state on the right, an arbitrary, orbifold invariant excitation of the 24 oscillators on the left, and level-matched internal momentum:

$$Z_{untw} = \frac{1}{2}\left(\frac{Z_{6,6}\left[{0 \atop 0}\right]\theta^2_{E_8[1]}(\tau)}{\eta^{24}(\tau)} + \frac{Z_{6,6}\left[{0 \atop 1}\right]\theta_{E_8[1]}(2\tau)}{\eta^8(\tau)\eta^8(2\tau)}\right)$$

- From this we need to extract the number of states with given $Q = (P_1 + P_2, P_3, P_4)$. For this, change basis from $(P_1, P_2)$ to

$$P_1 + P_2 = 2\Sigma + \wp\ , \quad P_1 - P_2 = 2\Delta - \wp$$

where $S, \Delta$ take values in the $E_8$ root lattice, and $\mathcal{P}$ is an element of the finite group $Z = \Lambda_r(E_8)/2\Lambda_r(E_8)$.

- In order to sum over the “unphysical charges” $\Delta$, introduce $E_8$ level-2 theta functions with characteristics:

$$\Theta_{E_8[2],\wp}(\tau) := \sum_{\Delta\in E_8(1)} e^{2\pi i\tau(\Delta-\frac{1}{2}\wp)^2}$$

and use

$$\theta^2_{E_8[1]}(\tau) = \sum_{\mathcal{P}\in E_8/2E_8} \theta_{E_8[2],\mathcal{P}}(\tau)\theta_{E_8[2],\mathcal{P}}(\tau)\ , \quad \theta_{E_8[1]}(2\tau) = \theta_{E_8[2],0}(\tau)$$

hence

$$Z_u = \frac{\theta^2_{E_8[2],\mathcal{P}}(\tau)}{\eta^{24}(\tau)} \pm \frac{1}{\eta^8(\tau)\eta^8(2\tau)} := q^{\Delta_\pm}\sum_{N=0}^{\infty} d^u_\pm(N)q^N$$

# CHL strings, cont.

- In the twisted sector, the situation is simpler:

$$Z_t = \frac{1}{2}\left(\frac{1}{\eta^{12}\theta_4^4} \pm \frac{1}{\eta^{12}\theta_3^4}\right) := q^{\Delta_\pm} \sum_{N=0}^{\infty} d_\pm^t(N) q^N$$

- Using the Rademacher formula, we find

$$\begin{aligned} \dim \mathcal{H}_{BPS}(Q) \quad &= \quad 2^{-5} \hat{I}_9 \left(4\pi\sqrt{Q^2/2}\right) \\ &+ \hat{I}_9 \left(4\pi\sqrt{Q^2/4}\right) \begin{cases} 15 \cdot 2^{-10} + 2^{-6} e^{2\pi i P\cdot\delta}, & \wp \in \mathcal{O}_1 \\ 2^{-10}, & \wp \in \mathcal{O}_{248} \\ -2^{-10}, & \wp \in \mathcal{O}_{3875} \\ 2^{-10} e^{i\pi Q^2}, & Q \in \Lambda_1 \end{cases} + \dots \end{aligned}$$

Hence we have agreement to all orders with OSV in all sectors. Subleading terms however are not captured by OSV, and depend crucially on the sector.

# An $N = 4$ exception to OSV

- Let us consider $typeII/K_3 \times T^2$ at the $Z_2$ orbifold point, and perform a further orbifold by the "quantum symmetry" acting as -1 on each twisted sector, combined with a shift along $T^2$: this gives a type II $N = 4$ model with 6+6 gauge fields.
- The heterotic dual is unclear; however, another dual description can be obtained by making a $Z_2$ orbifold of type II/$T^4 x T^2$ by $(-1)^{F_L}$ times a shift on $T^2$ This projects out all RR fields, leaving 6+6 vectors. In constrast to the previous (2,2) case, SUSY is realized as (4,0) on the worldsheet.

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- The amplitude $F_1$ can be computed at one-loop on the (2,2) case: one finds $F_1 \sim \log \theta_4(T)$, which has no perturbative part but only instantons: thus small black holes remain small, even with $R^2$ corrections !

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- Just as in the heterotic case, the (4,0) model admits a spectrum of DH states, enumerated by $\theta_i^4/\eta^{12}$. The microscopic degeneracies thus grow as $\hat{I}_5(2\pi\sqrt{2p^1 q_0})$, not matched by OSV !

# Absolute degeneracies vs. helicity supertraces

- We obtained agreement to all orders between the OSV prediction (at strong gravitational coupling) and the absolute degeneracy of DH states (at weak coupling). In general however, we expect that only a suitable index can be trusted in comparing weak and strong coupling results.
- The natural indexes to invoke are helicity supertraces:

$$\Omega_n = \mathrm{Tr}(-1)^F J_3^n$$

where $F$ is the target space fermion number, and $J_3$ one generator of the little group of a massive particle in D=3+1. For low $n$, and large supersymmetry, this index receives only contributions from short multiplets, while long (non BPS) multiplets cancel out.

- For $N = 4$ SUSY, the natural index for $1/2$ (resp. $1/4$) BPS states is $\Omega_4$ (resp. $\Omega_6$). In heterotic orbifold constructions, $\Omega_4$ is in fact equal to the absolute degeneracy of 1/2-BPS states, “explaining” agreement.
- For $N = 2$ SUSY, the natural index is $\Omega_2 \sim N_V - N_H$. As we shall see, in heterotic orbifolds this can be much smaller than the absolute degeneracy !

# A few words on $N = 2$ models

- A number of type II/CY - Het/$K3 \times T^2$ dual pairs are known, where OSV can be tested. While $F_{h>1}$ are now $\neq 0$, the degeneracies of small BH predicted by OSV, to all orders in $1/p^1 q_0$, at small $p^1/q_0$ are universally given by

$$\Omega_{OSV} = \hat{I}_{(n_V+2)/2}(4\pi \sqrt{Q^2/2})$$

- For heterotic asymmetric orbifolds with $N = 2$ supersymmetry, the DH states can be counted as before. In contrast to $N = 4$, in the untwisted sector DH states typically come in vector/hyper pairs, and the helicity supertrace $\Omega_2$ is much smaller than the OSV prediction. The absolute degeneracies agree with $\Omega_{OSV}$ at leading order only.
- In contrast, twisted states are all hypers, and have $\Omega_{abs} = \Omega_2$ in agreement to $\Omega_{OSV}$ to all orders in 1/Q.
- In a class of models such as Het/K3 with standard embedding, untwisted and twisted states cannot be distinguished, hence OSV gives the correct result to all orders.
- In other models such as FHSV, untwisted and twisted states can be distinguished by the modding of their charges, and OSV appears to fail in reproducing either $\Omega_{abs}$ or $\Omega_2$, unless some coarse-graining is made.

# Could the OSV formula be exact ?

- Go back to the benchmark case: exact degeneracies can be extracted from

$$1/\eta^{24} = \sum_{N=0}^{\infty} p_{24}(N) q^{N-1} := 1/\Delta(q)$$

by a contour integral:

$$p_{24}(N) = \frac{1}{2\pi i} \oint q^{-N} dq/\Delta(q) = \int dt\, t^{-14} \frac{\exp\left(\frac{\pi(N-1)}{t}\right)}{\Delta\left(e^{-4\pi t}\right)}$$

- By contrast, the OSV formula can be rewritten as

$$\Omega_{OSV}(p^1, q_0) \sim \int d\tau_1\, d\tau_2\, \tau_2^{-14} \frac{\exp\left(\frac{\pi(N-1)}{\tau_2}\right)}{|\Delta\left(e^{-2\pi\tau_2 + 2\pi i \tau_1}\right)|^2}$$

- The two agree asymptotically when $\Delta(q) \sim q$, but the OSV formula does not appear to make sense non-perturbatively !

# Reverse engineering

- Rather than extracting BH degeneracies from the topological amplitude, one may try to construct the BH partition function from our partial knowledge of exact degeneracies.
- In type II/$K3 \times T^2$, the lattices of electric charges are

$$
\begin{aligned}
\Lambda^{IIA}_{elec} &= D0(q_0) \oplus D2/T2(q_1) \oplus D2/\gamma_2(q_a) \oplus \dots \\
\Lambda^{IIA}_{mag} &= D6/K3 \times T^2(p^0) \oplus D4/K3(p^1) \oplus D4/T_2 \times \gamma_2(p^a) \oplus \dots
\end{aligned}
$$

Exact degeneracies are known for purely electric heterotic states , i.e. for vanishing $D2/T2$, $D4/T^2 \times \gamma^2$, $D6/K3 \times T^2$.

- Setting $p^0 = p^a = 0$, the BH partition function includes terms with $q^1 = 0$:

$$
Z'_{BH} = \sum_{q^0, q^a \in II^{3,19}} p_{24} \left( 1 + p^1 q_0 + \frac{1}{2} q_a C^{ab} q_b \right) e^{-\pi (q_0 \phi^0 + q_a \phi^a)}
$$

- Inserting the unity

$$1 = \sum_N \delta\left[N - 1 - \frac{1}{2} q_a C^{ab} q_b\right] = \sum_N \sum_{k^0=0}^{p^1-1} \frac{1}{p^1} e^{2\pi i k^0 (N-1-\frac{1}{2} q_a C^{ab} q_b)/p^1}$$

inside the sum, the sum over $N$ reconstructs the Dedekind function

$$Z'_{BH} = \frac{1}{p^1} \sum_{k^0=0}^{p^1-1} \frac{e^{-2\pi i \tau q_a C^{ab} q_b - \pi \phi^a q_a}}{\Delta(\tau)} \, , \quad \tau = \frac{i\phi^0 + 2k^0}{2p^1}$$

Doing a modular transformation on $\tau$ and a Poisson resummation on $q_a$ gives

$$Z'_{BH} = \sum_{k_0=0}^{p^1-1} \sum_{k^a \in II^{19,3}} Z_0(\phi^A + 2ik^A) \, , \quad Z_0(\phi^A) = \frac{\exp\left[-\frac{\pi}{2} \frac{p^1 C_{ab} \phi^a \phi^b}{\phi^0}\right]}{(p^1)^2 \, \Delta\left(\frac{2ip_1}{\phi^0}\right)}$$

- While $Z_0$ looks close to the topological string amplitude, it is in fact different: no $|\Delta|^2$, and the argument has no $\phi^1$ dependence !

- The sum over translations $\phi^A \to \phi^A + 2ik^A$ guarantees that the BH partition function has the expected periodicity due to the charge quantization. Yet much of the information in the topological string amplitude could be lost in the process of averaging !
- It is tempting to conjecture that the exact black hole partition function is a theta series whose general term is the topological string amplitude.
- Indeed, in a unrelated development, non-Gaussian theta series have been constructed based on cubic characters $\exp(I_3(X^A)/X_0)$ quite similar to CY prepotentials. It would be very interesting if invariance under monodromies in the CY moduli space could be realized in the same fashion.

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# Discussion

- The OSV conjecture for the partition function of BPS black holes has passed several non-trivial tests, leading to agreement with microscopic degeneracies to all orders in $1/Q^2$.
- For this to hold, a number of ambiguities had to be lifted: integration contour, holomorphic anomalies, identification of $\Omega_{OSV}$ with helicity supertraces, count states with arbitrary $J$.
- OSV is very successful in $N = 4$ models, less so in some $N = 2$ models. When $\chi \neq 0$, the saddle point lies at strong coupling of the pointlike instanton series, requiring a non-perturbative completion of the topological amplitude in this sector.
- At the non-perturbative level, a relation like "$Z_{BH} = |e^F|^2$" cannot hold, if only because the rhs is not periodic in $\phi$ modulo $2i$. This suggests that the BH partition function may instead be a theta series built on $e^F$, possibly with interesting automorphic properties.
- In a rather orthogonal approach, Sen was able to reproduce the BH entropy to all orders using a different ensemble, with a chemical potential $\mu$ for $Q^2$ rather than $Q$, and keeping non-holomorphic corrections. It would be interesting to relate the two approaches.