Instanton corrected hypermultiplet moduli spaces and black hole counting

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Introduction I

The moduli space of $D = 4, \mathcal{N} = 2$ (ungauged) SUGRA splits into a product $\mathcal{M}_V \times \mathcal{M}_H$. The vector multiplet part $\mathcal{M}_V$ is very well understood, but the hypermultiplet part $\mathcal{M}_H$ is still largely mysterious:

- In Type II/CY compactifications, $\mathcal{M}_V$ can be computed exactly in the (2,2) SCFT. On the contrary, $\mathcal{M}_H$ is subject to g_s corrections, especially instantons.
- In Het/$K3 \times T^2$, $\mathcal{M}_H$ can in principle be computed in the (0,4) SCFT, but this is hard in practice.
- Technical difficulty: $\mathcal{M}_V$ is a special Kähler manifold, conveniently described by a holomorphic prepotential. $\mathcal{M}_H$ is a quaternionic-Kähler manifold, not even Kähler.

Introduction II

Computing the exact QK metric $\mathcal{M}_H$ would have lots of applications:

- New CY topological invariants, higher rank Donaldson-Thomas invariants, NS5-D-brane bound states, ...
- A very useful packaging of black hole degeneracies, keeping track of the dependence on the moduli at infinity.
- New tests of Heterotic-type II duality, new K3 invariants, ...
- Possibly important for model building: the scalar potential in gauged supergravity generally depends on the hypermultiplet metric.

Beyond the QK metric, an infinite series of higher-derivative F-term interactions on $\mathcal{M}_H$ awaits to be computed...

Outline

1. Introduction
2. The hypermultiplet landscape
3. Twistor techniques for QK spaces
4. Instanton corrections to hypermultiplets

Outline

The hypermultiplet landscape I

Consider type IIA/ $\mathbb{R}^{1,3} \times X$:

$$\mathcal{M}_4 = \mathcal{SK}_K(X)_{2h^{1,1}} \times \mathcal{QK}_{cx}(X)_{4(h^{1,2}+1)}$$

- $\mathcal{SK}_K(X)$ parametrizes the complexified Kähler structure $J \in H^2(X,\mathbb{C})$. In the large volume limit, it is determined by the intersection product C_{abc} and $\chi(X)$. At finite volume it receives worldsheet instanton corrections: genus zero Gromov Witten invariants.
- $\mathcal{QK}_{cx}(X)$ describes parametrizes the complex structure of X, the Wilson lines of the RR forms on $H_{\rm odd}(X)$, and the axio-dilaton. It is well understood at zero string coupling, but gets one-loop correction and instanton corrections from D-branes$/H_{\rm odd}(X)$ and NS5$/X$ (see later)

The hypermultiplet landscape II

Consider now type IIA/ $\mathbb{R}^{1,2} \times S^1 \times X$:

$$\mathcal{M}_3 = \mathcal{QK}_K(X)_{4(h^{1,1}+1)} \times \mathcal{QK}_{cx}(X)_{4(h^{1,2}+1)}$$

- $\mathcal{QK}_{cx}(X)$ is identical to the HM moduli space in 4 dimensions.
- $\mathcal{QK}_K(X)$ parametrizes, in addition to the complexified Kähler structure J, the Wilson lines of the RR forms on $H_{\rm even}(X)$, the radius R of the circle and the NUT scalar, dual to the KK gauge field. At $R = \infty$, it is obtained by from $\mathcal{SK}_K(X)$ by the c-map,

$$\tilde{T}_{2h^{1,1}+3} \to \mathcal{QK}_K(X) \to \mathbb{R}^+ \times \mathcal{SK}_K(X)$$

where $\tilde{T}_{2h^{1,1}+3}$ is a twisted torus, a circle bundle over $T_{2h^{1,1}+2}$.

The hypermultiplet landscape III

Similarly, consider type IIB/ $\mathbb{R}^{1,3} \times Y$:

$$\mathcal{M}_4 = \mathcal{SK}_{cx}(Y)_{2h^{1,2}} \times \widetilde{\mathcal{QK}}_K(Y)_{4(h^{1,1}+1)}$$

- $\mathcal{SK}_{cx}(Y)$ parametrizes the complex structure of Y, via the periods $X^\Lambda = \int_{\gamma^\Lambda} \Omega, F_\Lambda = \int_{\gamma_\Lambda} \Omega = \partial F / \partial X^\Lambda$ of the holomorphic 3-form Ω. It has no quantum correction whatsoever.
- $\widetilde{\mathcal{QK}}_K(Y)$ parametrizes the complexified Kähler structure, the Wilson lines of the RR forms on $H_{\rm even}(Y)$, and the axio-dilaton. It is well understood at zero string coupling, but gets one-loop correction and instanton corrections from D-branes$/H_{\rm even}(Y)$ and NS5$/Y$ (see later).

The hypermultiplet landscape IV

Finally, consider now type IIB/ $\mathbb{R}^{1,2} \times S^1 \times Y$:

$$\mathcal{M}_3 = \widetilde{\mathcal{QK}}_{cx}(Y)_{4(h^{1,2}+1)} \times \widetilde{\mathcal{QK}}_K(Y)_{4(h^{1,1}+1)}$$

- $\widetilde{\mathcal{QK}}_K(Y)$ is identical to the HM moduli space in 4 dimensions.
- $\widetilde{\mathcal{QK}}_{cx}(Y)$ parametrizes, in addition to the complex structure, the Wilson lines of the RR forms on $H_{\rm odd}(X)$, the radius R of the circle and the NUT scalar, dual to the KK gauge field. At $R = \infty$, it is obtained by from $\mathcal{SK}_{cx}(Y)$ by the c-map,

$$\tilde{T}_{2h^{1,2}+3} \to \widetilde{\mathcal{QK}}_{cx}(Y) \to \mathbb{R}^+ \times \mathcal{SK}_{cx}(Y)$$

where $\tilde{T}_{2h^{1,2}+3}$ is a twisted torus, a circle bundle over $T_{2h^{1,2}+2}$.

The hypermultiplet landscape V

Using dualities, we can reduce these 4 QK manifolds to a single one:

- Good old mirror symmetry ($Y = \tilde{X}$): exchanges Kahler and cx structures:

$$\mathcal{SK}_K(X) = \mathcal{SK}_{cx}(\tilde{X})$$

- T-duality on S^1 ($Y = X$) : exchanges VM and HM, radius and coupling:

$$\mathcal{QK}_K(X) = \widetilde{\mathcal{QK}}_K(X), \qquad \mathcal{QK}_{cx}(X) = \widetilde{\mathcal{QK}}_{cx}(X)$$

- Generalized mirror symmetry:

$$\mathcal{QK}_K(X) = \mathcal{QK}_{cx}(\tilde{X})$$

- S-duality of type IIB, or lift IIA to M-theory on $X \times T^2$: $SL(2,\mathbb{Z})$ should act isometrically any of these spaces.

The hypermultiplet landscape VI

More slowly:

- T-duality implies that the 4D HM spaces $\mathcal{QK}_{cx}(X), \widetilde{\mathcal{QK}}_K(Y)$ at $g_s = 0$ are given by the c-map of $\mathcal{SK}_{cx}(X), \mathcal{SK}_K(Y)$.

  Cecotti Ferrara Girardello; Ferrara Sabharwal

- The 4D HM spaces are known to have a one-loop correction proportional to χ, inducing a further twist of $\tilde{T}^{2h+3}$ over the SK base. This predicts a one-loop correction to the 3D VM spaces $\mathcal{QK}_K(X), \widetilde{\mathcal{QK}}_{cx}(Y)$, coming from loops of gravitons along S^1.

  Antoniadis Minasian Theisen Vanhove; Robles-Llana, Saueressig, Vandoren

- Moreover, D-instanton on (resp. NS5-instanton) corrections to $\mathcal{QK}_{cx}(X)$ must equal contributions from black holes winding around S^1 (resp. Taub-NUT instantons) to $\widetilde{\mathcal{QK}}_{cx}(X)$. Thus $\mathcal{QK}_{cx}(X)$ looks like a very good way to package degeneracies of BPS black holes, keeping track of moduli dependence !

The hypermultiplet landscape VII

To summarize: to a given CY 3-fold X one may associate two QK spaces:

- $\mathcal{QK}_K(X)$, describing the complexified Kähler structure of X, together with stable objects in the derived category of coherent sheaves on X, and NS5.
- $\mathcal{QK}_{cx}(X)$, describing the complex structure of X, together with stable objects in the derived Fukaya category of special Lagrangian submanifolds (SLAG) on X, and NS5.
- Generalized (homological) mirror symmetry identifies $\mathcal{QK}_K(X) = \mathcal{QK}_{cx}(\tilde{X})$.
- $SL(2,\mathbb{Z})$ (and, if X is K3 fibered, $SL(3,\mathbb{Z})$ by Het/type II duality) must act isometrically on $\mathcal{QK}_K(X)$.

Outline

Twistor techniques for QK spaces I

- Recall that a $4d$-dimensional manifold $\mathcal{M}$ is QK if its holonomy is $Sp(d) \times Sp(1) \subset SO(4d)$. $\mathcal{M}$ is an Einstein space, in general non Kähler. The relevant spaces for SUGRA have negative curvature.
- QK manifolds $\mathcal{M}$ of dimension $4d$ are in (local) 1-1 correspondence with HK cones $\mathcal{S}$ of dimension $4d+4$: HK manifolds with a homothetic vector and a $SU(2)$ isometric action rotating the 3 complex structures.

Swann; De Wit Rocek Vandoren

- By Hitchin's theorem, HK manifolds $\mathcal{S}$ of dimension $4d+4$ are in 1-1 correspondence with complex spaces $\mathcal{Z}_\mathcal{S} = \mathcal{S} \times \mathbb{C}P^1$ equipped with a complex symplectic structure Ω (and some more data).
- The HKC condition restricts $\mathcal{Z}_\mathcal{S}$ to have a $\mathbb{C}^*$ action under which Ω is homogeneous. The complex symplectic structure Ω descends to a complex contact structure on $\mathcal{Z} = \mathcal{M} \ltimes \mathbb{C}P^1 = \mathcal{S}/\mathbb{C}^*$.

Lebrun Salamon; APSV; Ionas Neitzke

Twistor techniques for QK spaces II

- QK manifolds of dimension $4d$ are in (local) 1-1 correspondence with complex contact manifolds $\mathcal{Z}$ of dimension $4d+2$. $\mathcal{Z}$ is a $\mathbb{C}P^1$ bundle over $\mathcal{M}$, and carries a (Lorentzian) Kähler-Einstein metric:

$$ds^2_{\mathcal{Z}} = \frac{|D\zeta|^2}{(1+\zeta\bar{\zeta})^2} + \frac{\nu}{4}\,\mathrm{d}s^2_{\mathcal{M}}$$

where

$$D\zeta \equiv \mathrm{d}\zeta + p_+ - \mathrm{i}p_3\,\zeta + p_-\,\zeta^2$$

is the canonical (1,0)-form on $\mathcal{Z}$, $\vec{p}$ is the $Sp(1) = SU(2)$ part of the Levi-Civitta connection on $\mathcal{M}$, and $\nu \propto R(\mathcal{M}) < 0$ is a numerical constant.

Lebrun, Salamon

Twistor techniques for QK spaces III

- Locally in a patch U_i, one can always find a function $\Phi_{[i]}(x^\mu, \zeta)$, defined up to addition of a holomorphic function, such that

$$\mathcal{X}^{[i]} = 2\left(e^{\Phi_{[i]}} D\zeta\right)/\zeta\,,$$

  is a holomorphic one-form (i.e. $\bar\partial$ closed) on $\mathcal{Z}$, invariant under the real structure

$$\overline{\tau(\mathcal{X}^{[i]})} = -\mathcal{X}^{[\bar{i}]}\,,$$

  where τ is the antipodal map acting as $\tau : \zeta \to -1/\bar\zeta$.
- The "contact potential" $\Phi_{[i]}$ yields a Kähler potential for $ds^2_{\mathcal{Z}}$:

$$K^{[i]}_{\mathcal{Z}} = \log\frac{1+\zeta\bar\zeta}{|\zeta|} + \mathrm{Re}\,\Phi_{[i]}(x^\mu,\zeta)\,.$$

APSV

Twistor techniques for QK spaces IV

- Locally on U_i, there exist complex Darboux coordinates such that

$$\mathcal{X}^{[i]} = \mathrm{d}\alpha^{[i]} + \xi^{\Lambda}_{[i]}\, \mathrm{d}\tilde{\xi}^{[i]}_{\Lambda} \,.$$

- The global information is provided by complex contact transformations relating Darboux coordinates on $U_i \cap U_j$. These are generated by holomorphic functions $S^{[ij]}(\xi^{\Lambda}_{[i]}, \tilde{\xi}^{[j]}_{\Lambda}, \alpha^{[j]})$:

$$\xi^{\Lambda}_{[j]} = f_{ij}^{-2}\, \partial_{\tilde{\xi}^{[j]}_{\Lambda}} S^{[ij]} \,, \qquad \tilde{\xi}^{[i]}_{\Lambda} = \partial_{\xi^{\Lambda}_{[i]}} S^{[ij]} \,,$$

$$\alpha^{[i]} = S^{[ij]} - \xi^{\Lambda}_{[i]} \partial_{\xi^{\Lambda}_{[i]}} S^{[ij]} \,, \qquad e^{\Phi_{[i]}} = f_{ij}^2\, e^{\Phi_{[j]}} \,,$$

  where $f_{ij}^2 \equiv \partial_{\alpha^{[j]}} S^{[ij]}$, in such a way that $\mathcal{X}^{[i]} = f_{ij}^2\, \mathcal{X}^{[j]}$.

- $S^{[ij]}$ are subject to consistency conditions $S^{[ijk]}$, gauge equivalence under local contact transformations $S^{[i]}$, and reality constraints.

Twistor techniques for QK spaces V

- For generic choices of $S^{[ij]}$, the moduli space of solutions of the above gluing conditions, regular in each patch, is finite dimensional, and equal to (a circle bundle over) $\mathcal{M}$ itself.
- On each patch U_i, $u_m^{[i]} = (\xi^\Lambda_{[i]}, \tilde\xi_\Lambda^{[i]}, \alpha^{[i]})$ admit a Taylor expansion in ζ around ζ_i, whose coefficients are functions on $\mathcal{M}$. The functions $u_m^{[i]}(\zeta, x^\mu)$ parametrize the "twistor line" over $x^\mu \in \mathcal{M}$.
- The metric on $\mathcal{M}$ can be obtained by expanding $\mathcal{X}^{[i]}$ and $\mathrm{d}u_m^{[i]}$ around ζ_i, extracting the $SU(2)$ connection $\vec p$ and a basis of $(1,0)$ forms on $\mathcal{M}$ in almost complex structure $J(\zeta_i)$, and using $\mathrm{d}\vec p + \frac{1}{2}\,\vec p \times \vec p = \frac{\nu}{2}\,\vec\omega$.
- Deformations of $\mathcal{M}$ correspond to deformations of $S^{[ij]}$, so are parametrized by $H^1(\mathcal{Z}, \mathcal{O}(2))$.

Lebrun, Salamon

Twistor techniques for QK spaces VI

- Any (infinitesimal) isometry $\kappa_{\mathcal{M}}$ of $\mathcal{M}$ lifts to a holomorphic isometry $\kappa_{\mathcal{Z}}$ of $\mathcal{Z}$. The moment map construction provides an element of $H^0(\mathcal{Z}, \mathcal{O}(2))$, given locally by holomorphic functions

$$\mu_{[i]} = \kappa_{\mathcal{Z}} \cdot \mathcal{X}^{[i]} = e^{\Phi_{[i]}} \left(\mu_+ \zeta^{-1} - \mathrm{i}\mu_3 + \mu_- \zeta \right) .$$

  The moment map of the Lie bracket $[\kappa_1, \kappa_2]$ is the contact-Poisson bracket of the moment maps.

- Toric QK manifolds are those which admit $d+1$ commuting isometries. In this case, one can choose $\mu_{[i]}$ as the position coordinates. The transition functions must then take the form

$$S^{[ij]} = \alpha^{[j]} + \xi^{\Lambda}_{[i]} \tilde{\xi}^{[j]}_{\Lambda} - H^{[ij]} ,$$

  where $H^{[ij]}$ depends on $\xi^{\Lambda}_{[i]}$ only.

Twistor techniques for QK spaces VII

- More generally, one can consider "nearly toric QK", where $H^{[ij]}$ is a general function but its derivatives wrt to $\tilde\xi^{[j]}_\Lambda, \alpha^{[j]}$ are taken to be infinitesimal.
- The twistor lines can then be obtained by Penrose-type integrals. The formulae are simplest when $\partial_{\alpha^{[j]}} H^{[+j]} = 0$, and in the absence of "anomalous dimensions", e.g.

$$\xi^\Lambda_{[i]}(\zeta, x^\mu) = \zeta^\Lambda + \frac{Y^\Lambda}{\zeta} - \zeta \bar{Y}^\Lambda - \frac{1}{2} \sum_j \oint_{C_j} \frac{\mathrm{d}\zeta'}{2\pi \mathrm{i} \zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \left(\partial_{\tilde\xi^{[j]}_\Lambda} - \xi^\Lambda_{[i]} \, \partial_{\alpha^{[j]}} \right) H^{[+}$$

$$e^{\Phi_{[i]}} = \frac{1}{4} \sum_j \oint_{C_j} \frac{\mathrm{d}\zeta'}{2\pi \mathrm{i} \zeta'} \left(\zeta'^{-1} Y^\Lambda - \zeta' \bar{Y}^\Lambda \right) \partial_{\xi^\Lambda_{[j]}} H^{[+j]}(\xi(\zeta'), \tilde\xi(\zeta'))$$

Outline

The perturbative hypermultiplet moduli space I

- Consider the HM moduli space $\mathcal{M} = \widetilde{\mathcal{QK}}_K$ in type IIB compactified on Y. Recall that at tree level, $\mathcal{M} \sim \mathrm{c-map}(\mathcal{SK}_K)$. The latter is governed by the prepotential $F(X)$, given at large volume by

$$F(X^\Lambda) = -\kappa_{abc} \frac{X^a X^b X^c}{6X^0} + \frac{\zeta(3)(X^0)^2}{2(2\pi \mathrm{i})^3} \chi_Y - \frac{(X^0)^2}{(2\pi \mathrm{i})^3} \sum_{q_a > 0} n^{(0)}_{q_a} \mathrm{Li}_3 \left(e^{2\pi \mathrm{i} q_a \frac{q^a}{X^0}} \right)$$

- The twistor space of the c-map is governed by

$$H^{[0+]}_{\mathrm{tree}} = \frac{i}{2} F(\xi^\Lambda) \,, \quad H^{[0-]}_{\mathrm{tree}} = \frac{i}{2} \bar{F}(\xi^\Lambda)$$

Roček Vafa Vandoren

- The effect of the one-loop correction is to induce an "anomalous dimension" $c_\alpha = \frac{1}{96\pi} \chi_Y$ for the action coordinate α near $\zeta = 0$.

The perturbative hypermultiplet moduli space II

- As a result, the twistor lines are given at one loop by

$$\begin{aligned}
\xi^\Lambda &= \zeta^\Lambda + \tfrac{1}{2}\tau_2 \left(\zeta^{-1} z^\Lambda - \zeta\, \bar z^\Lambda\right) , \\
\rho_\Lambda &= \tilde\zeta_\Lambda + \tfrac{1}{2}\tau_2 \left(\zeta^{-1} F_\Lambda(z) - \zeta\, \bar F_\Lambda(\bar z)\right) , \\
\tilde\alpha &= \sigma + \tfrac{1}{2}\tau_2 \left(\zeta^{-1} W(z) - \zeta\, \bar W(\bar z)\right) - \tfrac{\mathrm{i}\chi_Y}{24\pi} \log \zeta ,
\end{aligned}$$

Neitzke BP Vandoren; Alexandrov; APSV

$$e^\Phi = \frac{\tau_2^2}{2}\, V(t^a) - \frac{\chi_Y \zeta(3)}{8(2\pi)^3}\, \tau_2^2 - \frac{\chi_Y}{192\pi}$$

$$+ \frac{\tau_2^2}{4(2\pi)^3} \sum_{q_a\gamma^a \in H_2^+(Y)} n_{q_a}^{(0)} \mathrm{Re}\left[\mathrm{Li}_3\left(X\right) + 2\pi q_a t^a\, \mathrm{Li}_2\left(X\right)\right]$$

$$W(z) \equiv F_\Lambda(z)\zeta^\Lambda - z^\Lambda \tilde\zeta_\Lambda , \quad X = e^{2\pi \mathrm{i} q_a z^a} , \quad z^a = b^a + \mathrm{i} t^a ,$$

$$\rho_\Lambda \equiv -2\mathrm{i}\tilde\xi_\Lambda^{[0]} , \quad \tilde\alpha \equiv 4\mathrm{i}\alpha^{[0]} + 2\mathrm{i}\tilde\xi_\Lambda^{[0]}\xi^\Lambda ,$$

Enforcing S-duality and electric-magnetic duality I

- In the absence of one-loop and worldsheet instanton corrections, $\mathcal{M}$ admits an isometric action of $SL(2,\mathbb{R})$. This can be shown by producing global sections of $H^0(\mathcal{Z},\mathcal{O}(2))$ satisfying the $SL(2,\mathbb{R})$ algebra under (contact) Poisson bracket:

$$\xi^0 \mapsto \frac{a\xi^0+b}{c\xi^0+d}\,, \qquad \xi^a \mapsto \frac{\xi^a}{c\xi^0+d}\,,$$

$$\tilde\xi_a \mapsto \tilde\xi_a + \frac{\mathrm{i}\,c}{4(c\xi^0+d)}\kappa_{abc}\xi^b\xi^c\,,$$

$$\begin{pmatrix}\tilde\xi_0\\ \alpha\end{pmatrix} \mapsto \begin{pmatrix} d & -c\\ -b & a\end{pmatrix}\begin{pmatrix}\tilde\xi_0\\ \alpha\end{pmatrix} + \frac{\mathrm{i}c\,\kappa_{abc}\xi^a\xi^b\xi^c}{12(c\xi^0+d)^2}\begin{pmatrix} c(c\xi^0+d)\\ -[c(a\xi^0+b)+2]\end{pmatrix}.$$

Berkovits Siegel; Robles-Llana Roček Saueressig Theis Vandoren; APSV

Enforcing S-duality and electric-magnetic duality II

- This descends to the standard action of $SL(2,\mathbb{R})$ on $\mathcal{M}$,

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}\,, \qquad t^a \mapsto t^a |c\tau + d|\,, \qquad c_a \mapsto c_a\,,$$

$$\begin{pmatrix} c^a \\ b^a \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} c^a \\ b^a \end{pmatrix}\,, \qquad \begin{pmatrix} c_0 \\ \psi \end{pmatrix} \mapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} c_0 \\ \psi \end{pmatrix}$$

where the type IIB fields c^0, c^a, c_a, c_0, ψ are related to the type IIA variables $\zeta^\Lambda, \tilde\zeta_\Lambda, \sigma$ by the "generalized mirror map"

$$\zeta^0 = \tau_1\,, \qquad \zeta^a = -(c^a - \tau_1 b^a)\,,$$

$$\tilde\zeta_a = c_a + \frac{1}{2}\kappa_{abc}\, b^b (c^c - \tau_1 b^c)\,, \quad \tilde\zeta_0 = c_0 - \frac{1}{6}\kappa_{abc}\, b^a b^b (c^c - \tau_1 b^c)\,,$$

$$\sigma = -2(\psi + \frac{1}{2}\tau_1 c_0) + c_a(c^a - \tau_1 b^a) - \frac{1}{6}\kappa_{abc}\, b^a c^b (c^c - \tau_1 b^c)\,.$$

Gunther Herrmann Louis; Berkooz BP; APSV

Enforcing S-duality and electric-magnetic duality III

- The contact potential $e^{\Phi} = \frac{\tau_2^2}{2} V(t^a)$ is not invariant, but transforms so that $K_{\mathcal{Z}}$ undergoes a Kähler transformation,

$$e^{\Phi} \mapsto \frac{e^{\Phi}}{|c\tau + d|}\,, \quad K_{\mathcal{Z}} \mapsto K_{\mathcal{Z}} - \log(|c\xi^0 + d|)\,, \quad \mathcal{X}^{[i]} \to \frac{\mathcal{X}^{[i]}}{c\xi^0 + d}$$

- The one-loop term and worldsheet instanton corrections break $SL(2,\mathbb{R})$ continuous S-duality. A discrete subgroup $SL(2,\mathbb{Z})$ can be restored by summing over images:

$$\mathrm{Li}_k(e^{2\pi \mathrm{i} q_a z^a}) \to \sum_{m,n}{}' \frac{\tau_2^{k/2}}{|m\tau + n|^k} e^{-S_{m,n}}\,,$$

where $S_{m,n} = 2\pi q_a |m\tau + n|\, t^a - 2\pi \mathrm{i} q_a (mc^a + nb^a)$ is the action of a (m,n)-string wrapped on $q_a \gamma^a$.

Robles-Llana Roček Saueressig Theis Vandoren

Enforcing S-duality and electric-magnetic duality IV

- The tree-level $2\zeta(3)\chi_Y/g_s^2$ and $\zeta(2)\chi_Y$ are unified together with D-instantons, while the worldsheet instantons are unified with Euclidean D- string instantons.
- After Poisson resummation on $n \to q_0$, we get a sum over D(-1)-D1 bound states,

$$e^\Phi = \cdots + \frac{\tau_2}{8\pi^2} {\sum_{q_\Lambda}}' n_{q_a}^{(0)} \sum_{m=1}^{\infty} \frac{|k_\Lambda z^\Lambda|}{m} \cos\left(2\pi m\, q_\Lambda \zeta^\Lambda\right) K_1\left(2\pi m\, |q_\Lambda z^\Lambda| \tau_2\right)$$

where $z^0 = 1, q_0 \in \mathbb{Z}, q_a\gamma^a \in H_2^+(Y), n_0^{(0)} = -\chi_Y/2$.

Robles-Llana Saueressig Theis Vandoren

Enforcing S-duality and electric-magnetic duality V

- From the point of view of type IIA on the mirror CY X, $D(-1)$ and $D1$ correspond to $D2$ wrapped on A-cycles in $H_3(X,\mathbb{Z})$. B-cycles can be restored by symplectic invariance:

$$e^{\Phi} = \dots + \frac{\tau_2}{8\pi^2} \sum_{\gamma} n_\gamma \sum_{m=1}^{\infty} \frac{|W_\gamma|}{m} \cos\left(2\pi m \Theta_\gamma\right) K_1\left(2\pi m |W_\gamma|\right)$$

$$W_\gamma \equiv \frac{1}{2}\tau_2 \left(q_\Lambda z^\Lambda - p^\Lambda F_\Lambda \right) , \quad \Theta_\gamma \equiv q_\Lambda \zeta^\Lambda - p^\Lambda \tilde{\zeta}_\Lambda$$

  where n_γ are a priori new topological invariants of X. However this result can only hold in the "one instanton" approximation.

- The exponent $|W_\gamma| \pm \mathrm{i}\Theta_\gamma$ agrees with the classical action of D2-branes wrapped on a SLAG γ, or D5-branes with a coherent sheaf F.

The hypermultiplet twistor space I

- The contact structure on the twistor space can be obtained by inserting an elementary symplectomorphism generated by

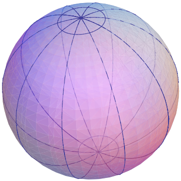
$$S_\gamma^{[ij]}(\xi^\Lambda_{[i]}, \tilde\xi_\Lambda^{[j]}, \alpha^{[j]}) = \alpha^{[j]} + \xi^\Lambda_{[i]}\, \tilde\xi_\Lambda^{[j]} + \frac{\mathrm{i}}{2(2\pi)^2}\, n_\gamma\, \mathrm{Li}_2\left(\mathcal{X}_\gamma\right) \, .$$

Gaiotto Moore Neitzke

across the "BPS ray" $\ell(\gamma)$,

$$\ell(\gamma) = \{\zeta :\ \pm W_\gamma/\zeta \in \mathrm{i}\mathbb{R}^-\} \, ,$$

$$\mathcal{X}_\gamma = e^{-2\pi\mathrm{i}(q_\Lambda \xi^\Lambda_{[i]} + 2\mathrm{i} p^\Lambda \tilde\xi_\Lambda^{[j]})}$$



- The BPS rays and the invariants n_γ in general depend on the point in $\mathcal{SK}(X)$.

The hypermultiplet twistor space II

- BPS rays $\ell(\gamma_1)$ and $\ell(\gamma_2)$ cross at lines of marginal stability. The wall crossing formula

$$\prod^{\curvearrowleft}_{\substack{\gamma=n\gamma_1+m\gamma_2 \\ m>0,n>0}} U_\gamma^{n^-(\gamma)} = \prod^{\curvearrowright}_{\substack{\gamma=n\gamma_1+m\gamma_2 \\ m>0,n>0}} U_\gamma^{n^+(\gamma)} \,,$$

  ensures that the consistency of the twistor space across the LMS.

  Gaiotto Neitzke Moore; Kontsevich Soibelman; Joyce; ...

- The metric is regular across the LMN. Physically, single instanton contributions on one side of the wall get replaced by multi-instanton configurations on the other side.

Counting BH and NS5-branes I

- If indeed $n_{p,q}$ counts the number of BH microstates, the instanton series will be severely divergent. It is conceivable that the finite radius of the circle puts a cut-off on allowed charges, or that only polar states contribute...
- We know of one example where the instanton measure and BPS degeneracy differ: R^4 couplings in $D = 9$ type II string theories. The D(-1) instanton measure $n(N)$ is given by the $U(N)$ matrix integral, while the index degeneracy $\Omega(N)$ of N D0-branes is given by the Witten index of the $U(N)$ Matrix at zero temperature:

$$\Omega(N) = 1 = \left(1 + \sum_{d|N, d<N} \frac{1}{d^2}\right) - \sum_{d|N, d<N} \frac{1}{d^2} = n(N) + b(N)$$

The difference $b(N)$ comes from a "bulk contribution" to the index due to flat directions in the potential. *Sethi Stern; Green Gutperle*

NS5-brane or NUT contributions I

- In contrast to D-instantons, NS5-brane instantons should induce genuine contact transformations, with $S^{[ij]} \propto e^{\mathrm{i}k\alpha^{[j]}} F_k(\xi, \tilde{\xi})$. It is not clear a priori what function F_k to consider.
- One might hope to determine the NS5 instantons by $SL(2, \mathbb{Z})$ duality from the D5-instantons. This is difficult due to the complicated transformation rule of $\tilde{\xi}_\Lambda, \alpha$, and the fact that e^Φ becomes ζ-dependent.
- Enforcing a larger duality group, e.g. $SL(3, \mathbb{Z})$ as apparent in the dual heterotic string on $K3 \times T^3$, may allow to shortcut this route and obtain NS5-brane contributions from perturbative corrections.

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NS5-brane or NUT contributions II

- When the NS5-brane charge k is non-zero, electric and magnetic translations no longer commute: $[p^\Lambda, q_\Sigma] = k\delta^\Lambda_\Sigma$. As a result, the Fourier coefficients become wave functions:

$$F_k(\xi, \tilde{\xi}) = \sum_{l^\Lambda \in \Gamma_e/(2|k|\Gamma_e)} \sum_{n^\Lambda \in \Gamma_e + l^\Lambda} \Psi_{l^\Lambda}(\xi^\Lambda + n^\Lambda, k)\, e^{4\pi i k\, n^\Lambda \tilde{\xi}_\Lambda}$$

- To relate Ψ on different patches, the contact transformations must be quantized, consistently with wall crossing: the quantum dilogarithm is a natural candidate for this task...
- Does Ψ bear any connection to the (generalized) topological amplitude ?

Conclusion I

- Twistors give a powerful parametrization of QK manifolds. Determining the exact twistor space is hard, for lack of a consistent framework for non-perturbative string theory. Recent developments in mathematics are suggestive...
- The exact metric on $\mathcal{QK}_{K,cx}(X)$ seems to offer a very convenient packaging of the degeneracies of 4D black holes, although the issue of divergence remains to be understood.
- In some cases with a high degree of symmetry, one may hope that automorphy will fix the hypermultiplet metric exactly.
- One may also consider higher derivative $\tilde{F}_g$-type corrections to the hypers, suggestive of a generalized topological wave function.