# Black holes, instantons and twistors

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Based on work with Günaydin, Neitzke, Waldron,
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ULB-VUB-KUL Seminar,
Nov 19, 2008, Leuven

# Outline

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# Exact BPS black hole degeneracies I

- Explaining the microscopic origin of Bekenstein-Hawking entropy of black holes is a pass/fail test for any theory of quantum gravity. String theory has been very successful for (near) BPS black holes to LO (leading order), and in some cases to NLO as $Q \to \infty$.

*Strominger Vafa,...; Cardoso de Wit Mohaupt,...*

- For BPS BH preserving 4 supercharges in $D = 4, \mathcal{N} = (4|8)$ SUGRA in certain duality orbits, it has even been possible to obtain the microscopic degeneracies at finite $Q$ exactly. Those are beautifully encoded as Fourier coefficients of certain Siegel modular forms, related to certain Borcherds-Kac-Moody algebras.

*Dijkgraaf, Verlinde, Verlinde; ...*

# Exact BPS black hole degeneracies II

- For BPS BH preserving 4 supercharges in $D = 4, \mathcal{N} = 2$ SUGRA, the story is much less understood. The Ooguri-Strominger-Vafa conjecture $Z(p, \zeta) = |\Psi_{top}(p + \mathrm{i}\zeta)|^2$ is suggestive but raises more questions than answers.
- Part of the difficulty lies in the strong dependence of the BPS spectrum on the value of the moduli at infinity: while the indexed degeneracies are locally constant, they may jump on lines of marginal stability (LMS), where the decay $\Gamma \to \Gamma_1 \oplus \Gamma_2$ is allowed. This is familiar e.g. from $D = 4, \mathcal{N} = 2$ Seiberg-Witten gauge theories.

*Bilal Ferrari*

- On the SUGRA side, this jump corresponds to multi-centered solutions becoming unbound. Wall-crossing formulae give a powerful constraint on the exact BPS spectrum. In the simplest case,

$$\Delta\Omega(\Gamma, t) = (-1)^{\langle \Gamma_1, \Gamma_2 \rangle} \langle \Gamma_1, \Gamma_2 \rangle \, \Omega(\Gamma_1, t) \, \Omega(\Gamma_2, t)$$

*Denef; Denef, Moore; Konsevich Soibelman*

- LMS exist for $\mathcal{N} = 4$ SUGRA, but their patterns are much simpler than in $\mathcal{N} = 2$. In particular, the same Siegel modular form controls the degeneracies in all regions of moduli space, although the "contour prescription" to extract the Fourier coefficients differ.

*Sen; Cheng Verlinde, ...*

# BH partition function and 3D effective action I

- In general, one would be interested in the black hole partition function $Z(t;\zeta,\tilde{\zeta}) = \sum \Omega(p,q;t) e^{\mathrm{i}(q\zeta+p\tilde{\zeta})}$ at arbitrary values of the moduli $t$ at infinity, and with chemical potentials $(\zeta^I, \tilde{\zeta}_I)$ conjugate to all charges.
- Moreover, it is artificial to focus on one-particle states: while the one-particle spectrum jumps across a LMS, the many-particle spectrum is smooth.
- There is an even more intrinsic object to consider: reduce the $D=4$ SUGRA theory on an Euclidean circle. The partition function of the $D=3$ theory, or any coupling $F$ in the LEEA, is a function of the radius $\beta = e^U l_p$, the moduli $t$ at infinity, the Wilson lines $\zeta^I, \tilde{\zeta}_I$ of the 4D gauge fields and their duals, and the NUT scalar $\sigma$.

# 4D black holes vs 3D instantons I

- Any black hole solution in 4D wrapping along the circle yields an instanton in 3D, with classical action $S_{p,q} = e^U |Z_{p,q}| + \mathrm{i}(q\zeta + p\tilde{\zeta})$.
- There are also instantons in 3D with non-vanishing NUT charge $k$ which do not lift to bona fide BH in 4D, but their action goes like $S_k = \mathrm{i}k\sigma + \dots$.
- Thus, one may read off the 4D BPS degeneracies $\Omega(p, q; t)$ by Fourier decomposing $\int F(U, t, \zeta, \tilde{\zeta}, \sigma) d\sigma$ wrt $(\zeta, \tilde{\zeta})$.
- The power of this approach is that the 3D theory typically has a much larger group of symmetries $G_3$ than the 4D duality group $G_4$. While $G_4$ leaves the BH entropy invariant, $G_3$ mixes states with different entropy, and provides a spectrum generating symmetry, hopefully strong enough to fix the degeneracies entirely.

*Gunaydin Neitzke BP Waldron*

# 4D black holes vs 3D instantons II

- Incidentally, the continuous group $G_3(\mathbb{R})$ has been used as a solution generating symmetry to construct all spherically symmetric solutions in $D = 4$. Indeed, a spherically symmetric BH in 4D corresponds to a geodesic on the 3D moduli space $K_3 \backslash G_3$, hence to a one-parameter subgroup of $G_3$.

*Breitenlohner Gibbons Maison; Cvetic Youm*

- More recently, it was shown that BPS black holes correspond to BPS geodesics, afforded by the special holonomy of $\mathcal{M}_3$, and extremal black holes correspond to null geodesics with nilpotent charge of degree 3.

*Gunaydin Neitzke BP Waldron, Neitzke BP Vandoren, Gaiotto Li Padi*

# 4D black holes vs 3D instantons III

- For e.g., in $\mathcal{N}=4$, $G_4 = SL(2)\times SO(6,n_v)$, $G_3 = SO(8,n_v+2)$. The $\nabla^2 R^2$ couplings in $\mathcal{N}=4$ should be $G_3$-invariant and should count 1/4-BPS BH.
- Note that $G_3$ does not contain $Sp(2)\times SO(6,n_v)$. There may still be a correspondence between Siegel modular forms and $SO(8,n_v+2)$ automorphic forms...
- A crucial assumption that the instanton measure equals the BH degeneracy. Known exceptions to this fact arise for threshold bound states. If the assumption is true, there is still a potential danger that the instanton sum will be severely divergent... Let us set aside these worries for now.

# Outline

- We now restrict to $\mathcal{N}=2$ supergravity. The BPS-saturated coupling is the metric on the 3D moduli space, which splits into the product of two quaternionic-Kähler manifolds

$$\mathcal{M}_3 = \mathcal{M}_V^{(4n_V+4)} \times \mathcal{M}_H^{(4n_H)}$$

- Since the radius $U$ is in $\mathcal{M}_V$, the hypermultiplet moduli space $\mathcal{M}_H$ is identical to the one in 4 dimensions. In the limit $U \to \infty$, $\mathcal{M}_V$ reduces to the "c-map" of the 4D VM moduli space,

$$\mathcal{M}_V^{(4n_V+4)} = \mathbb{R}_U^+ \times \mathcal{M}_V^{(2n_v)} \times T^{2n_v+2}_{\zeta^I,\tilde{\zeta}_I} \times S^1_\sigma$$

Note that $S^1_\sigma$ is non trivially fibered over $T_{\zeta^I,\tilde{\zeta}_I}$, with $c_1 \propto d\zeta^I \wedge d\tilde{\zeta}_I$.

*Cecotti Ferrara Girardello; Ferrara Sabharwal*

# BHs and instantons in $\mathcal{N} = 2$ SUGRA II

- $\mathcal{M}_V$ receives perturbative corrections from KK states on $S^1$, and instanton corrections from 4D BH (and states with non-zero NUT charge). In type IIA on CY $X$, these are D0-D2-D4-D6 branes on $H^{even}(X)$, and KKM on $X \times S^1$.

- Similarly, in type IIB on the same CY $X$, $\mathcal{M}_H$ in 4D has a $c$-map structure at weak coupling (computable using the mirror CY $Y$), but it receives one-loop and instanton corrections, from D(-1)-D1-D3-D5 branes on $H^{even}(X) \times S^1$, and NS5 on $X$.

*Antoniadis Minasian Theisen Vanhove; Robles-Llana, Saueressig, Vandoren*

- T-duality along $S^1$ maps IIA to IIB (on the same CY $X$), and exchanges the VM and HM moduli spaces :

$$\mathcal{M}_V^A = \mathcal{M}_H^B \,, \qquad \mathcal{M}_H^A = \mathcal{M}_V^B$$

  Thus, the problem of counting 4D BH is equivalent to that of computing instanton corrections to the 4D hypermultiplet space !
- Before attacking this problem, let us review some general techniques for quaternionic-Kähler spaces...

# Outline

# Twistor space, Swann bundle I

- QK metrics are difficult to describe, since they are usually not Kähler. HK metrics are Kähler, but the Kähler potential must satisfy some complicated differential equations. Both of them are however admit an algebraic (holomorphic) description via twistors.
- Recall that a $4d$-dimensional manifold is QK if its holonomy is $Sp(d) \times Sp(1)$, and HK if its holonomy is in $Sp(d)$.
- Given any QK manifold $\mathcal{M}$, one may construct $\mathbb{C}^2 \sim \mathbb{R}^+ \times SU(2)$ bundle over $\mathcal{M}$ by canceling the $Sp(1)$ part of the connection: $\mathcal{S}$ is a HK manifold with an $SU(2)$ isometric action and homothetic Killing vector, known as the Swann bundle or HK cone $\mathcal{S}$.

*Salamon; Swann; de Wit Rocek Vandoren*

- Any HK manifold $\mathcal{S}$ carries an complex symplectic structure, since $\Omega = \omega_1 + i\omega_2$ is holomorphic wrt to $J_3$.
- More generally, one may consider the twistor space $\mathcal{Z}_{\mathcal{S}} = \mathbb{C}P^1 \times \mathcal{S}$, equipped with the complex symplectic form

$$\Omega^{[0]}(\zeta) = \omega^+ - \mathrm{i}\zeta\,\omega^3 + \zeta^2\,\omega^- \;,$$

with $\omega^\pm = -\frac{1}{2}(\omega^1 \mp \mathrm{i}\omega^2)$, holomorphic wrt to the complex structure

$$J(\zeta,\bar\zeta) = \frac{1-\zeta\bar\zeta}{1+\zeta\bar\zeta}\,J^3 + \frac{\zeta+\bar\zeta}{1+\zeta\bar\zeta}\,J^2 + \mathrm{i}\frac{\zeta-\bar\zeta}{1+\zeta\bar\zeta}\,J^1$$

- $\Omega^{[0]}$ is regular at $\zeta = 0$, but has a pole at $\zeta = \infty$. Since it is only defined up to overall factor, one may instead consider

$$\Omega^{[\infty]}(\zeta) \equiv \zeta^{-2}\,\Omega^{[0]}(\zeta) = \omega^- - \mathrm{i}\omega^3/\zeta + \omega^+/\zeta^2 \;,$$

# Twistor space, Swann bundle III

- $\Omega$ is real wrt to the antipodal map $\zeta \mapsto -1/\bar{\zeta}$,

$$\Omega^{[\infty]}(\zeta) = \overline{\Omega^{[0]}(-1/\bar{\zeta})}$$

- More generally, $\Omega$ defines a section of $\Lambda^2 T^*_F(2)$, or "real global $\mathcal{O}(2)$ section" for short:

$$\Omega^{[i]} = f_{ij}^2 \Omega^{[j]} \mod d\zeta \;, \qquad \Omega^{[\bar{i}]}(\zeta) = \overline{\Omega^{[i]}(-1/\bar{\zeta})}$$

  where $f_{ij}$ are the transition functions of the $\mathcal{O}(1)$ bundle on $\mathbb{C}P^1$.
- Knowing $\Omega$, one may compute the HK metric by expanding around $\zeta = 0$ (or any other point).

*Hitchin Karlhede Lindström Roček*

# Twistor space, Swann bundle IV

- Locally, one can choose complex Darboux coordinates $\nu^I_{[i]}(\zeta)$ and $\mu_I^{[i]}(\zeta)$ on $\mathcal{Z}_{\mathcal{S}}$, regular in patch $U_i$, such that $\Omega^{[i]} = d\mu_I^{[i]} \wedge d\nu^I_{[i]}$.
- On the overlap of two patches $U_i \cap U_j$, they must be related by a symplectomorphism,

$$\mu_I^{[i]} = \partial_{\nu^I_{[i]}} S^{[ij]} \,, \quad \nu^I_{[j]} = \partial_{\mu_I^{[j]}} S^{[ij]} \,, \quad S^{[ij]} = S^{[ij]}(\nu^I_{[i]}, \mu_I^{[j]}, \zeta)$$

- $(\nu_{[i]}, \mu^{[i]})$ are ambiguous up to a local symplectomorphism, regular in $U_i$. Moreover, in $U_i \cap U_j \cap U_k$, $S^{[ij]}, S^{[jk]}, S^{[ik]}$ are subject to consistency relations: $S^{[ij]}$ defines a class in $H^1(\mathcal{Z}_{\mathcal{S}})$.

*Alexandrov BP Saueressig Vandoren; Lindström Roček*

# Twistor space, Swann bundle V

- Any triholomorphic isometry of $\mathcal{S}$ yields a triplet of moment maps $\vec{\mu}_\kappa = (v, \bar{v}, x)$, such that $\kappa \cdot \vec{\omega} = d\vec{\mu}_\kappa$. This triplet is best viewed as a real global section of $\mathcal{O}(2)$:

$$\eta = \frac{v}{\zeta} + x - \bar{v}\zeta$$

- In the presence of $d$ commuting tri-holomorphic isometries $\kappa^I$, one may choose $\nu_{[i]} = f_{i0}^2 \zeta \eta^I$ as our "position" coordinates. $S^{[ij]}$ must now be of the form

$$S^{[ij]} = \nu^I_{[i]} \mu_I^{[j]} + H^{[ij]}(\eta^I_{[i]}, \zeta)$$

such that, on $U_i \cap U_j$,

$$\mu_I^{[i]} - \mu_I^{[j]} = \partial_{\eta^I} H^{[ij]}$$

# Twistor space, Swann bundle VI

- This gluing condition can be solved in general,

$$\mu_I^{[i]}(\zeta) = \frac{\mathrm{i}}{2}\,\rho_I + \sum_j \oint_{C_j} \frac{\mathrm{d}\zeta'}{2\pi\mathrm{i}\,\zeta'}\,\frac{\zeta+\zeta'}{2(\zeta'-\zeta)}\,\partial_{\eta^I} H^{[0j]}(\zeta'),$$

  where $\kappa^I = \partial_{\rho_I}$ generate the tri-holomorphic isometries.
- This reproduces the standard Legendre transform construction of toric HK metrics, with generalized prepotential $H(\eta^I,\zeta)$:

$$\mathcal{L} = \oint \frac{d\zeta}{2\pi\mathrm{i}\zeta} H(\eta,\zeta)\ , \quad \chi(v^I,\bar{v}^I,w_I,\bar{w}_I) = \langle \mathcal{L} - x^I(w_I+\bar{w}_I)\rangle_{x^I}$$

- Perturbations away from toric metrics are described by holomorphic functions $H_{(1)}^{[ij]}(\nu_{[i]},\mu^{[i]},\zeta)$. Eg: $H=\eta^2/\zeta + m\,\eta\log\eta$ produces Taub-NUT, deformed into Atiyah-Hitchin by $H_{(1)} = \eta e^{\mu}$.

*Alexandrov Saueressig BP Vandoren I*

# Twistor space, Swann bundle VII

- When $\mathcal{M}$ is QK, $\mathcal{S}$ is HKC. In this case, superconformal invariance requires $S^{[ij]}$ to be (quasi)homogeneous of degree one in $\nu^I_{[i]}$ and $\zeta$-independent.
- Coordinates on $Z_\mathcal{M}$ are given by projectivizing coordinates on $Z_\mathcal{S}$, singling out one index $\flat$ ($c^{[i]}$ is an anomalous dimension):

$$\xi^\Lambda_{[i]} = \nu^\Lambda_{[i]}/\nu^\flat_{[i]}\ , \quad \tilde\xi^{[i]}_\Lambda = \mu^{[i]}_\Lambda\ , \quad \alpha^{[i]} = \mu^{[i]}_\flat + c^{[i]} \log \nu^\flat_{[i]}$$

- The complex homogeneous symplectic form on $Z_\mathcal{S}$ descends to a complex contact structure on $Z_\mathcal{M}$:

$$\Omega = d(\nu^I d\mu_I) = d(\nu^\flat X)\ , \quad X \equiv d\alpha + \xi^\Lambda d\tilde\xi_\Lambda$$

while $S^{[ij]}(\xi, \tilde\xi, \alpha)$ now generate contact transformations.

# Twistor space, Swann bundle VIII

- The coordinates $\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha$ as functions of $x^\mu \in \mathcal{M}$ and $z = \frac{\bar{\pi}_2\zeta + \pi^1}{-\bar{\pi}_1\zeta+\pi^2}$ now define the contact twistor lines. They are singular at $z = 0$ and $z = \infty$, corresponding to the zeros of $\nu^\flat$.
- The contact twistor lines encode the QK metric on $\mathcal{M}$ via

$$X = e^\Phi \left( \frac{dz}{z} + \frac{p_+}{z} - \mathrm{i} p_3 + p_- z \right) \; , \quad dp_3 + 2\mathrm{i} p_+ \wedge p_- = \omega_3$$

- For toric QK manifold, $S^{[ij]} = \alpha + \xi^\Lambda \tilde{\xi}_\Lambda - H^{[ij]}(\xi^\Lambda)$ recovers the (less) standard Legendre transform construction. Deformations away from toric QK manifolds are again described by holomorphic functions $H^{[ij]}_{(1)}(\xi, \tilde{\xi}, \alpha)$.
- One could by-pass the Swann space and its twistor space and work with $Z_\mathcal{M}$ only. Still, the HK potential on $\mathcal{S}$ is a useful object, as it must be invariant under all isometries.

*Alexandrov Saueressig BP Vandoren II*

# Outline

# The perturbative hypermultiplet moduli space I

- In the absence of instanton corrections, the symplectic structure on $\mathcal{Z}_\mathcal{S}$ is governed by

$$H_{\rm pert} = \frac{i}{2}\frac{F(\eta^\Lambda)}{\eta^\flat} + c\,\eta^\flat \log \eta^\flat$$

where $F$ is the prepotential of the SK base, e.g. in IIB on a CY $Y$,

$$F(X^\Lambda) = -\kappa_{abc}\frac{X^a X^b X^c}{6X^0} + \frac{1}{2}\frac{\zeta(3)(X^0)^2}{(2\pi \mathrm{i})^3}\chi_Y - \frac{(X^0)^2}{(2\pi \mathrm{i})^3}\sum_{k_a>0} n_{k_a}\,\mathrm{Li}_3\left(e^{2\pi \mathrm{i} k_a X^a/X^0}\right)\,,$$

and $c$ originates from one-loop correction: $c = \frac{1}{96\pi}\chi_Y$.

*Roček Vafa Vandoren; Robles Llana Saueressig Vandoren*

# Enforcing S-duality and electric-magnetic duality I

- The one-loop term and worldsheet instanton corrections break the $SL(2,\mathbb{R})$ continuous S-duality symmetry of the metric, acting holomorphically on $\mathcal{S}$ as

$$\begin{pmatrix} \nu^0 \\ \nu^\flat \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \nu^0 \\ \nu^\flat \end{pmatrix} , \qquad \nu^a \mapsto \nu^a$$

- Symmetry under the discrete S-duality $SL(2,\mathbb{Z})$ can be restored by summing over images. The tree-level $2\zeta(3)\chi_Y/g_s^2$ and $\zeta(2)\chi_Y$ are unified together with D-instantons in a standard $SL(3)$ Eisenstein series, while the worldsheet instantons are unified with Euclidean D- string instantons.

*Robles-Llana Roček Saueressig Theis Vandoren; Green Gutperle, Kiritsis BP*

# Enforcing S-duality and electric-magnetic duality II

- From the point of view of type IIA on the mirror CY $X$, $D(-1)$ and $D1$ correspond to $D0$ and $D2$ wrapped on A-cycles in $H_3(X,\mathbb{Z})$:

$$S = \eta^I \mu_I + H_{\rm pert} + \eta^\flat \sum_{q_\Lambda} n_{q_\Lambda} \sum_n \frac{1}{n^2} e^{2\pi i n q_\Lambda \frac{\eta^\Lambda}{\eta^\flat}}$$

  where $n_{q^\Lambda} = n_{q^a}$ are the genus 0 Gopakumar-Vafa invariants of $X$.
- Restoring symplectic invariance and mapping back to IIB, one obtains $D(-1) - D1 - D3 - D5$ instanton effects in type IIB. At the linear (one-instanton) order,

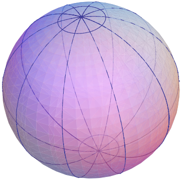
$$S = \eta^I \mu_I + H_{\rm pert} + \eta^\flat \sum_{p,q} n_{p^\Lambda, q_\Lambda} \mathrm{Li}_2 \left( e^{2\pi i (q_\Lambda \frac{\eta^\Lambda}{\eta^\flat} - p^\Lambda \mu_\Lambda)} \right) + \dots$$

*Alexandrov BP Saueressig Vandoren III*

# Enforcing S-duality and electric-magnetic duality III

- Beyond the one-instanton approximation, the structure is much more complicated. At perturbative level $\mathcal{Z}_{\mathcal{M}}$ is a twisted torus bundle $(\mathbb{C}^\times)^{2n_v+2}_{\xi^\Lambda,\tilde\xi_\Lambda} \ltimes \mathbb{C}^\times_\alpha$.

- Each instanton induces a contact transformation $\mathrm{Li}_2[e^{2\pi i(q_\Lambda\xi^\Lambda - p^\Lambda\tilde\xi_\Lambda))}]$ across the meridian $\arg z = |Z_{p,q}|$ on $\mathbb{C}P^1$.



*Gaiotto Neitzke Moore; Alexandrov Saueressig BP Vandoren III*

- This structure is very reminiscent of the KS wall-crossing formula, as we now review.

# The Kontsevich-Soibelman wall-crossing formula I

- Kontsevich and Soibelman show that across a LMS, the infinite non-commutative products

$$\prod_{\arg(Z_{p,q})\nearrow} U_{p,q}^{\Omega_+(p,q)} = \prod_{\arg(Z_{p,q})\searrow} U_{p,q}^{\Omega_-(p,q)} ,$$

where $\Omega_\pm$ are "motivic GW invariants", $U_{p,q}$ are formal group elements

$$U_{p,q} = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n^2} e_{np^\Lambda, nq_\Lambda}\right)$$

and $e_{p,q}$ satisfy the Lie algebra

$$\left[e_{p,q}, e_{p',q'}\right] = (-1)^{p^\Lambda q'_\Lambda - p'^\Lambda q_\Lambda} \left(p^\Lambda q'_\Lambda - p'^\Lambda q_\Lambda\right) e_{p+p',q+q'} \ .$$

# The Kontsevich-Soibelman wall-crossing formula II

- Up to the sign, which can be absorbed by a choice of "quadratic refinement", $U_{p,q}$ can be viewed as a symplectomorphism of the complex torus $(\mathbb{C})^{\times 2n_V+2}$:

$$e_{p,q} = e^{i(q_\Lambda \xi^\Lambda - p^\Lambda \tilde{\xi}_\Lambda)} \ , \quad [*,*] = \{*,*\}_{PB}$$

- Indeed, in the context of 4D/3D $\mathcal{N}=2$ gauge theories, the KS formula with $n_{p,q} \equiv \Omega(p,q)$ ensures that the full instanton-corrected metric on the 3D moduli space is well defined and continuous across the LMS.
- Physically, single-instanton contribution on one side of the LMS is replaced by multi-instanton contributions on the other side.

*Gaiotto Neitzke Moore*

# Generalization to SUGRA I

Generalizing SYM $\to$ SUGRA is challenging:

- If $n_{p,q} \sim e^{S_{BH}(p,q)}$, there are severe convergence issues. If, on the other hand, $n_{p,q}$ has support on polar states, like the standard Donaldson-Thomas invariants, the series may be convergent, but the connection to BH counting may be lost.
- Moreover, NS5-brane instantons need to be incorporated. One might hope to get them by $SL(2,\mathbb{Z})$ duality from the D5-instantons. This is difficult due to the complicated transformation rule of $\mu_\Lambda$,

$$\begin{pmatrix} \mu_0 \\ \mu_\flat \end{pmatrix} \mapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} \mu_0 \\ \mu_\flat \end{pmatrix} + \frac{\kappa_{abc}\nu^a\nu^b\nu^c}{6(\nu^\flat)^2} \begin{pmatrix} c^2/(c\nu^0 + d\nu^\flat) \\ -[c^2(a\nu^0 + b\nu^\flat) + 2c\nu^\flat]/(c\nu^0 + d\nu^\flat)^2 \end{pmatrix}$$

# Generalization to SUGRA II

- Enforcing a larger duality group, e.g. $SL(3, \mathbb{Z})$ as apparent in the dual heterotic string on $K3 \times T^3$ , may allow to shortcut this route and obtain NS5-brane contributions from perturbative corrections.

*Halmagyi BP*

- When the NS5-brane charge $k$ is non-zero, electric and magnetic translations no longer commute: $[p^\Lambda, q_\Sigma] = k\delta^\Lambda_\Sigma$. One has to resort to Landau-type wave functions, non-Abelian Fourier coefficients. From the mathematical point of view, it seems natural to replace $\mathrm{Li}_2$ by the quantum dilogarithm, with $\hbar \sim k$...
- Eventually, one may hope that the exact hypermultiplet 3D metric, hence the 4D BH spectrum, will be entirely fixed by modular invariance under $G_3(\mathbb{Z})$...

# Conclusion and open problems I

- Counting 4D black holes by computing instanton corrections in 3D seems very promising. If so, 3D U-dualities can act as spectrum generating symmetries for 4D black holes ! For $\mathcal{N} = 4, 8$, this suggests new relations between Siegel modular forms and automorphic forms of $SO(8, n_V + 2, \mathbb{Z})$ and $E_{8(8)}(\mathbb{Z})$.

*Gunaydin Neitzke BP Waldron*

- For $\mathcal{N} = 2$, we are back to the problem of computing the exact metric on the hypermultiplet moduli space in 4D ! While the exact metric may be too difficult to compute, it may be possible to determine the exact complex symplectic (or contact) structure on the twistor space.

# Conclusion and open problems II

- One may also contemplate a generalized topological wave function computing the higher derivative $\tilde{F}_g$-type corrections to the hypers. It is tempting to speculate that it defines a "triholomorphic" function on $\mathcal{S}$, giving a one-variable generalization of the standard topological amplitude.

*Antoniadis Gava Narain Taylor; Gunaydin Neitzke BP*