Quantum Attractor Flows

Boris Pioline

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Main references

- Ooguri Strominger Vafa [hep-th/0405146]
- Ooguri Verlinde Vafa [hep-th/0502211]
- BP [hep-th/0506228]
- Gunaydin, Neitzke, BP and Waldron [hep-th/0512296]
- more to appear

The OSV Conjecture I

- Static, spherically symmetric BPS black holes in $N = 2$ SUGRA have a "BPS" Bekenstein-Hawking-Wald entropy

$$S_{BHW}(p^I, q_I) = \langle \mathcal{F}(p^I, \phi^I) + q_I \phi^I \rangle_\phi$$

where the "topological free energy" $\mathcal{F}$ is related to the generalized prepotential $F(X^I, W^2)$ by

$$\mathcal{F} = -\mathrm{Im} F(X^I = p^I + i\phi^I, W^2 = 2^8)$$

Cardoso De Wit Mohaupt; Ooguri Strominger Vafa

- This automatically incorporates the attractor equations $\mathrm{Re}(X^I) = p^I, \mathrm{Re}(F_I) = q_I$ which govern the value of the scalars at horizon.

The OSV Conjecture II

- For Type II compactified on a CY 3-fold Y, the generalized prepotential $F(X^I, W^2)$ is computed by the topological string via

$$\Psi_{top} = e^{\frac{i\pi}{2}F}$$

Bershadsky Cecotti Ooguri Vafa; Antoniadis Gava Narain Taylor

- By a bold extrapolation, Ooguri, Strominger and Vafa (OSV) proposed that the actual number of micro-states is in fact

$$\Omega(p^I, q_I) \sim \int d\phi^I \, |\Psi_{top}(p^I + i\phi^I)|^2 \, e^{\phi^I q_I} \qquad (*)$$

to all orders at large charges, and perhaps non-perturbatively.

Dabholkar; Dabholkar Denef Moore BP

OSV conjecture and symplectic invariance

- This proposal may seem to treat electric and magnetic charges differently, fortunately it does not !
- For $\Omega(p,q)$ to be independent of the choice of polarization, Ψ_{top} should transform in the metaplectic representation of the electric-magnetic duality group $Sp(2n_v,\mathbb{R})$:

$$\begin{aligned} (p,q) \to (q,-p) \quad &: \quad \Psi_{top}(p) \to \text{Fourier}[\Psi_{top}](p) \\ (p,q) \to (p,q+p) \quad &: \quad \Psi_{top}(p) \to e^{ip^2}[\Psi_{top}](p) \\ (p,q) \to (p+q,q) \quad &: \quad \Psi_{top}(p) \to e^{-i\partial_p^2}[\Psi_{top}](p) \end{aligned}$$

- Indeed, Ψ_{top} is best viewed as a state in a Hilbert space, represented by different wave functions in different polarizations.

The topological amplitude as a quantum state

- In the "holomorphic" (Hodge) polarization, it satisfies the BCOV anomaly equations.

$$\begin{aligned} H &= \lambda^{-1}\Omega(t,\bar{t}) + z^i D_{t^i}\Omega(t,\bar{t}) + cc \,, \\ \omega &= e^{-K}(d\lambda^{-1} \wedge d\bar{\lambda}^{-1} + g_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}) \end{aligned}$$

- In the "real" polarization, implicit in OSV, it transforms metaplectically under change of symplectic basis,

$$H = p^I \gamma_I + q_I \gamma^I \,, \quad \omega = dp^I \wedge dq_I$$

Witten; BCOV; Dijkgraaf Vonk Verlinde; Verlinde

- The recent confusion about symplectic invariance is largely due to the difficulty of computing the real-polarized Ψ_{top} explicitly.

OSV conjecture and Wigner function

- Performing a Wick rotation $\phi^I = i\chi^I$, the rhs of (*)

$$\Omega(p^I, q_I) \sim \int d\chi^I \Psi^*_{top}(p^I + \chi^I)\Psi_{top}(p^I - \chi^I)e^{i\chi^I q_I}$$

is recognized as the (polarization-independent) Wigner distribution in phase-space, associated to the quantum state $\Psi_{top}(p^I)$.

- Even more suggestively, defining

$$\Psi_{p,q}(\chi) := e^{iq\chi}\Psi_{top}(\chi - p) := V_{p,q} \cdot \Psi_{top}(\chi)$$

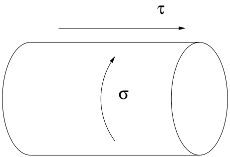
this is rewritten as an overlap of two wave functions

$$\Omega(p, q) \sim \int d\chi \, \Psi^*_{p,q}(\chi) \, \Psi_{p,q}(\chi)$$

OSV conjecture and channel duality I

- This is reminiscent of the familiar open/closed duality for CFT on the cylinder,

$$\mathrm{Tr} e^{-\pi t H_{open}} = \langle B | e^{-\frac{\pi}{t} H_{closed}} | B \rangle$$



- Indeed, the near-horizon geometry $AdS_2 \times S^2$ has the topology of a cylinder, and may be quantized in two ways, equivalent by AdS/CFT:

OSV conjecture and channel duality II

(global or Poincaré)

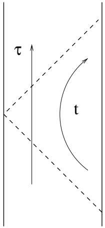
$\updownarrow$

Conformal Quantum Mechanics

Radial quantization

$\updownarrow$

Quantum Attractor Flow



Ooguri Vafa Verlinde;Dijkgraaf Gopakumar Ooguri Vafa; Gukov Saraikin Vafa

- The topological amplitude is interpreted as a particular wave function for the radial attractor flow, in a “mini-superspace” approximation where only spherically symmetric geometries are retained.

The black hole / universe wave function

- The idea of mini-superspace radial quantization of black holes was in fact much studied by the gr-qc community, but yielded little insight on the nature of black hole micro-states.

Cavaglia de Alfaro Filippov; Kuchar; Thiemann Kastrup; Breitenlohner Hellmann

- One novelty here is that one works in a SUSY context, for which the "mini-superspace" truncation to spherically symmetric geometries, and omission of D-term interactions, has some chance of being exact.
- Further interest possibly arises from the relation between black hole attractor equations and SUSY vacua in flux compactifications.

WHICH black hole / universe wave function ?

- Q: What physical principle, if any, picks out Ψ_{top} from the ∞-dimensional BPS Hilbert space ?
- A (plausible): (3-dimensional) U-duality picks out a unique "automorphic" wave function, whose Fourier coefficients should produce the exact black hole degeneracies.

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Outline

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1. Introduction
2. Radial quantization and geodesic motion
3. Very special supergravities
4. The automorphic black hole wave function
5. Conclusions and open problems

Stationary solutions and KK* reduction I

- Stationary solutions in 4D can be parameterized in the form

$$ds_4^2 = -e^{2U}(dt+\omega)^2 + e^{-2U} ds_3^2 \ , \quad A_4^I = \zeta^I dt + A_3^I$$

where $ds_3, U, \omega, A_3^I, \zeta^I$ and the 4D scalars $z^i \in \mathcal{M}_4$are independent of time. The D=3+1 theory reduces to a field theory in three Euclidean dimensions.

- In contrast to the usual KK ansatz,

$$ds_4^2 = e^{2U}(dy+\omega)^2 + e^{-2U} ds_{2,1}^2 \ , \quad A_4^I = \zeta^I dy + A_3^I$$

where the fields are independent of y, we reduce along a time-like direction.

Stationary solutions and KK* reduction II

- For the usual KK reduction to 2+1D, the one-forms (A_3^I, ω) can be dualized into pseudo-scalars $(\tilde{\zeta}_I, a)$, where a is the twist (or NUT) potential. The 4D Einstein-Maxwell equations reduce to 3D gravity + scalars living in a Riemannian space

$$\mathcal{M}_3 = \frac{Sl(2)}{U(1)}\Big|_{U,a} \times \mathcal{M}_4 \ltimes \mathbb{R}^{2n_v}\Big|_{\zeta^I,\tilde{\zeta}_I}$$

- The KK* reduction is simply related to the KK reduction by letting $(\zeta^I, \tilde{\zeta}_I) \to i(\zeta^I, \tilde{\zeta}_I)$. As a result, the scalar fields live in a pseudo–Riemannian space $\mathcal{M}_3^*$, with non-positive definite signature.

Breitenlohner Gibbons Maison; Hull Julia

Stationary solutions and KK* reduction III

- $\mathcal{M}_3^*$ always has $2n+2$ isometries corresponding to the shifts of $\zeta^I, \tilde{\zeta}_I, a, U$, satisfying the graded Heisenberg algebra

$$\begin{aligned} [p^I, q_J] &= 2\delta^I_J\, k \\ \left[m, p^I\right] = p^I,\ [m, q_I] &= q_I\ , [m, k] = 2k \end{aligned}$$

- The notation anticipates the identification of the corresponding conserved charges with the electric and magnetic charges q_I and p_I, NUT charge k and ADM mass m.

Attractor flow and geodesic motion I

- Now, restrict to spherically symmetric solutions, with spatial slices

$$ds_3^2 = N^2(\rho) d\rho^2 + r^2(\rho) d\Omega_2^2$$

- The sigma-model action becomes, up to a total derivative (g_{ij} is the metric on $\mathcal{M}_3^*$):

$$S = \int d\rho \left[\frac{N}{2} + \frac{1}{2N} \left(\dot{r}^2 - r^2 g_{ij} \dot{\phi}^i \dot{\phi}^j \right) \right]$$

- The lapse N can be set to one, but it imposes the Hamiltonian constraint

$$H_{WDW} = (p_r)^2 - \frac{1}{r^2} g^{ij} p_i p_j - 1 \equiv 0$$

Attractor flow and geodesic motion II

- Solutions are thus massive geodesics on the cone $\mathbb{R}^+ \times \mathcal{M}_3^*$. The problem separates into geodesic motion on $\mathcal{M}_3^*$, times conformal motion along r.
- Extremal black holes have flat 3D slices, so we may choose the "extremal gauge" $N = 1, r = \rho$ from the outset: the solutions are light-like geodesics on $\mathcal{M}_3^*$, with affine parameter $\tau = 1/r$.
- For the purpose of defining observables such as the horizon area, $A_H = e^{-2U} r^2|_{U \to -\infty}$ and ADM mass $M_{ADM} = r(e^{2U} - 1)|_{U \to 0}$, keeping the variable r may be convenient.

Geodesic motion and conserved charges I

- The isometries of $\mathcal{M}_3$ imply conserved Noether charges, whose Poisson bracket reflect the Lie algebra of the isometries:

$$
\begin{aligned}
[p^I, q_J] &= 2\delta^I_J\, k \\
\left[m, p^I\right] = p^I,\ [m, q_I] &= q_I\ , [m, k] = 2k
\end{aligned}
$$

- If $k \neq 0$, the off-diagonal term in the 4D metric

$$ds_4^2 = -e^{2U}(dt + k\cos\theta d\phi)^2 + e^{-2U}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]$$

  implies the existence of closed time-like curves around ϕ direction, near $\theta = 0$.
- Bona fide 4D black holes arise in the "classical limit" $k \to 0$, which meshes well with the Wigner form of the OSV conjecture. Keeping $k \neq 0$ will allow us to greatly extend the symmetry.

BPS black holes and BPS geodesics I

- Reducing the full $D = 4$ SUGRA on the stationary, spherically symmetric ansatz, we find the Lagrangian for a superparticle propagating on $R^+ \times \mathcal{M}_3^*$: in the extremal gauge,

$$S = \int dr \, \left[g_{ij} \dot{\phi}^i \dot{\phi}^j + \psi^a \dot{\psi}_a + R_{abcd} \psi^a \psi^b \psi^c \psi^d \right]$$

invariant under SUSY variations

$$\delta \phi^i = O(\psi) \, , \quad \delta \psi^a = V_i^{a\alpha} \dot{\phi}^i \epsilon_\alpha + O(\psi^2)$$

where $V_i^{a\alpha}$ are 1-forms on $\mathcal{M}_3^*$, and $P^{a\alpha} = V_i^{a\alpha} \dot{\phi}^i$ is the momentum of the fiducial particle on $\mathcal{M}_3^*$.

BPS black holes and BPS geodesics II

- The BH solution preserves SUSY iff there exists $\epsilon_\alpha \neq 0$ such that $\delta\psi^a = 0$. This implies $P^2 = 0$, i.e. the geodesic is light-like, or the black hole is extremal.
- In $N > 2$ SUGRA, black hole solutions may preserve different amounts of SUSY, depending on the number of solutions to $P\epsilon = 0$.

c-map and c*-map I

- The reduction of tree-level $4D\ N=2$ SUGRA coupled to vector multiplets [*hypers go along for the ride*] to 2+1 dimensions is known as the $c-map$: $\mathcal{M}_3$ is a quaternionic-Kähler space, entirely determined by the tree-level prepotential in 4 dimensions.

$$ds^2 = 2(dU)^2 + g_{i\bar{j}}(z,\bar{z})dz^i dz^{\bar{j}} + \frac{1}{2}e^{-4U}\left(da + \zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I\right)^2$$
$$-e^{-2U}\left[(\mathrm{Im}\mathcal{N})_{IJ}d\zeta^I d\zeta^J + (\mathrm{Im}\mathcal{N}^{-1})^{IJ}\left(d\tilde{\zeta}_I + (\mathrm{Re}\mathcal{N})_{IK}d\zeta^K\right)\left(d\tilde{\zeta}_J + (\right.\right.$$

Ferrara Sabharwal; de Wit Van Proyen Vanderseypen

- The manifold $\mathcal{M}_3^*$ obtained by analytic continuation $(\zeta^I, \tilde{\zeta}_I) \to i(\zeta^I, \tilde{\zeta}_I)$ is sometimes called "para-quaternionic-Kahler manifold".

Cortes Mayer Mohaupt Saueressig

$N = 2$ attractor flow and geodesic motion on c*-map I

- The fermionic variation is controlled by the quaternionic vielbein $V^{\alpha A}$

$$\delta\psi^\Gamma = V_i^{\alpha\Gamma} \dot{\phi}^i \epsilon_\alpha + O(\psi^2)$$

where V is a $2 \times 2n$ pseudo-real matrix of 1-forms, which may be expressed in terms of the conserved quantities p^I, q_I, k.

- The BPS condition $V^{\alpha\Gamma}\epsilon_\alpha = 0$ implies the attractor flow equations

$$\left.\begin{array}{rcl} r^2 \frac{dU}{dr} & = & e^U |Z| \\ r^2 \frac{dz^i}{dr} & = & 2 e^U g_{i\bar{j}} \partial_{\bar{j}} |Z| \end{array}\right\} , \qquad Z = e^{K/2}(q_I X^I - p^I F_I)$$

or rather, their generalization to non-zero NUT charge.

Gutperle Spalinski; Gunaydin Neitzke BP Waldron

The quantum attractor mechanism I

- The standard way to quantize geodesic motion of a particle on $R^+ \times \mathcal{M}_3^*$ is to replace the classical trajectories by wave functions in $L_2(R^+ \times \mathcal{M}_3^*)$, satisfying the WDW equation

$$\left[-\frac{\partial^2}{\partial r^2} + \frac{\Delta}{r^2} - 1\right] \Psi(r, U, z^i, \bar{z}^{\bar{i}}, \zeta^I, \tilde{\zeta}_I, a) = 0$$

  where Δ is the Laplace-Beltrami operator on $\mathcal{M}_3^*$.

- After quantizing the fermions ψ^a, the wave function is therefore a section of some bundle on $\mathcal{M}_3^*$, or equivalently a set of differential forms on $\mathcal{M}_3^*$.

The quantum attractor mechanism II

- We are really interested in the BPS Hilbert space, satisfying the stronger constraint

$$\exists \epsilon / \ \epsilon^\alpha P_{a\alpha} = 0 \quad \Rightarrow \quad \exists \epsilon / \ \epsilon^\alpha \frac{\partial}{\partial X^{a\alpha}} \Psi = 0$$

- In $N = 2$, this restricts Ψ to be a holomorphic function (section of sheaf cohomology, rather) on the twistor space $T = M_3 \bowtie \mathbb{P}_1$ where $\mathbb{P}_1 = \{\epsilon_1/\epsilon_2\}$.

Physical interpretation of the wave function

- As in quantum cosmology, the wave function is independent of the "time" variable ρ, and some other variable should be chosen as a "clock". It is natural to use U as the "radial clock", since it goes from $-\infty$ at the horizon to 0 at spatial infinity.
- Observables are defined at a fixed value of U. We expect the wave function to become more and more peaked around the attractor values of the moduli and of the horizon area as $U \to -\infty$.
- The natural inner product is obtained by using the Klein-Gordon inner product (or Wronskian) at fixed values of U. Unfortunately, it is famously known NOT to be positive definite.
- A possible way out is "third quantization", where the wave function Ψ becomes itself an operator... this may describe the possible black hole fragmentation near the horizon...

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The universal sector I

- It is instructive to investigate the "universal sector", which encodes the scale U, the graviphoton electric and magnetic charges, and the NUT charge k (truncating all moduli away):

$$H_{WDW} = \frac{1}{8}(p_U)^2 - \frac{1}{4}e^{2U}\left[(p_{\tilde{\zeta}} - k\zeta)^2 + (p_\zeta + k\tilde{\zeta})^2\right] + \frac{1}{2}e^{4U}k^2$$

  Gauge conditions are $U = \zeta = \tilde{\zeta} = a = 0$ at $\tau = 0$.

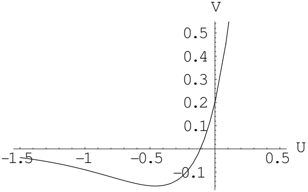
- The motion in the $(\tilde{\zeta}, \zeta)$ plane is the Landau problem of a charged particle in a constant magnetic field. The electric and magnetic charges don't commute:

$$p = p_{\tilde{\zeta}} + \zeta k\ , \quad q = p_\zeta - \tilde{\zeta} k\ , \quad [p, q] = k$$

The universal sector II

- The motion in the U direction is governed effectively by

$$H = \frac{1}{8}(p_U)^2 + \frac{1}{2}e^{4U}k^2 - \frac{1}{4}e^{2U}\left[p^2 + q^2 - 4kJ\right]$$



- Since V_α^A is a 2×2 matrix, the BPS property is equivalent to extremality:

$$H = \frac{1}{2}\left|p_U + ike^{2U}\right|^2 - \frac{1}{4}e^{2U}\left|p + iq\right|^2 = 0$$

The universal sector III

- At spatial infinity, p_U becomes equal to the ADM mass, and J vanishes; hence the BPS mass relation

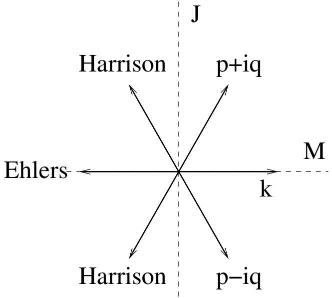
$$M^2 + k^2 = p^2 + q^2$$

- At the horizon $U \to -\infty$, $\tau \to \infty$, the last term is irrelevant and one recovers $AdS_2 \times S_2$ geometry with area

$$A = 2\pi(p^2 + q^2) = 2\pi\sqrt{(p^2 + q^2)^2}$$

$SU(2,1)$: Geodesic motion and co-adjoint orbits I

- The universal sector corresponds to the symmetric space $SU(2,1)/Sl(2,\mathbb{R}) \times U(1)$:



Kinnersley

$SU(2,1)$: Geodesic motion and co-adjoint orbits II

- The corresponding Noether charges can be arranged in a matrix Q valued in the (dual) Lie algebra $su(2,1)$, such that

$$\mathrm{Tr}(Q) = 0 \,, \quad \mathrm{Tr}(Q^2) = H \,, \quad \det(Q) = 0$$

- At fixed value of the Casimir H, different trajectories are related by the (co-)adjoint action $Q \to hQh^{-1}$. For generic H, Q is a diagonalizable element, whose orbit is of the form $SU(2,1)/U(1) \times U(1)$.

$SU(2,1)$: BPS geodesics and nilpotent orbits I

- 3×3 matrices satisfy the identity

$$Q^3 - \mathrm{Tr}(Q) Q^2 + [\mathrm{Tr}(Q^2) - (\mathrm{Tr} Q)^2] Q - \det(Q) = 0$$

- BPS solutions have $H = 0 \quad \Rightarrow Q^3 = 0$ hence Q is no longer diagonalizable. Its orbit is instead $SU(2,1)/P$ where P is the parabolic subgroup which stabilizes Q: this is known as a nilpotent orbit.
- Stated in terms of the (co-)adjoint representation, the BPS condition reads

$$Ad(Q)^5 = 0$$

which holds more generally, as we'll see.

Outline

Very special SUGRA and Jordan algebras I

Recall that there is an interesting class of $N = 2$ supergravities whose moduli spaces are symmetric spaces. They are associated to Jordan algebras J of degree 3:

- $\mathbb{R}$: $N = x^3$
- $\mathbb{R} \oplus \Gamma$: $N = x_1 x_a Q^{ab} x_b$
- 3×3 hermitean matrices $X = \begin{pmatrix} \alpha_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \alpha_2 & x_1 \\ x_2 & \bar{x}_1 & \alpha_3 \end{pmatrix}$ with

  $\alpha_i \in \mathbb{R}, x_i \in \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$

$$N = \alpha_1 \alpha_2 \alpha_3 - \sum_{i=1,2,3} \alpha_i (x_i \bar{x}_i) + 2\mathrm{Re}(x_1 x_2 x_3)$$

Very special supergravities

Q	$D=5$	$D=4$	$D=3$	$D=3^*$
8		$\frac{SU(n,1)}{SU(n)\times U(1)}$	$\frac{SU(n+1,2)}{SU(n+1)\times SU(2)\times U(1)}$	$\frac{SU(n+1,2)}{SU(n,1)\times Sl(2)\times U(1)}$
8	$\mathbb{R}\times\frac{SO(n-1,1)}{SO(n-1)}$	$\frac{SO(n,2)}{SO(n)\times SO(2)}\times\frac{Sl(2)}{U(1)}$	$\frac{SO(n+2,4)}{SO(n+2)\times SO(4)}$	$\frac{SO(n+2,4)}{SO(n,2)\times SO(2,2)}$
8		$\varnothing$	$\frac{SU(2,1)}{SU(2)\times U(1)}$	$\frac{SU(2,1)}{Sl(2)\times U(1)}$
8	$\varnothing$	$\frac{Sl(2)}{U(1)}$	$\frac{G_{2(2)}}{SO(4)}$	$\frac{G_{2(2)}}{SO(2,2)}$
8	$\frac{Sl(3)}{SO(3)}$	$\frac{Sp(6)}{SU(3)\times U(1)}$	$\frac{F_{4(4)}}{USp(6)\times SU(2)}$	$\frac{F_{4(4)}}{Sp(6)\times Sl(2)}$
8	$\frac{Sl(3,C)}{SU(3)}$	$\frac{SU(3,3)}{SU(3)\times SU(3)\times U(1)}$	$\frac{E_{6(+2)}}{SU(6)\times SU(2)}$	$\frac{E_{6(+2)}}{SU(3,3)\times Sl(2)}$
24	$\frac{SU^*(6)}{USp(6)}$	$\frac{SO^*(12)}{SU(6)\times U(1)}$	$\frac{E_{7(-5)}}{SO(12)\times SU(2)}$	$\frac{E_{7(-5)}}{SO^*(12)\times Sl(2)}$
8	$\frac{E_{6(-26)}}{F_4}$	$\frac{E_{7(-25)}}{E_6\times U(1)}$	$\frac{E_{8(-24)}}{E_7\times SU(2)}$	$\frac{E_{8(-24)}}{E_{7(-25)}\times Sl(2)}$
10			$\frac{Sp(2n,4)}{Sp(2n)\times Sp(4)}$	?
12			$\frac{SU(n,4)}{SU(n)\times SU(4)}$	?
16	$\mathbb{R}\times\frac{SO(n-5,5)}{SO(n-5)\times SO(5)}$	$\frac{Sl(2)}{U(1)}\times\frac{SO(n-4,6)}{SO(n-4)\times SO(6)}$	$\frac{SO(n-2,8)}{SO(n-2)\times SO(8)}$	$\frac{SO(n-2,8)}{SO(n-4,2)\times SO(2,6)}$
18			$\frac{F_{4(-20)}}{SO(9)}$	?
20		$\frac{SU(5,1)}{SU(5)\times U(1)}$	$\frac{E_{6(-14)}}{SO(10)\times SO(2)}$	$\frac{E_{6(-14)}}{SO^*(10)\times SO(2)}$
32	$\frac{E_{6(6)}}{USp(8)}$	$\frac{E_{7(7)}}{SU(8)}$	$\frac{E_{8(8)}}{SO(16)}$	$\frac{E_{8(8)}}{SO^*(16)}$

Very special SUGRA

- In 5D, the scalars take values in the real symmetric space

$$M_3 = \{N(X) = 1\} = \frac{\text{Lorentz}(J)}{\text{Aut}(J)}$$

where Lorentz(J) is the invariance group of the cubic norm $N(X)$.

Gunaydin Sierra Townsend

- Upon reduction to 4D, one obtains a $N = 2$ SUGRA with cubic prepotential

$$F = N(X)/X^0 = C_{ABC} X^A X^B X^C / X^0$$

The 4D moduli space is a (special Kähler) symmetric space

$$M_4 = \frac{\text{Conf}(J)}{\text{Lorentz}^c(J) \times U(1)} , \quad K = -\log[N(z^A - \bar{z}^A)]$$

Very special SUGRA

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Gunaydin Sierra Townsend

- Upon reduction to 4D, one obtains a $N = 2$ SUGRA with cubic prepotential

$$F = N(X)/X^0 = C_{ABC} X^A X^B X^C / X^0$$

The 4D moduli space is a (special Kähler) symmetric space

$$M_4 = \frac{\text{Conf}(J)}{\text{Lorentz}^c(J) \times U(1)} , \quad K = -\log[N(z^A - \bar{z}^A)]$$

Very Special Black Holes I

- Applying the general attractor formulae to the cubic prepotential $F = N(X)/X^0$, and using its remarkable invariance under Legendre transform

$$\langle \frac{N(X)}{X^0} + Y_0 X^0 + Y_A X^A \rangle_X = -\frac{N(Y)}{Y^0}$$

one finds that 4D BPS black holes have entropy

$$S_{BH}(p,q) = \sqrt{I_4(p,q)}$$

where I_4 is the quartic invariant of the 4D U-duality group Conf(J).

$$I_4(p,q) = 4p^0 N(q_A) - 4q_0 N(p^A) + 4\frac{\partial N(q_A)}{\partial q_A}\frac{\partial N(p^A)}{\partial p^A} - (p^0 q_0 + p^A q_A)^2$$

Very special SUGRA - $D = 3$ I

- Upon compactification to 3D, the scalar manifold is a symmetric quaternionic-Kahler manifold

$$\mathcal{M}_3 = \frac{\mathrm{QConf}(J)}{\mathrm{Conf}^c(J) \times SU(2)} \,, \quad \mathcal{M}_3^* = \frac{\mathrm{QConf}(J)}{\mathrm{Conf}(J) \times Sl(2)}$$

The 3D U-duality group $G_3 = \mathrm{QConf}(J)$ is called the quasi-conformal group of J, because it admits an action on $2n+1$ variables (p^I, q_I, k) which leaves the "quartic light-cone" invariant:

$$\Delta(Q, Q') = I_4\left(p^I - p'^I, q^I - q'^I\right) + 2\left(k - k' + p'^I q_I - p^I q'_I\right)^2 = 0$$

Gunaydin Koepsell Nicolai; Gunaydin Neitzke BP Waldron

- In fact, $K = -\log[\Delta(Z, \bar{Z})]$ determines the Kähler potential on the twistor space of the QK space $\mathcal{M}_3$.

The quasiconformal realization I

- In more detail, $QConf(J)$ admits the 5-graded decomposition

$$G_{-2} \oplus G_{-1} \oplus [Conf(J) \times R]_0 \oplus \{p^I, q_I\}_{+1} \oplus \{k\}_{+2}$$

where $G_{-2} \oplus \mathbb{R}_0 \oplus G_2 = Sl(2)_{\text{Ehlers}}$.

- Since $P = G_{-2} \oplus G_{-1} \oplus G_0$ is a parabolic subgroup of $G_3 = \text{QConf}(J)$, there is an action of G_3 on $G/P = \{p^I, q_I, k\}$. The Heisenberg algebra acts in the usual way,

$$(p^I, q_I, k) \to (p^I + \epsilon^I, q_I + \eta_I, k + \epsilon^I \eta_I)$$

while the grade -2 generator acts as

$$\delta(p^I, q_I, k) = \left(\frac{\partial I_4}{\partial q_I} - kp^I, -\frac{\partial I_4}{\partial p^I} - kq_I, I_4 - 2k^2 \right)$$

This may be deformed by a character of P, leading to the quaternionic discrete series representation of G.

Gross Wallach

Attractor flow for very special SUGRA

- Classically, the momentum P of a particle on G/H, or the Noether charge Q, is valued in $G_1 \oplus G_{-1}$. The BPS condition implies that it can be conjugated into G_1 by an Ehlers transformation. Equivalently,

$$[Ad(Q)]^5 = 0$$

  Thus, the SUSY phase space is a nilpotent coadjoint orbit of the 3D U-duality group $QConf(J)$.
- Quantum-mechanically, the wave functions of BPS BH are holomorphic functions on the twistor space of $\mathcal{M}_3^*$. Thus, they transform in the quasiconformal representation on (p^I, q_I, k).
- The topological amplitude has to live in an even smaller Hilbert space, since (in the real polarization) it depends only on p^I – at least naively.

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Small representations I

- The action of QConf(J) preserves the orbit of (p^I, q_I) under the 4D U-duality group. These orbits are characterized by the number of independent charges:

Ferrara Gunaydin

dim	Constraint on (p,q)	♯charges
$2n_v+1$	$I_4 \neq 0$	4
$2n_v$	$I_4 = 0$	3
$(5n_v-2)/3$	$\partial I_4(p,q) = 0$	2
n_v+1	$\partial \otimes \partial\vert_{\mathrm{Conf}(J)} I_4(p,q) = 0$	1

- The action of QConf(J) on the smallest orbit is the minimal representation of $G_3 = \mathrm{QConf}(J)$, on n_v+1 variables (p^I, k). Morally, it is the space of tri-holomorphic functions on the hyperKähler cone of $\mathcal{M}_3$!

Gunaydin Koepsell Nicolai; Kazhdan BP Waldron; Gunaydin Pavlyk

Minimal representation and spherical vector I

- The minimal representation is the natural habitat of the topological string amplitude, or rather a generalization thereof which includes the extra charge k !
- More precisely, in order to embed the BPS Hilbert space $\mathcal{H}$ inside the big Hilbert space $L_2(G/H)$, one should find a H-invariant ("spherical") vector f_H inside $\mathcal{H}$ so that

$$f \to \Psi(g) = \langle f, \rho(g) f_H \rangle$$

- The spherical vector f_H was computed for a totally different motivation (for split groups), and does recover the topological amplitude in some limit

$$\lim_{\beta\to\infty} e^{\beta H_\omega} f_H = e^{iN(\chi^A)/\chi^0}$$

BP Kazhdan Waldron

Holomorphic anomaly and minimal representation

- We now have evidence that, for very special SUGRA at least, the BCOV holomorphic anomaly equations follow from identities in the Joseph ideal of the minimal representation, e.g.

$$2\hbar J_I^- - W_- Y_I^+ - \frac{1}{\sqrt{3}} C_{IJK} Y_-^J Y_-^K \quad \equiv \quad 0$$

involving generators of the Fourier-Jacobi group $G_4 \ltimes$ Heise.

- This is totally analogous to the way the heat equation for the Jacobi theta series

$$\left[\partial_\tau - \frac{1}{4\pi}\partial_v^2\right] \theta_1(\tau, v) = 0$$

arises by restriction of the metaplectic representation of $Sp(4)$.

Non-perturbative Ψ_{top} and mirror symmetry I

- After compactification along Euclidean time, the total moduli space factorizes into two quaternionic-Kahler spaces $\mathcal{M}_3^V \times \mathcal{M}_4^H$, exchanged under T-duality along the thermal direction.
- In particular, the thermal compactification of the 4D vector-multiplet couplings F_g leads to 3D couplings on $\mathcal{M}_3^V$ isomorphic to the 4D hypermultiplet couplings $\tilde{F}_g$ on $\mathcal{M}_4^V$,

$$\sum_{h=0}^{\infty} \tilde{F}_h(X,S)\ \partial\partial S\ \partial\partial S\ (\partial Z)^{2h-2}$$

Antoniadis Gava Narain Taylor; Antoniadis Pioline Taylor

- Contrary to F_h, which arises only at h-loop, $\tilde{F}_h$ receives instanton corrections in 4D, so depends on X^I as well as S. Moreover, it has to be invariant under $Sl(2,Z)_{IIB}$.

Non-perturbative Ψ_{top} and mirror symmetry II

- Similarly, the reduction of F_h to 3D depends on U, and receives instanton corrections from 4D black-holes winding around the time direction. Moreover, it is invariant under $Sl(2,Z)_M$ which flips the Euclidean-time and eleven-th dimension.
- Thus, the notion of "non-perturbative topological amplitude" relevant for 4D black hole counting depends on one additional parameter, the IIB string coupling, and should more properly be viewed as a *tri-holomorphic function on the hyperKähler cone over the quaternionic space* $\mathcal{M}_3$.
- Said like this, no wonder that the non-perturbative topological amplitude counts 4D black holes ! What's unclear yet is how it reduces to the modulus square of the perturbative Ψ_{top} in the weak coupling limit.

$N = 2$ very special SUGRA vs. $N = 4, 8$ SUGRA I

- The above construction applies most directly to $N = 2$ SUGRA, where the U-duality groups in 5D, 4D, 3D are in their rank 2,3,4 real form. SUGRA with $N > 2$ can be obtained by going to other real forms.
- For example, $N = 8$ SUGRA is based on U-duality groups $E_{6(6)}$, $E_{7(7)}$, $E_{8(8)}$ in the split ("maximally non compact") real forms. They can be obtained from the exceptional $N = 2$ SUGRA with U-duality groups $E_{6(-26)}, E_{7(-25)}, E_{8(-24)}$ by replacing the compact octonions by split octonions, whose norm $x\bar{x}$ has signature (4,4) rather than (8,0).
- The dimension of the moduli spaces changes, but the structure of the attractor equations and black hole entropy are unaffected:

$$p^I = \mathrm{Re}(X^I) \ , \quad p_I = \mathrm{Re}(F_I)$$

where (X^I, F_I) are symplectic sections on $E_{7(7)}/SU(8)$.

$N = 2$ very special SUGRA vs. $N = 4, 8$ SUGRA II

- After analytic continuation of the quasiconformal representation to $E_{8(8)}$, we obtain unipotent reps of dimension 57,56,46,29 corresponding to the BPS Hilbert space of 1/8 BPS, small 1/8 BPS, 1/4 BPS and 1/2 BPS black holes !
- Since the maximal compact group changes, the spherical vector however will be different.

Outline

The automorphic attractor wave function I

- We have found some evidence that (a one-parameter generalization of) the topological string amplitude can be viewed as a particular "spherical" vector f_H in the Hilbert space of BPS black holes, carrying an unitary action of $G_3(\mathbb{R})$.
- Moreover, we have noticed that in order for the OSV conjecture to be consistent with U-duality, the wave function had to be invariant under the metaplectic action of $G_4(\mathbb{Z})$.
- This suggests that we should pick out the unique vector $f_{G_3(\mathbb{Z})}$ invariant under the 3D U-duality group $G_3(Z)$. This incorporates 4-dimensional U-duality invariance, as well as charge quantization.

The automorphic attractor wave function II

- This is in fact a general procedure to construct automorphic forms:

$$\theta_G(g) = \langle f_{G(\mathbb{Z})}, \rho(g) f_H \rangle$$

It is natural to propose that suitable Fourier coefficients of θ_G will predict the exact BPS black hole degeneracies in 4 dimensions.

Automorphic forms for freshmen I

- E.g, the Jacobi theta series

$$\theta(\tau) = \sum_{m \in Z} e^{i\pi m^2 \tau}$$

fits into this frame: τ is an element of $Sl(2)/U(1)$, ρ is the metaplectic representation

$$E_+ = x^2 \ , \quad E_0 = x\partial_x + \partial_x x \ , \quad E_- = \partial_x^2 \ ,$$

f_K is the ground state of the harmonic oscillator, and $f_{G(\mathbb{Z})}$ is the "Dirac comb" distribution $\sum_{m \in \mathbb{Z}} \delta(x - m)$.

BP Waldron Les Houches lecture

Automorphic forms for freshmen II

- $f_{G(\mathbb{Z})}$ can be obtained adelically by finding the spherical vector over all p-adic fields $\mathbb{Q}_p$, and taking the product over all primes p:

$$\sum_{m \in \mathbb{Z}} \delta(x-m) = \prod_{p \in \mathbb{Z}} \gamma_p(x) \ , \quad \gamma_p(x) = \begin{cases} 1 & \text{if} \quad x \in \mathbb{Z}_p \\ 0 & \text{if} \quad x \notin \mathbb{Z}_p \end{cases}$$

Indeed, $\gamma_p(x)$ is invariant under p-adic Fourier transform !

Black hole degeneracies and Fourier coefficients I

- In the general theory of automorphic forms, Fourier coefficients are associated to choices of parabolic subgroups $P = LN$ of G, and are indexed by characters ξ of P:

$$\hat{\theta}(\xi) = \int_{N(\mathbb{R})/N(\mathbb{Z})} \xi(g)\, \theta_G(g)\, dg$$

- Choosing the character $\xi_{p,q} = e^{i(q_I \zeta^I + p^I \tilde{\zeta}_I)}$ of the Heisenberg parabolic P, one finds

$$\hat{\theta}(p,q) = \int d\zeta^I\ e^{iq_I\zeta^I}\ f^*_{G(\mathbb{Z})}(p^I - \zeta^I, 0)\, f_{K(\mathbb{R})}(p^I + \zeta^I, 0)$$

which is tantalizingly close to the OSV formula !

Black hole degeneracies and Fourier coefficients II

- Said differently, the automorphic attractor wave function is obtained by choosing the real spherical vector at infinity, and the adelic spherical vector at the horizon. The Fourier coefficients are by construction invariant under $G_4(\mathbb{Z})$.

Outline

Some open problems

- Incorporate higher derivative corrections
- Relation to the DVV “genus 2” formula
- Relation to Gaiotto-Strominger and Denef-Moore
- Investigate not so special N=2 theories
- Rotating and multi-centered black holes in 4D