Three Ways Across the Wall

Boris Pioline

CERN & LPTHE, Jussieu

Based on
1011.1258, 1103.0261,1103.1887
with J. Manschot and A. Sen

The man who could walk through walls

Marcel Aymé, *Le passe-muraille*, 1943

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“When he left [his mistress’ room], Dutilleul passed through the walls of the house and felt an unusual rubbing sensation against his hips and shoulders. He felt as though he were moving through some gel-like substance that was growing thicker (...) Dutilleul was immobilized inside the wall. He is there to this very day, imprisoned in the stone.”

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- More often than not, bound states decay into multi-particle states across certain codimension-one walls in moduli space: a way to learn about their elementary constituents !
- Using semi-classical methods, one may sometimes determine the spectrum at weak coupling. Understanding these decays systematically is important to extrapolate to strong coupling.

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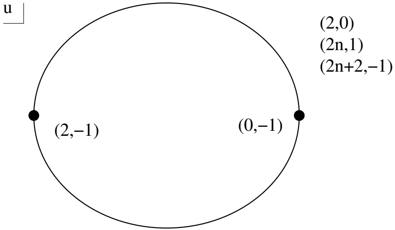
- While the number of BPS states may change erratically, the BPS index $\Omega(\gamma, t) = \mathrm{Tr}(-1)^F$ is constant – at least away from the walls.
- The jump $\Delta\Omega$ across the wall is determined by certain universal wall-crossing formulae, some of which have been discovered independently in the math literature.

  Joyce Song 2008; Kontsevich Soibelman 2008

Wall-crossing in gauge theories

- E.g., in $D = 4$, $N = 2$ SQCD with $G = SU(2)$ (Seiberg-Witten) on the Coulomb branch,



All BPS states in the weak coupling region are bound states of the magnetic monopole $(0, -1)$ and dyon $(2, -1)$. Those are immortal, i.e. exist everywhere on the Coulomb branch.

Seiberg Witten 1994; Bilal Ferrari 1996

Bound states as multi-centered solutions

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- Near the wall, the centers become farther apart, and behave like point particles interacting by (Newton), Coulomb, Lorentz and scalar forces.
- The degeneracy of the bound state (hence the jump in Ω) is determined by the SUSY quantum mechanics of these point particles, together with the internal degeneracies carried by each center.

Denef 2002; Manschot BP Sen 2010; Lee Yi 2011

Wall-crossing and multi-instantons

- Similar wall-crossing phenomena take place for instanton corrections to certain (BPS, F-term) couplings in the effective action. One-instanton effects are discontinuous across certain walls in the one-instanton approximation, but multi-instanton effects should conspire to ensure continuity of the coupling.

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- These two phenomena are identical for 4D, $\mathcal{N} = 2$ gauge theories / string vacua compactified on a circle: the effective action receives instanton correction from 4D monopoles / black holes winding around the circle. The continuity of the effective action is ensured by the KS wall-crossing formula !

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- Such couplings are a very useful book-keeping device for 4D black hole degeneracies, consistent with wall-crossing and dualities !

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- Bound states are labelled by their electric and magnetic charges q_Λ, p^Λ, by their mass M and spin J_3.
- The charge vector $\gamma = (p^\Lambda, q_\Lambda)$ takes values in a lattice equipped with an integer antisymmetric pairing, corresponding to the angular momentum carried by the electromagnetic field:

$$\langle \gamma, \gamma' \rangle \equiv q_\Lambda \, p'^\Lambda - q'_\Lambda \, p^\Lambda \in \mathbb{Z}$$

Dirac 1931; Schwinger 1966; Zwanziger 1968

States with $\langle \gamma, \gamma' \rangle \neq 0$ are 'mutually non-local'.

BPS states and BPS index

- In models with $\mathcal{N} = 2$ supersymmetries, the mass of any state is bounded from below by the BPS bound

$$M \geq |Z(\gamma, t)| \ , \qquad Z(\gamma, t) = e^{\mathcal{K}/2}(q_\Lambda X^\Lambda - p^\Lambda F_\Lambda)$$

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Witten Olive 1978

- Two short multiplets might combine into a long multiplet and desaturate the BPS bound, but the index Ω stays constant under this process:

$$\Omega(\gamma; t) = \mathrm{Tr}_{\mathcal{H}'_\gamma(t)}(-1)^{2J_3}$$

Walls of marginal stability

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- The decay of BPS bound states is constrained by the triangular inequality

$$M(\gamma_1 + \gamma_2) = |Z(\gamma_1 + \gamma_2)| = |Z(\gamma_1) + Z(\gamma_2)| \leq M(\gamma_1) + M(\gamma_2)$$

The decay is energetically possible only when the central charges are aligned, i.e. on the wall of marginal stability

$$W(\gamma_1, \gamma_2) = \{t \,/\, \arg[Z(\gamma_1, t)] = \arg[Z(\gamma_2, t)]\} \subset \mathcal{M}$$

Cecotti Vafa 1992; Seiberg Witten 1994

Primitive wall-crossing from two-centered solutions I

- For $\langle \gamma_1, \gamma_2 \rangle \neq 0$, there exists a two-centered BPS solution of charge $\gamma = \gamma_1 + \gamma_2$:

$$|x_1 - x_2| = \sqrt{G_4} \frac{\langle \Gamma_1, \Gamma_2 \rangle}{2} \frac{|Z(\Gamma_1 + \Gamma_2, t)|}{\mathrm{Im}(Z(\Gamma_1, t) \bar{Z}(\Gamma_2, t))}$$

Denef 2002

- The solution exists only on one side of the wall. As t approaches the wall, the distance r_{12} diverges and the bound state decays into its constituents γ_1 and γ_2.

Primitive wall-crossing from two-centered solutions

- Near the wall, the two monopoles can be treated as pointlike particles with charge γ_i and $\Omega(\gamma_i)$ internal degrees of freedom, interacting via Newton, Coulomb, etc forces.

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Primitive wall-crossing formula (Denef Moore 2007)

$$\Delta\Omega(\gamma_1+\gamma_2) = \pm \underbrace{|\langle\gamma_1,\gamma_2\rangle|}_{\text{angular momentum}} \times \underbrace{\Omega(\gamma_1)}_{\text{internal states of 1}} \times \underbrace{\Omega(\gamma_2)}_{\text{internal states of 2}}$$

Multi-centered solutions

- On the same wall, many other bound states will decay: those represented by multi-centered BPS solutions with charges $\alpha_i = M_i\gamma_1 + N_i\gamma_2$, with $M_i \geq 0, N_i \geq 0$ and $(M_i, N_i) \neq 0$.
- Stationary BPS solutions with n centers at $\vec{r} = \vec{r}_i$ exist whenever

Denef's equations (Denef 2000)

$$\forall i \;:\; \sum_{j\neq i} \frac{\alpha_{ij}}{|\vec{r}_i - \vec{r}_j|} = c_i(t)$$

Here $\alpha_{ij} \equiv \langle \alpha_i, \alpha_j \rangle$, $c_i = 2\,\mathrm{Im}\left[e^{-i\phi} Z(\alpha_i, t)\right]$, $\phi = \arg[Z(\sum_i \alpha_i, t)]$.

BPS phase space

- For fixed charges α_i and moduli t, the space of solutions modulo overall translations is a compact symplectic manifold $\mathcal{M}_n$ of dimension $2n-2$, invariant under $SO(3)$:

$$\omega = \frac{1}{2} \sum_{i<j} \alpha_{ij} \sin\theta_{ij} \, \mathrm{d}\theta_{ij} \wedge \mathrm{d}\phi_{ij} \; , \quad \vec{J} = \frac{1}{2} \sum_{i<j} \alpha_{ij} \frac{\vec{r}_{ij}}{|r_{ij}|}$$

de Boer El Showk Messamah Van den Bleeken 2008

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  1. $\Omega(\gamma_i)$ internal states at each center
  2. $g(\{\alpha_i\})$ external states obtained by geometric quantization of $\mathcal{M}_n$

Non-primitive wall-crossing (naive)

- For fixed total charge $\gamma = M\gamma_1 + N\gamma_2$, the index $\Omega(\gamma)$ includes contributions from all *n*-centered solutions with charges $\alpha_i = M_i\gamma_1 + N_i\gamma_2$ such that $(M, N) = \sum_i (M_i, N_i)$. All these solutions disappear at once across the wall.

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- Naively, the jump of the index across the wall should be

$$\Delta\Omega(\gamma) = \sum_{n\geq 2} \sum_{\gamma=\alpha_1+\dots+\alpha_n} g(\{\alpha_i\}) \prod_{i=1}^{n} \Omega(\alpha_i)$$

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- This however ignores the issue of statistics.

Non-primitive wall-crossing (correct)

- Taking Bose-Fermi statistics into account, the formula for $\Delta\Omega(\gamma)$ is cumbersome (e.g. it involves products of $\Omega(\alpha_i)$ with $\gamma \neq \sum \alpha_i$).
- The correct formula is obtained by replacing $\Omega \to \bar{\Omega}$ where

$$\bar{\Omega}(\gamma) \equiv \sum_{d|\gamma} \frac{1}{d^2} \Omega(\gamma/d)$$

Joyce Song

and introducing a Boltzmann symmetry factor:

Non-primitive wall-crossing formula

$$\Delta\bar{\Omega}(\gamma) = \sum_{n\geq 2} \sum_{\gamma=\alpha_1+\dots+\alpha_n} \frac{g(\{\alpha_i\})}{|\mathrm{Sym}(\{\alpha_i\})|} \prod_{i=1}^{n} \bar{\Omega}(\alpha_i)$$

Manschot BP Sen 2010

Outline

Geometric quantization and localization

- Given a symplectic manifold $(\mathcal{M}, \omega)$, geometric quantization produces a graded Hilbert space $\mathcal{H}$, the space of harmonic spinors for the Dirac operator *D* coupled to ω. If $\mathcal{M}$ is compact, $\mathcal{H}$ is finite dimensional.

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- More generally, the refined index $g(\{\alpha_i\}, y) \equiv \mathrm{Tr}(-y)^{2J_3}$ in the SUSY quantum mechanics. is equal to the equivariant index of D.
- Since $\mathcal{M}_n$ admits a $U(1)$ action, the equivariant index can be computed by localization:

$$\mathrm{Ind}(D) = \lim_{y \to 1} \mathrm{Ind}(D, y) \,, \quad \mathrm{Ind}(D, y) = \sum_{\text{fixed pts}} \mathrm{Jac}(p)\, y^{2J_3(p)}$$

Atiyah Bott, Berline Vergne

Symplectic volume and equivariant index

- In the limit $\omega \gg 1$, this reduces to the Duistermaat-Heckman formula for the (equivariant) symplectic volume:

$$\mathrm{Vol}(\mathcal{M}_n, y) \equiv \int_{\mathcal{M}_n} \omega^{n-1}\, y^{2J_3} = \sum_{\text{fixed pts}} \mathrm{Jac}'(p)\, y^{2J_3(p)}$$

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- E.g. for $n = 2$, $\mathcal{M}_2 = S^2$, $J_3 = \alpha_{12} \cos\theta$:

$$\mathrm{Vol}(\mathcal{M}_2, y) = \frac{1}{2\log y} \left(\underbrace{y^{+\alpha_{12}}}_{\text{North pole}} - \underbrace{y^{-\alpha_{12}}}_{\text{South pole}} \right) \stackrel{y\to 1}{\longrightarrow} \alpha_{12}$$

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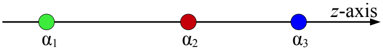
$$\mathrm{Ind}(\mathcal{M}_2, y) = \frac{y^{+\alpha_{12}} - y^{-\alpha_{12}}}{(y - 1/y)} = \mathrm{Tr}_{j=\frac{1}{2}(\alpha_{12}-1)}\, y^{2J_3}$$

The Coulomb branch formula

- For any n, the fixed points of the action of J_3 are collinear multi-centered configurations along the z-axis:



$$\forall i\,,\quad \sum_{j\neq i} \frac{\alpha_{ij}}{|z_i - z_j|} = c_i\,,\qquad J_3 = \frac{1}{2}\sum_{i<j} \alpha_{ij}\,\mathrm{sign}(z_j - z_i)\,.$$

- These fixed points are isolated, and labelled by permutations σ:

Coulomb branch wall-crossing formula

$$g(\{\alpha_i\}, y) = \frac{(-1)^{\sum_{i<j}\alpha_{ij}+n-1}}{(y-y^{-1})^{n-1}} \sum_{\sigma} s(\sigma)\, y^{\sum_{i<j}\alpha_{\sigma(i)\sigma(j)}}\;,\qquad s(\sigma) = 0, \pm 1$$

Manschot, BP, Sen 2010

An example: 3-body decay

- E.g. for $n = 3$ with $\alpha_{12} > \alpha_{23}$, there are 4 collinear configurations:

$$g(\alpha_i, y) = \frac{(-1)^{\alpha_{13}+\alpha_{23}+\alpha_{12}}}{(y-1/y)^2} \times$$

$$\Big[\underbrace{y^{\alpha_{13}+\alpha_{23}+\alpha_{12}}}_{(123)} - \underbrace{y^{-\alpha_{13}-\alpha_{23}+\alpha_{12}}}_{(312)} - \underbrace{y^{\alpha_{13}+\alpha_{23}-\alpha_{12}}}_{(213)} + \underbrace{y^{-\alpha_{13}-\alpha_{23}-\alpha_{12}}}_{(321)} \Big]$$

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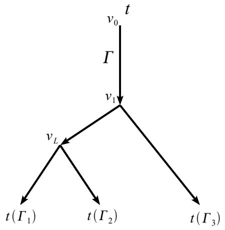
In the limit $y \to 1$,

- $$g(\alpha_1, \alpha_2, \alpha_3) = (-1)^{\alpha_{13}+\alpha_{23}+\alpha_{12}} \alpha_{12} (\alpha_{13} + \alpha_{23}) = \pm \langle \alpha_1, \alpha_2 \rangle \langle \alpha_1 + \alpha_2, \alpha_3 \rangle$$



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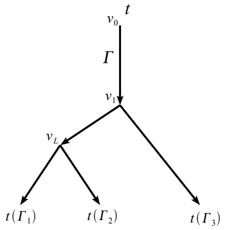
- $$g(\alpha_1, \alpha_2, \alpha_3) = (-1)^{\alpha_{13}+\alpha_{23}+\alpha_{12}} \, \alpha_{12} \, (\alpha_{13} + \alpha_{23}) = \pm \langle \alpha_1, \alpha_2 \rangle \, \langle \alpha_1 + \alpha_2, \alpha_3 \rangle$$



- A similar formula holds for $\alpha_{12} < \alpha_{23}$

An example: 3-body decay

- E.g. for $\gamma = \gamma_1 + 2\gamma_2$, three types of bound states contribute:

$$\begin{aligned}\Delta\Omega(\gamma) =& (-1)^{\gamma_{12}}\,\gamma_{12}\,\bar\Omega(\gamma_2)\,\bar\Omega(\gamma_1+\gamma_2) + 2\gamma_{12}\,\bar\Omega(2\gamma_2)\,\bar\Omega(\gamma_1)\\ &+ \tfrac{1}{2}(\gamma_{12})^2\,\bar\Omega(\gamma_2)\,\bar\Omega(\gamma_2)\,\bar\Omega(\gamma_1)\\ =& (-1)^{\gamma_{12}}\,\gamma_{12}\,\Omega(\gamma_2)\,\Omega(\gamma_1+\gamma_2) + 2\gamma_{12}\,\Omega(2\gamma_2)\,\Omega(\gamma_1)\\ &+ \tfrac{1}{2}\gamma_{12}\,\Omega(\gamma_2)\,(\gamma_{12}\Omega(\gamma_2)+1)\,\Omega(\gamma_1)\end{aligned}$$

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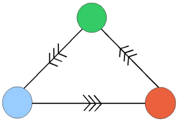
- This is a somewhat trivial example of 'semi-primitive wall-crossing' with $\gamma = \gamma_1 + N\gamma_2$, without genuine 3-body interactions.

Outline

Quiver Matrix Mechanics

- In the weak coupling limit, the centers can be realized as D-branes interacting via open strings. At low energy, this is described by a Matrix Quantum Mechanics, with field content specified by a quiver with n nodes $\{1 \dots n\}$ of dimension 1 and $\langle \alpha_i, \alpha_j \rangle$ arrows from i to j.



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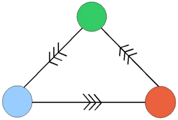

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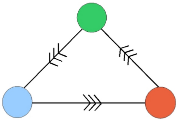

- The Matrix Quantum Mechanics admits a Coulomb branch where the D-branes are well-separated, described by Denef's equations above. It also has a Higgs branch where all D-branes coincide.
- If all α_i lie on a 2-dimensional lattice spanned by γ_1, γ_2, the quiver has no oriented closed loop, and one expects a 1-1 map between states on the Higgs branch and on the Coulomb branch.

Higgs branch formula

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Higgs branch wall-crossing formula

$$g(\{\alpha_i\}, y) = \frac{\pm 1}{(y - 1/y)^{n-1}} \sum_{\sigma} N(\{\alpha_i\}, \sigma)\, y^{\sum_{i<j} \alpha_{\sigma(i)\sigma(j)}}$$

$$N(\{\alpha_i\}, \sigma) = \prod_{\substack{k=2\dots n \\ \sigma(k)<\sigma(k-1)}} \Theta(\langle \gamma, \sum_{i=k}^{n} \alpha_{\sigma(i)} \rangle) \prod_{\substack{k=2\dots n \\ \sigma(k)>\sigma(k-1)}} \Theta(\langle \sum_{i=k}^{n} \alpha_{\sigma(i)}, \gamma \rangle)$$

Reineke 2003; Manschot BP Sen 2010

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- Amazingly, this agrees with the Coulomb branch formula ! Using the Boltzmann trick $\Omega \to \bar{\Omega}$, non-Abelian quivers are reduced to Abelian ones !

Outline

The Kontsevich-Soibelman algebra

- Consider the Lie algebra $\mathcal{A}$ spanned by abstract generators $\{e_\gamma, \gamma \in \Gamma\}$, satisfying the commutation rule

$$[e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle \gamma_1, \gamma_2 \rangle} \langle \gamma_1, \gamma_2 \rangle \, e_{\gamma_1+\gamma_2} \ .$$

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- For a given charge vector γ and moduli t, consider the operator $U_\gamma(t)$ in the Lie group $\exp(\mathcal{A})$

$$U_\gamma(t) \equiv \exp\left(\Omega(\gamma; t) \sum_{d=1}^{\infty} \frac{e_{d\gamma}}{d^2} \right) = \exp\left(\sum_{n=1}^{\infty} \bar{\Omega}(n\gamma; t) \, e_{n\gamma} \right)$$

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- The operators e_γ / U_γ can be realized as Hamiltonian vector fields / symplectomorphisms of a twisted torus.

The Kontsevich-Soibelman formula

- The ordered product $\overset{\curvearrowleft}{\prod}_\gamma U_\gamma(t)$ must be constant, hence the

Kontsevich-Soibelman wall-crossing formula

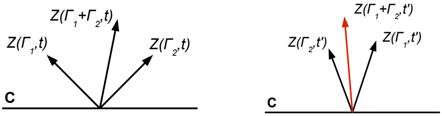
$$\overset{\curvearrowleft}{\prod}_{M,N} U^{+}_{M\gamma_1+N\gamma_2} = \overset{\curvearrowright}{\prod}_{M,N} U^{-}_{M\gamma_1+N\gamma_2} ,$$

Starting from the l.h.s and reordering the product using the Baker-Campbell-Hausdorff (BCH) formula, one may express $\Omega^-(\gamma)$ in terms of $\Omega^+(\gamma)$.

- Both sides may be infinite, but only a finite number of factors contribute to $\Delta\Omega(M\gamma_1 + N\gamma_2)$ for fixed M, N.

Primitive wall-crossing from the KS formula

- For example, the primitive wall-crossing formula follows from



$$U^{+}_{\gamma_1} \cdot U^{+}_{\gamma_1+\gamma_2} \cdot U^{+}_{\gamma_2} = U^{-}_{\gamma_2} \cdot U^{-}_{\gamma_1+\gamma_2} \cdot U^{-}_{\gamma_1}$$

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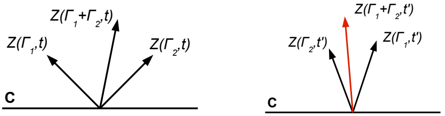

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- Wall-crossing in Seiberg-Witten theory is summarized by

$$U_{2,-1} \cdot U_{0,1} = U_{0,1} \cdot U_{2,1} \cdot U_{4,1} \dots U^{(-2)}_{2,0} \dots U_{4,-1} \cdot U_{2,-1}$$

Denef Moore; Dimofte Gukov Soibelman

Refined wall-crossing formula

- The KS formula admits a quantum deformation which determines the wall-crossing of the refined index $\Omega(\gamma, y)$.

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$$\Delta\bar{\Omega}(\gamma, y) = \sum_{n\geq 2} \sum_{\gamma=\alpha_1+\dots+\alpha_n} \frac{g_{\rm KS}(\{\alpha_i\}, y)}{|{\rm Sym}(\{\alpha_i\})|} \prod_{i=1}^{n} \bar{\Omega}(\alpha_i, y) \, ,$$

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  for some universal coefficients $g_{\rm KS}(\{\alpha_i\}, y)$.
- The fact that $g_{\rm KS}(\{\alpha_i\}, y)$ is also given by the Coulomb or Higgs branch formula is non-trivial, but has recently been shown by induction.

Sen 2012

Outline

Wall-crossing and instantons

- The operators U_γ beg for a physics explanation. One possibility is to consider the effective action of the $\mathcal{N} = 2$ gauge/string theory compactified on a circle, i.e. on $\mathbb{R}^3 \times S^1(R)$.

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- In gauge theories with rank r, $\mathcal{M}_3$ is a $4r$-dimensional torus bundle over the Coulomb branch of the 4D gauge theory. The fibers of the torus parametrize the holonomies $\zeta^\Lambda, \tilde\zeta_\Lambda$ of the Abelian gauge fields $\mathcal{A}^\Lambda$ and their magnetic duals $\tilde{A}_\Lambda$ along S_1.

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- For $R \to \infty$, the metric is 'semi-flat', i.e. flat along the torus fibers. For any R, the metric must be hyperkähler.

Multi-instanton corrections in gauge theories

- For R finite, there are instanton corrections to the semi-flat metric from four-dimensional BPS states winding around the circle,

$$g \sim g_{\rm sf} + \sum_{\gamma} \Omega(\gamma, t)\, e^{-R|Z(\gamma,t)| + \mathrm{i}(q_\Lambda \zeta^\Lambda - p^\Lambda \tilde\zeta_\Lambda)} + \text{multi-instantons}$$

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Gaiotto Moore Neitzke 2008

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- In this context, the operators U_γ appear as gluing functions for the twistor space $\mathcal{Z}$ of $\mathcal{M}_3$, a complex symplectic manifold which encodes the HK metric.

Multi-instanton corrections in string theory

- In string theory vacua, the story is similar except that $\mathcal{M}_3$ is a quaternion-Kähler space which includes the radius R and twist potential σ. The twistor space $\mathcal{Z}$ is a complex contact manifold.

Alexandrov BP Saueressig Vandoren 2008

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- By T-duality along S^1, the problem of computing $\mathcal{M}_3$ is mapped to that of computing the instanton corrected hypermultiplet moduli space in $D = 4$…

Conclusion

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- The wall-crossing formula ensures that instanton corrections to certain BPS couplings in the effective action on $\mathbb{R}^3 \times S^1$ are continuous across the wall.
- Similar techniques can be used to subtract, at any point in moduli space, contributions from multi-centered black holes, and zoom in on elementary, single-centered black holes – for which AdS/CFT applies.

Let’s cross the wall !

Laius I

Today, I would like to teach you - for those who don't already know - how to painlessly walk across walls. There is a fastly growing body of Literature on this important topic, starting with a seminal paper (or rather a short story) by the french writer Marcel Aymé almost 70 years ago, entitled "Le passe-muraille", or "The man who could walk through walls" - from which this picture is taken.

The story is that of an office clerk, named Dutilleul, who discovers at the age of 42 that he has the ability to walk through walls. At first he doesn't quite see what to make out of this skill, but soon enough he discovers that he can use it to scare his boss to death by popping up his head through the wall of his office, or to rob any bank or jewelry that he likes. He may even let himself be caught by the police, the thickness of the prison wall being all the more enjoyable to him.

Laius I

Finally he gets caught by love, and after visiting his mistress who was kept behind closed doors by her jealous husband, his talent suddenly disappears, and he gets stuck in the wall of his mistress' house, where he is imprisoned to this day. This story serves as a warning to all of us who practice wall-crossing, which is still a risky business.

My goal is that by the end of this seminar, you will all grasp the basics of wall-crossing, and the more adventurous of you will be able to exit this room through this wall.

My lecture will be mostly based on 2 papers with Jan Manschot and Ashoke Sen about a year ago, and a proceedings called 'Four ways across the wall'. Four, not three, but as you start practicing wall-crossing, and other related activies, you quickly get confused about space, time and even numbers.