Rankin-Selberg methods for String Amplitudes

Boris Pioline

CERN & LPTHE

Fields Institute, Toronto, Sep 18, 2013

based on work with C. Angelantonj and I. Florakis,

arXiv:1110.5318,1203.0566,1304.4271,and work in progress

Modular integrals and BPS amplitudes I

.

- In closed string theory, an interesting class of amplitudes are given by a modular integral

$$\mathcal{A} = \int_{\mathcal{F}} \mathrm{d}\mu\, \Gamma_{d+k,d}\, \Phi(\tau)\ , \quad \mathrm{d}\mu = \frac{\mathrm{d}\tau_1 \mathrm{d}\tau_2}{\tau_2^2}$$

  - $\mathcal{F} = \Gamma \backslash \mathcal{H}$: fundamental domain of the modular group $\Gamma = SL(2,\mathbb{Z})$ on the Poincaré upper half plane $\mathcal{H}$;
  - $\Gamma_{(d+k,d)} = \tau_2^{d/2} \sum q^{\frac{1}{2}p_L^2}\, \bar{q}^{\frac{1}{2}p_R^2}$: a Siegel-Narain series for an even self-dual lattice of signature $(d+k,d)$;
  - $\Phi(\tau)$: an (almost, weakly) holomorphic modular form of weight $w = -k/2$, which I will call the elliptic genus

Modular integrals and BPS amplitudes II

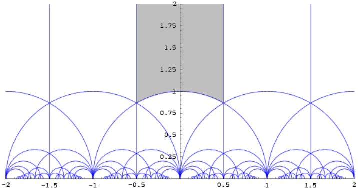
- Such modular integrals arise in one-loop computations of certain BPS-saturated amplitudes, such as F^2, R^2, F^4, R^4, after integrating over the location of the vertex operators.
- More general one-loop amplitudes are given by similar integrals, but $\Phi(\tau)$ is no longer (almost) holomorphic, hence much harder to compute.
- $\mathcal{A}$ provides a function on the moduli space of lattices,

$$G_{d+k,d} = \frac{O(d+k,d)}{O(d+k)\times O(d)} \ni (g_{ij}, B_{ij}, Y_i^a) \ ,$$

which is invariant T-duality, i.e. under the automorphism group $\mathcal{O}(\Gamma_{d+k,d})$: an example of Theta correspondence.

Unfolding trick

- In the physics literature, the time-honored way to evaluate such integrals has been the unfolding trick or orbit method:



$$\int_{\Gamma\backslash\mathcal{H}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f|_0\gamma = \int_{\Gamma_\infty\backslash\mathcal{H}} f$$

$$f|_w\gamma(\tau) = (c\tau+d)^{-w} f\left(\frac{a\tau+b}{c\tau+d}\right)$$

- E.g for $d=1$, representing $\Gamma_{1,1} = R\sum_{m,n} e^{-\pi R^2|m-n\tau|^2/\tau_2}$,

$$\begin{aligned}\int_{\mathcal{F}} \Gamma_{1,1} =& R\int_{\mathcal{F}} \mathrm{d}\mu + R\int_{\mathcal{S}} \mathrm{d}\mu \sum_{m\neq 0} e^{-\pi R^2 m^2/\tau_2} \\ =& \frac{\pi}{3}R + \frac{\pi}{3}R^{-1}\end{aligned}$$

Unfolding trick, revisited

- For higher dimensional lattices, the theta series $\Gamma_{d+k,d}$ involves several different orbits of $SL(2,\mathbb{Z})$. The orbit decomposition breaks manifest invariance under the automorphism group $O(\Gamma_{d+k,d})$.
- I will present an alternative method for computing such modular integrals, which keeps T-duality manifest at all stages. The method is inspired by the Rankin-Selberg method commonly used in number theory.
- The result is typically expressed as a field theory amplitude with an infinite number of BPS states running through the loop.
- The method is in principle applicable to higher genus amplitudes, though for the most part I will focus on genus one.

Rankin-Selberg method I

- Consider the completed non-holomorphic Eisenstein series

$$E^\star(\tau;s) = \zeta^\star(2s) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \tau_2^s|\gamma = \tfrac{1}{2}\,\zeta^\star(2s) \sum_{(c,d)=1} \frac{\tau_2^s}{|c\,\tau+d|^{2s}}$$

  where $\zeta^\star(s) \equiv \pi^{-s/2}\,\Gamma(s/2)\,\zeta(s) = \zeta^\star(1-s)$.
- $E^\star(\tau;s)$ is convergent for $\mathrm{Re}(s) > 1$, and has a meromorphic continuation to all s, invariant under $s \mapsto 1-s$, with simple poles at $s=0,1$ with constant residue:

$$E^\star(\tau;s) = \frac{1}{2(s-1)} + \tfrac{1}{2}\left(\gamma - \log(4\pi\,\tau_2\,|\eta(\tau)|^4)\right) + \mathcal{O}(s-1)\,,$$

Rankin-Selberg method (cont.)

- For any modular function $F(\Omega)$ of rapid decay, consider the Rankin-Selberg transform

$$\mathcal{R}^\star(F,s) = \int_{\mathcal{F}} \mathrm{d}\mu\, E^\star(\tau;s)\, F(\tau)$$

- By the unfolding trick, $\mathcal{R}^\star(F,s)$ is proportional to the Mellin transform of the constant term $F_0(\tau_2) = \int_{-1/2}^{1/2} \mathrm{d}\tau_1\, F(\tau)$,

$$\begin{aligned}
\mathcal{R}^\star(F;s) =& \zeta^\star(2s) \int_{\mathcal{S}} \mathrm{d}\mu\, \tau_2^s\, F(\tau) \\
=& \zeta^\star(2s) \int_0^\infty \mathrm{d}\tau_2\, \tau_2^{s-2}\, F_0(\tau_2)\,,
\end{aligned}$$

Rankin-Selberg method (cont.)

- It inherits the meromorphicity and functional relations of $E^\star$, e.g. $\mathcal{R}^\star(F;s) = \mathcal{R}^\star(F;1-s)$.
- Since the residue of $E^\star(\tau;s)$ at $s=0,1$ is constant, the residue of $\mathcal{R}^\star(F;s)$ at $s=1$ is proportional to the modular integral of F,

$$\mathrm{Res}_{s=1}\mathcal{R}^\star(F;s) = \frac{1}{2}\int_{\mathcal{F}} \mathrm{d}\mu\, F$$

Rankin-Selberg-Zagier method I

- This was extended by Zagier to the case where $F^{(0)}$ is of power-like growth $F^{(0)}(\tau) \sim \varphi(\tau_2)$ at the cusp: the renormalized integral

$$\mathrm{R.N.} \int_{\mathcal{F}} \mathrm{d}\mu\, F(\tau) = \lim_{\mathcal{T}\to\infty} \left[\int_{\mathcal{F}_\mathcal{T}} \mathrm{d}\mu\, F(\tau) - \hat{\varphi}(\mathcal{T}) \right]$$

$$\varphi(\tau_2) = \sum_\alpha c_\alpha \tau_2^\alpha\,, \quad \hat{\varphi}(\mathcal{T}) = \sum_{\alpha\neq 1} c_\alpha \frac{\tau_2^{\alpha-1}}{\alpha-1} + \sum_{\alpha=1} c_\alpha \log \tau_2$$

is related to the Mellin transform of the (regularized) constant term

$$\mathcal{R}^\star(F;s) = \zeta^\star(2s) \int_0^\infty \mathrm{d}\tau_2\, \tau_2^{s-2} \left(F^{(0)} - \varphi \right)\,,$$

via

$$\mathrm{R.N.} \int_{\mathcal{F}} \mathrm{d}\mu\, F(\tau) = 2\,\mathrm{Res}_{s=1} \mathcal{R}^\star(F;s) + \delta$$

Rankin-Selberg-Zagier method II

- δ is a scheme-dependent correction which depends only on the leading behavior $\varphi(\tau_2)$,

$$\delta = 2\,\mathrm{Res}_{s=1}\left[\zeta^\star(2s)\,h_\mathcal{T}(s) + \zeta^\star(2s-1)\,h_\mathcal{T}(1-s)\right] - \hat{\varphi}(\mathcal{T})\,,$$

  where $h_\mathcal{T}(s) = \int_0^\mathcal{T} \mathrm{d}\tau_2\,\varphi(\tau_2)\,\tau_2^{s-2}$.
- The Rankin-Selberg transform $\mathcal{R}^\star(F;s)$ is itself equal to the renormalized integral

$$\mathcal{R}^\star(F;s) = \mathrm{R.N.}\int_\mathcal{F} \mathrm{d}\mu\,F(\tau)\,\mathcal{E}^\star(s;\tau)$$

- According to this prescription, $\mathrm{R.N.}\int_\mathcal{F} \mathrm{d}\mu\,\mathcal{E}^\star(\tau;s) = 0$!

Epstein series from modular integrals

- The RSZ method applies immediately to integrals with $\Phi = 1$:

$$\mathcal{R}^\star(\Gamma_{d,d}; s) = \zeta^\star(2s) \int_0^\infty \mathrm{d}\tau_2\, \tau_2^{s+d/2-2} \sum_{p_L^2 - p_R^2 = 0}{}' e^{-\pi\tau_2 (p_L^2 + p_R^2)}$$

$$= \zeta^\star(2s) \frac{\Gamma(s + \frac{d}{2} - 1)}{\pi^{s+\frac{d}{2}-1}} \mathcal{E}_V^d(g, B; s + \tfrac{d}{2} - 1)$$

where $\mathcal{E}_V^d(g, B; s)$ is the constrained Epstein series

$$\mathcal{E}_V^d(g, B; s) \equiv \sum_{\substack{(m_i, n^i) \in \mathbb{Z}^{2d} \setminus (0,0) \\ m_i n^i = 0}} \mathcal{M}^{-2s}, \qquad \mathcal{M}^2 = p_L^2 + p_R^2$$

- This is identified as a sum over all BPS states of momentum m_i and winding n^i, with mass

$$\mathcal{M}^2 = (m_i + B_{ik} n^k) g^{ij} (m_j + B_{jl} n^l) + n^i g_{ij} n^j$$

subject to the BPS condition $m_i n^i = 0$. Invariance under $O(\Gamma_{d,d})$ is manifest.

- The constrained Epstein Zeta series $\mathcal{E}^d_V(g, B; s)$ converges absolutely for $\mathrm{Re}(s) > d$. The RSZ method shows that it admits a meromorphic continuation in the s-plane satisfying

$$\mathcal{E}^{d\star}_V(s) = \pi^{-s}\,\Gamma(s)\,\zeta^\star(2s - d + 2)\,\mathcal{E}^d_V(s) = \mathcal{E}^{d\,\star}_V(d - 1 - s)\,,$$

with a simple pole at $s = 0, \frac{d}{2} - 1, \frac{d}{2}, d - 1$ (double poles if $d = 2$).

Epstein series and BPS state sums II

- The residue at $s=\frac{d}{2}$ produces the modular integral of interest:

$$\mathrm{R.N.} \int_{\mathcal{F}} \mathrm{d}\mu\, \Gamma_{d,d}(g,B) = \frac{\Gamma(\frac{d}{2}-1)}{\pi^{\frac{d}{2}-1}}\, \mathcal{E}^{d}_{V}\left(g,B;\tfrac{d}{2}-1\right)$$

rigorously proving an old conjecture of Obers and myself (1999).

- For $d=2$, the BPS constraint $m_i n^i = 0$ can be solved, leading to

$$\mathcal{E}^{2\star}_{V}(T,U;s) = 2\, E^{\star}(T;s)\, E^{\star}(U;s)$$

hence to Dixon-Kaplunovsky-Louis famous result (1989)

$$\int_{\mathcal{F}} \left(\Gamma_{2,2}(T,U)-\tau_2\right)\, d\mu = -\log\left(T_2\, U_2\, |\eta(T)\,\eta(U)|^4\right) + \mathrm{cte}$$

Relation with other constructions

- The differential equations

$$0 = \left[\Delta_{\mathrm{SO}(d,d)} - 2\,\Delta_{\mathrm{SL}(2)} + \tfrac{1}{4}\,d(d-2)\right]\ \mathit{\Gamma}_{d,d}(g,B)$$
$$0 = \left[\Delta_{\mathrm{SL}(2)} - \tfrac{1}{2}\,s(s-1)\right]\ E^{\star}(\tau;s)\,,$$

imply that $\mathcal{E}^{d\star}_{V}(s)$ is an eigenmode of the Laplace-Beltrami operator on the Grassmannian $G_{d,d}$ with eigenvalue $s(s-d+1)$, and more generally, of all $O(d,d)$ invariant differential operators.

- $\mathcal{E}^{d\star}_{V}(g,B;s)$ is proportional to the Langlands-Eisenstein series of $O(d,d)$ with infinitesimal character $\rho - 2s\alpha_1$.
- The residue at $s=\frac{d}{2}$ is the minimal theta series, attached to the minimal representation of $SO(d,d)$ (functional dimension $2d-3$).

Ginzburg Rallis Soudry; Kazhdan BP Waldron; Green Vanhove Miller

Modular integrals with unphysical tachyons I

- For many cases of interest, the integrand is NOT of moderate growth at the cusp, rather it grows exponentially, due to the heterotic unphysical tachyon, $\Phi(\tau) \sim 1/q^\kappa + \mathcal{O}(1)$ with $\kappa = 1$.
- In mathematical terms, $\Phi(\tau) \in \mathbb{C}[\hat{E}_2, E_4, E_6, 1/\Delta]$ is an almost, weakly holomorphic modular form with weight $w = -k/2 \leq 0$.
- The RSZ method fails, however the unfolding trick could still work provided $\Phi(\tau)$ can be represented as a uniformly convergent Poincaré series with seed $f(\tau)$ is invariant under $\Gamma_\infty : \tau \to \tau + n$,

$$\Phi(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} f(\tau)|_w \gamma$$

- Convergence requires $f(\tau) \ll \tau_2^{1-\frac{w}{2}}$ as $\tau_2 \to 0$. The choice $f(\tau) = 1/q^\kappa$ works for $w > 2$ but fails for $w \leq 2$.

Various Poincaré series representations I

- One option is to insert a non-holomorphic convergence factor à la Hecke-Kronecker, i.e. choose a seed $f(\tau) = \tau_2^{s-\frac{w}{2}}\, q^{-\kappa}$

$$E(s,\kappa,w) \equiv \frac{1}{2} \sum_{(c,d)=1} \frac{(c\tau+d)^{-w}\, \tau_2^{s-\frac{w}{2}}}{|c\tau+d|^{2s-w}}\, e^{-2\pi i\kappa\, \frac{a\tau+b}{c\tau+d}}$$

Selberg;Goldfeld Sarnak; Pribitkin

- This converges absolutely for $\mathrm{Re}(s) > 1$, but analytic continuation to desired value $s = \frac{w}{2}$ is tricky, and in general non-holomorphic.
- Moreover, $E(s,\kappa,w)$ is not an eigenmode of the Laplacian, rather

$$\left[\Delta_w + \tfrac{1}{2}\, s(1-s) + \tfrac{1}{8}\, w(w+2)\right]\, E(s,\kappa,w) = 2\pi\kappa\, (s - \tfrac{w}{2})\, E(s{+}1,\kappa,w)$$

Niebur-Poincaré series I

- We shall use another regularization which does not require analytic continuation: the Niebur-Poincaré series

$$\mathcal{F}(s,\kappa,w) = \frac{1}{2} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathcal{M}_{s,w}(-\kappa\tau_2)\, e^{-2\pi i\kappa\tau_1}\, |_w\, \gamma$$

Niebur; Hejhal; Bruinier Ono Bringmann...

where $\mathcal{M}_{s,w}(y)$ is proportional to a Whittaker function, so that

$$\left[\Delta_w + \frac{1}{2}\, s(1-s) + \frac{1}{8}\, w(w+2)\right]\, \mathcal{F}(s,\kappa,w) = 0$$

- The seed $f(\tau) = \mathcal{M}_{s,w}(-\kappa\tau_2)\, e^{-2\pi i\kappa\tau_1}$ is uniquely determined by

$$f(\tau) \sim_{\tau_2\to 0} \tau_2^{\,s-\frac{w}{2}}\, e^{-2\pi i\kappa\tau_1} \qquad f(\tau) \sim_{\tau_2\to\infty} \frac{\Gamma(2s)}{\Gamma(s+\frac{w}{2})}\, q^{-\kappa}$$

ensuring that $\mathcal{F}(s,\kappa,w)$ converges absolutely for $\mathrm{Re}(s) > 1$.

Niebur-Poincaré series II

- Under raising and lowering operators,

$$D_w = \frac{\mathrm{i}}{\pi}\left(\partial_\tau - \frac{\mathrm{i}w}{2\tau_2}\right) , \qquad \bar{D}_w = -\mathrm{i}\pi\, \tau_2^2 \partial_{\bar\tau} ,$$

  the NP series transforms as

$$\begin{aligned} D_w \cdot \mathcal{F}(s,\kappa,w) &= 2\kappa\,(s+\tfrac{w}{2})\,\mathcal{F}(s,\kappa,w+2) ,\\ \bar{D}_w \cdot \mathcal{F}(s,\kappa,w) &= \frac{1}{8\kappa}(s-\tfrac{w}{2})\,\mathcal{F}(s,\kappa,w-2) . \end{aligned}$$

- Under Hecke operators,

$$H_{\kappa'} \cdot \mathcal{F}(s,\kappa,w) = \sum_{d|(\kappa,\kappa')} d^{1-w}\,\mathcal{F}(s,\kappa\kappa'/d^2,w) .$$

- For congruence subgroups of $SL(2,\mathbb{Z})$, one can similarly define NP series $\mathcal{F}_{\mathfrak{a}}(s,\kappa,w)$ for each cusp.

Niebur-Poincaré series III

- For $s = 1 - \frac{w}{2}$, the value relevant for weakly holomorphic modular forms, the seed simplifies to

$$f(\tau) = \Gamma(2-w)\,\left(q^{-\kappa} - \bar{q}^{\kappa}\sum_{\ell=0}^{-w}\frac{(4\pi\kappa\tau_2)^{\ell}}{\ell!}\right)$$

- For $w < 0$, the value $s = 1 - \frac{w}{2}$ lies in the convergence domain, but $\mathcal{F}(1-\frac{w}{2},\kappa,w)$ is in general NOT holomorphic, but rather a weakly harmonic Maass form,

$$\Phi = \sum_{m=-\kappa}^{\infty} a_m\,q^m + \sum_{m=1}^{\infty} m^{w-1}\,\bar{b}_m\,\Gamma(1-w,4\pi m\tau_2)\,q^{-m}$$

- For any such form, $\bar{D}\Phi = \tau_2^{2-w}\bar{\Psi}$ where $\Psi = \sum_{m\geq 1} b_m q^m$ is a holomorphic cusp form of weight $2-w$, the shadow of the Mock modular form $\Phi^{-} = \sum_{m=-\kappa}^{\infty} a_m\,q^m$.

Niebur-Poincaré series IV

- If $|w|$ is small enough, the negative frequency coefficients b_m vanish and Φ is in fact a weakly holomorphic modular form:

w	$\mathcal{F}(1-\frac{w}{2},1,w)$
0	$j+24$
-2	$3!\,E_4E_6/\Delta$
-4	$5!\,E_4^2/\Delta$
-6	$7!\,E_6/\Delta$
-8	$9!\,E_4/\Delta$
-10	$11!\,\Phi_{-10}$
-12	$13!/\Delta$
-14	$15!\,\Phi_{-14}$

  where Φ_{-10} and Φ_{-14} are Mock modular forms with shadow $2.8402... \times \Delta$ and $1.3061... \times E_4\,\Delta$.

Niebur-Poincaré series V

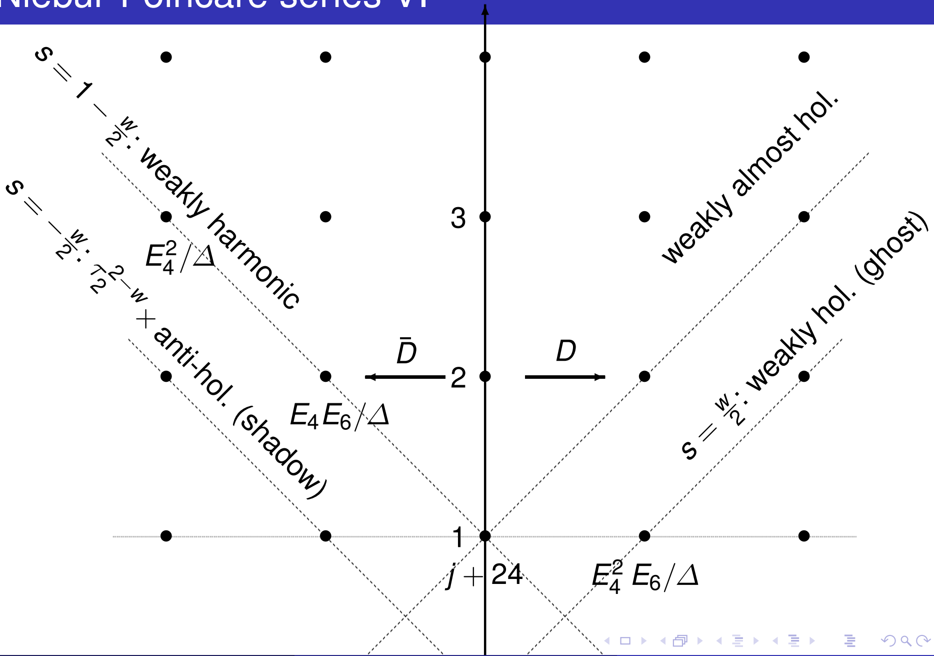
- Theorem (Bruinier) : any weakly holomorphic modular form of weight $w \leq 0$ with polar part $\Phi = \sum_{-\kappa \leq m < 0} a_m\, q^m + \mathcal{O}(1)$ is a linear combination of Niebur-Poincaré series

$$\Phi = \frac{1}{\Gamma(2-w)} \sum_{-\kappa \leq m < 0} a_m\, \mathcal{F}(1 - \tfrac{w}{2}, m, w) + a'_0\, \delta_{w,0}$$

  (The same holds for congruence subgroups of $SL(2,\mathbb{Z})$, including contributions from all cusps)

- Weakly almost holomorphic modular forms of weight $w \leq 0$ can similarly be represented as linear combinations of $\mathcal{F}(1 - \frac{w}{2} + n, m, w)$ with $-\kappa \leq m < 0, 0 \leq n \leq p$ where p is the depth. This fails for positive weight, as such forms are not necessarily harmonic !

Niebur-Poincaré series VI



Unfolding the modular integral

- By Bruinier's thm, any modular integral is a linear combination of

$$\mathcal{I}_{d+k,d}(s,\kappa) = \mathrm{R.N.} \int_{\mathcal{F}} \mathrm{d}\mu\, \Gamma_{d+k,d}(G,B,Y)\, \mathcal{F}(s,\kappa,-\tfrac{k}{2})$$

- Using the unfolding trick, one arrives at the BPS state sum

$$\begin{aligned}\mathcal{I}_{d+k,d}(s,\kappa) =& (4\pi\kappa)^{1-\frac{d}{2}}\, \Gamma(s+\tfrac{2d+k}{4}-1) \\ & \times \sum_{\mathrm{BPS}} {}_2F_1\left(s-\tfrac{k}{4}\,,\, s+\tfrac{2d+k}{4}-1\,;\, 2s\,;\, \tfrac{4\kappa}{p_{\mathrm{L}}^2}\right) \left(\frac{p_{\mathrm{L}}^2}{4\kappa}\right)^{1-s-\frac{2d+k}{4}}\end{aligned}$$

Bruinier; Angelantonj Florakis BP

where $\sum_{\mathrm{BPS}} \equiv \sum_p \delta(p_{\mathrm{L}}^2 - p_{\mathrm{R}}^2 - 4\kappa)$. This converges absolutely for $\mathrm{Re}(s) > \frac{2d+k}{4}$ and can be analytically continued to $\mathrm{Re}(s) > 1$ with a simple pole at $s = \frac{2d+k}{4}$.

Unfolding the modular integral

- For values $s = 1 - \frac{w}{2} + n$ relevant for almost holomorphic modular forms, the summand can be written using elementary functions, e.g.

$$\mathcal{I}_{2+k,2}(1+\tfrac{k}{4},\kappa) = -\,\Gamma(2+\tfrac{k}{2}) \sum_{\mathrm{BPS}} \left[\log\left(\frac{p_{\mathrm{R}}^2}{p_{\mathrm{L}}^2}\right) + \sum_{\ell=1}^{k/2} \frac{1}{\ell} \left(\frac{p_{\mathrm{L}}^2}{4\kappa}\right)^{-\ell} \right]$$

- The result is manifestly $O(\Gamma_{d+k,d})$ invariant, and requires no choice of chamber in Narain modular space. Singularities on $G_{d+k,d}$ arise when $p_L^2 = 0$ for some lattice vector.

Fourier-Jacobi expansion I

- For $d=2, k=0$, the Fourier expansion in T_1 (or U_1) is obtained by solving the BPS constraint. E.g. for $\kappa=1$, all solutions to $m_1 n^1 + m_2 n^2 = 1$ are

$$\begin{cases} m_1 = b + dM,\ n^1 = -c \\ m_2 = a + cM,\ n^2 = d \end{cases} \quad ,\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash SL(2,\mathbb{Z}) \ , M \in \mathbb{Z}$$

- After Poisson resumming over M, the sum over γ neatly produces a Niebur-Poincaré series in U,

$$\begin{aligned} \mathcal{I}(s,1) =& 2^{2s}\sqrt{4\pi}\Gamma(s-\tfrac12)T_2^{1-s}\mathcal{E}(U;s) \\ & +4\sum_{N>0}\sqrt{\frac{T_2}{N}}K_{s-\frac12}(2\pi N T_2)\left[e^{2\pi \mathrm{i} N T_1}\mathcal{F}(s,N,0;U)+\mathrm{cc}\right] \end{aligned}$$

- Moreover, recall $\mathcal{F}(s,N,0) = H_N \cdot \mathcal{F}(s,1,0)$...

Fourier-Jacobi expansion II

- For $s = 1$, relevant for weakly holomorphic modular forms, one recovers the usual Borcherds products,

$$\begin{aligned}\mathcal{A} =& 8\pi \,\mathrm{Res}_{s=1} \left[T_2^{1-s} \mathcal{E}(s;U) \right] + 2 \sum_{N>0} \left[\frac{q_T^N}{N} H_N^{(U)} \cdot [j(U) + 24] + \mathrm{cc} \right] \\ =& -24 \log \left[T_2 U_2 |\eta(T)\eta(U)|^4 \right] - 2 \sum_{M,N} c(MN) [\log(1 - q_T^N q_U^M) + \mathrm{c.c}] \\ =& -24 \log \left[T_2 U_2 |\eta(T)\eta(U)|^4 \right] - \log |j(T) - j(U)|^4\end{aligned}$$

where we have used $\mathcal{F}(1,1,0;U) = j(U) + 24$, $j(U) = \sum c(M) q^M$.

Borcherds; Harvey Moore

Fourier-Jacobi expansion III

- For $s = 1 + n$, relevant for almost holomorphic modular forms of depth $p \geq n$, we can use

$$\begin{aligned} D_T^n\, q_T^N &= 2\,(-2N)^n \sqrt{NT_2}\, K_{n+\frac{1}{2}}(2\pi N T_2)\, e^{2\pi i N T_1} \\ D_T^n 1 &= (2n)!\,(-2\pi T_2)^{-n}/n! \\ D_U^n\, \mathcal{F}(n+1,\kappa,-2n;U) &= (2\kappa)^n\, n!\, \mathcal{F}(n+1,\kappa,0;U) \\ D_U^n\, E(n+1,-2n;U) &= (2\pi)^n\, \mathcal{E}(U;n+1)/n! \end{aligned}$$

to express $\mathcal{I}_{2,2}(n+1,1)$ as the iterated derivative of a generalized prepotential formally of weight $(-2n,-2n)$,

$$\mathcal{I}_{2,2}(n+1,1) = 4\,\mathrm{Re}\left[\frac{(-D_T D_U)^n}{n!} f_n(T,U)\right]$$

Fourier-Jacobi expansion IV

- The resulting prepotential is holomorphic in T but harmonic in U,

$$
\begin{aligned}
f_n(T,U) =& 2\,(2\pi)^{2n+1}\,E(n+1,-2n;U) \\
&+ \sum_{N>0} \frac{2q_T^N}{(2N)^{2n+1}} \mathcal{F}(n+1,N,-2n;U)
\end{aligned}
$$

- One can turn f_n into a holomorphic function $\tilde{f}_n(T,U)$ by replacing $E(n+1,-2n;U)$ and $\mathcal{F}(n+1,N,-2n;U)$ by their analytic parts without affecting the real part of its iterated derivative.

Gangl Zagier

- The generalized holomorphic prepotential $\tilde{f}_n(T,U)$ now transforms as an Eichler integral of weight $(-2n,-2n)$ under $SL(2,\mathbb{Z})_T \times SL(2,\mathbb{Z})_U \ltimes (T \leftrightarrow U)$.

Fourier-Jacobi expansion V

- The generalized Yukawa coupling $\partial_T^{2n+1}\tilde{f}_n$ is an ordinary modular form of weight $(2n+2, -2n)$, e.g for $n = 1$

$$\partial_T^3 \tilde{f}_1 \propto \sum_{N>0} q_T^N H_N^{(U)} \cdot \frac{E_4(U)E_6(U)}{\Delta(U)} = \frac{E_4(T)E_4(U)E_6(U)}{\Delta(U)[j(T) - j(U)]}$$

- The case $n = 1$ describes the standard prepotential appearing in string vacua with $\mathcal{N} = 2$ supersymmetry. Its modular anomaly was discussed by Antoniadis, Ferrara, Gava, Narain, Taylor in 1995, which is the first occurrence of Eichler integrals in string theory !
- The case $n = 2$ has appeared in the context of 1/4-BPS amplitudes in Het/K_3.

Lerche Stieberger 1998

.

Rankin-Selberg method at higher genus I

- String amplitudes at genus $h \leq 3$ take the form

$$\mathcal{A}_h = \int_{\mathcal{F}_h} \mathrm{d}\mu_h \, \varGamma_{d+k,d,h}(G,B,Y;\Omega)\, \Phi(\Omega) \; , \quad \mathrm{d}\mu_h = \frac{\mathrm{d}\Omega_1 \mathrm{d}\Omega_2}{[\det \Omega_2]^{h+1}}$$

  - $\mathcal{F}_h$ is a fundamental domain of the action of $\Gamma = Sp(2h,\mathbb{Z})$ on Siegel's upper half plane $\{\Omega = \Omega^t \in \mathbb{C}^{h\times h}, \Omega_2 > 0\}$
  - $\varGamma_{d+k,d,h}$ a Siegel-Narain theta series of signature $(d+k,d)$

$$\varGamma_{d+k,d,h} = [\det \Omega_2]^{d/2} \sum_{(\Gamma_{d+k,d})^h} e^{\mathrm{i}\pi \mathrm{Tr}(\Omega P_L P_L^t) - \mathrm{i}\pi \mathrm{Tr}(\bar{\Omega} P_R P_R^t)}$$

  - $\Phi(\Omega)$ a Siegel modular form of weight $-k/2$.

- We would like to generalize the previous methods to the case where $\Phi(\Omega)$ is an almost holomorphic modular form with poles inside $\mathcal{F}_h$, such as $1/\chi_{10}$. As a first step, take $k=0$, $\Phi=1$.

Rankin-Selberg method at higher genus II

- The genus h analog of $\mathcal{E}^\star(s;\tau)$ is the non-holomorphic Siegel-Eisenstein series

$$\mathcal{E}^\star_h(s;\Omega) = \zeta^\star(2s) \prod_{j=1}^{[h/2]} \zeta^\star(4s-2j) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} |\Omega_2|^s|\gamma$$

where $\Gamma_\infty = \{\begin{pmatrix} A & B \\ 0 & A^{-t}\end{pmatrix}\} \subset \Gamma$, $|\Omega_2| = |\det \mathrm{Im}\Omega|$.

- The sum converges absolutely for $\mathrm{Re}(s) > \frac{h+1}{2}$ and can be meromorphically continued to the full s plane. The analytic continuation is invariant under $s \mapsto \frac{h+1}{2} - s$, and has a simple pole at $s = \frac{h+1}{2}$ with constant residue $r_h = \frac{1}{2}\prod_{j=1}^{[h/2]} \zeta^\star(2j+1)$

Rankin-Selberg method at higher genus III

- For any modular function $F(\Omega)$ of rapid decay, the Rankin-Selberg transform can be computed by unfolding the integration domain against the sum,

$$\mathcal{R}^{\star}_h(F;s) = \int_{\mathcal{F}_h} \mathrm{d}\mu_h\, F(\Omega)\, \mathcal{E}^{\star}_h(\Omega,s)$$
$$= \zeta^{\star}(2s) \prod_{j=1}^{[h/2]} \zeta^{\star}(4s-2j) \int_{GL(h,\mathbb{Z})\backslash\mathcal{P}_h} \mathrm{d}\Omega_2\, |\Omega_2|^{s-h-1}\, F_0(\Omega_2)$$

where $\mathcal{P}_h$ is the space of positive definite real matrices, and $F_0(\Omega_2) = \int_0^1 \mathrm{d}\Omega_1 F(\Omega)$ is the constant term of F.

- The residue at $s = \frac{h+1}{2}$ is proportional to the average of F,

$$\mathrm{Res}_{s=\frac{h+1}{2}} \mathcal{R}^{\star}_h(F;s) = r_h \int_{\mathcal{F}_h} F\ .$$

Rankin-Selberg method at higher genus IV

- The Siegel-Narain theta series is not a cusp form, instead its zero-th Fourier mode is

$$\Gamma^{(0)}_{d,d,h}(g,B;\Omega) = |\Omega_2|^{d/2} \sum_{(m_i^\alpha, n^{i\alpha})\in\mathbb{Z}^{2d\times h}, m_i^{(\alpha} n^{i\beta)}=0} e^{-\pi \mathrm{Tr}(M^2\Omega_2)}$$

  where

$$M^{2;\alpha\beta} = (m_i^\alpha + B_{ik} n^{k\alpha}) g^{ij} (m_j^\beta + B_{jl} n^{l\beta}) + n^{i\alpha} g_{ij} n^{j\beta}$$

  Terms with $\mathrm{Rk}(m_i^\alpha, n^{i\alpha}) < h$ do not decay rapidly at $\Omega_2 \to \infty$. For $d < h$, this is always the case.

- The Siegel-Eisenstein series $\mathcal{E}^\star_h(\Omega, s)$ similarly has non-decaying constant term of the form $\sum_T e^{-\mathrm{Tr}(T\Omega_2)}$ with $\mathrm{Rk}(T) < h$.

Rankin-Selberg method at higher genus V

- The regularized Rankin-Selberg transform is obtained by subtracting non-suppressed terms, and yields a field theory-type amplitude, with BPS states running in the loops,

$$\mathcal{R}_h(\Gamma_{d,d,h};s) = \int_{GL(h,\mathbb{Z})\backslash\mathcal{P}_h} \frac{\mathrm{d}\Omega_2}{|\Omega_2|^{h+1-s-\frac{d}{2}}} \sum_{\mathrm{BPS}} e^{-\pi \mathrm{Tr}(M^2\Omega_2)}$$

$$= \Gamma_h(s - \tfrac{h+1-d}{2}) \sum_{\mathrm{BPS}} \left[\det M^2\right]^{\frac{h+1-d}{2}-s}$$

$$\sum_{\mathrm{BPS}} = \sum_{\substack{(m_i^\alpha, n^{i\alpha})\in\mathbb{Z}^{2d\times h},\\ m_i^{(\alpha} n^{i\beta)}=0, \det M^2\neq 0}} , \qquad \Gamma_h(s) = \pi^{\frac{1}{4}h(h-1)} \prod_{k=0}^{h-1} \Gamma(s - \tfrac{k}{2})$$

Rankin-Selberg method at higher genus VI

- This is recognized as the Langlands-Eisenstein series of $SO(d,d,\mathbb{Z})$ with infinitesimal character $\rho - 2(s-\frac{h+1-d}{2})\lambda_h$, associated to $\Lambda^h V$ where V is the defining representation,

$$\mathcal{R}_h(\Gamma_{d,d};s) \propto \mathcal{E}^{SO(d,d)}_{\Lambda^h V}(s-\tfrac{h+1-d}{2}) \qquad (h>d)$$

- For $h=d$, $\Lambda^h V = S^2 \oplus C^2$ where S, C are spinor representations,

$$\mathcal{R}_h(\Gamma_{h,h,h};s) \propto \mathcal{E}^{SO(h,h)}_{S}(2s-1) + \mathcal{E}^{SO(h,h)}_{C}(2s-1)$$

- The modular integral of $\Gamma_{d,d,h}$ is proportional to the residue of $\mathcal{R}_h(\Gamma_{d,d,h};s)$ at $s=\frac{h+1}{2}$, up to a scheme dependent term δ. For $d<h$, the entire result comes from δ.

Rankin-Selberg method at higher genus VII

- For $d = 1$, any h,

$$\mathcal{A}_h = \mathcal{V}_h(R^h + R^{-h}) \, , \quad \mathcal{V}_h = \int_{\mathcal{F}_h} \mathrm{d}\mu_h = 2 \prod_{j=1}^{h} \zeta^\star(2j)$$

- For $h = d = 2$, either by computing the BPS sum, or by unfolding the Siegel-Narain theta series, one finds

$$\begin{aligned} \mathcal{R}_2^\star(\Gamma_{2,2}, s) =& 2\zeta^\star(2s)\zeta^\star(2s-1)\zeta^\star(2s-2) \\ & \times [\mathcal{E}_1^\star(T; 2s-1) + \mathcal{E}_1^\star(U; 2s-1)] \end{aligned}$$

hence

$$\mathcal{A}_2 = 2\zeta^\star(2)\,[\mathcal{E}_1^\star(T;2) + \mathcal{E}_1^\star(U;2)]$$

proving the conjecture by Obers and BP (1999).

Rankin-Selberg method at higher genus VIII

- For $h = d = 3$,

$$\mathcal{R}_3^\star(\Gamma_{3,3};s) = \zeta^\star(2s)\,\zeta^\star(2s-1)\,\zeta^\star(2s-2)\,\zeta^\star(2s-3)\\ \left[\mathcal{E}_S^{\star,SO(3,3)}(2s-1) + \mathcal{E}_C^{\star,SO(3,3)}(2s-1)\right]$$

hence

$$\mathcal{A}_3 = 2\zeta^\star(2)\zeta^\star(4)\,\left[\mathcal{E}_S^{\star,SO(3,3)}(3) + \mathcal{E}_C^{\star,SO(3,3)}(3)\right]$$

Conclusion - Outlook

- Modular integrals can be efficiently computed using Rankin-Selberg type methods. The result is expressed as a field theory amplitude with BPS states running in the loop.
- T-duality and singularities from enhanced gauge symmetry are manifest. Fourier-Jacobi expansions can be obtained in some cases by solving the BPS constraint.
- The RSZ method also works at higher genus, at least for $h = 2, 3$. For computing modular integrals with $\Phi \neq 1$ it will be important to develop Poincaré series representations for Siegel modular forms with poles at Humbert divisors, such as $1/\Phi_{10}$.
- Non-BPS amplitudes where Φ is not almost weakly holomorphic are challenging ! So are amplitudes with $h \geq 4$!