Rankin-Selberg methods for String Amplitudes

Boris Pioline

CERN & LPTHE

String Math 2013, Simons Institute

based on work with C. Angelantonj and I. Florakis,

arXiv:1110.5318,1203.0566,1304.4271,and work in progress

Modular integrals and BPS amplitudes I

.

- In closed string theory, an interesting class of amplitudes are given by a modular integral

$$\mathcal{A} = \int_{\mathcal{F}} \mathrm{d}\mu \, \Gamma_{d+k,d} \, \Phi(\tau) \ , \quad \mathrm{d}\mu = \frac{\mathrm{d}\tau_1 \mathrm{d}\tau_2}{\tau_2^2}$$

  - $\mathcal{F} = \Gamma \backslash \mathcal{H}$: fundamental domain of the modular group $\Gamma = SL(2,\mathbb{Z})$ on the Poincaré upper half plane $\mathcal{H}$;
  - $\Gamma_{(d+k,d)} = \tau_2^{d/2} \sum q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2}$: a theta series for an even self-dual lattice of signature $(d+k,d)$, known as Narain's lattice partition function;
  - $\Phi(\tau)$: an (almost, weakly) holomorphic modular form of weight $w = -k/2$, known as the elliptic genus

Modular integrals and BPS amplitudes II

- Such integrals arise in a variety of BPS-saturated amplitudes:
  - Gauge thresholds, $R^2 F^{2h-2}$ in Het$/K3 \times T^2$ at one-loop

    Dixon Kaplunovsky Louis; Harvey Moore; Antoniadis Gava Narain Taylor
  - F^4 couplings in Het$/T^d$ at one-loop

    Bachas Fabre Kiritsis Obers Vanhove
  - R^4 couplings in type II/T^d at one-loop ($\Phi = 1$)

    Green Vanhove; Kiritsis BP
  - R^2 couplings in type $II/K3 \times T^2$ at one-loop

    Harvey Moore; Gregori Kiritsis Kounnas Obers Petropoulos BP
  - F^4 couplings in type $II/T^4/\mathbb{Z}_N$ at tree-level

    Obers BP
  - $\nabla^4 R^4$ couplings in $D = 11$ $SUGRA/T^d$ at two-loops

    Green Vanhove Russo
- These terms are strongly constrained by supersymmetry, and offer precise tests of string dualities.

Theta correspondances

- From a mathematical point of view, modular integrals give a theta correspondence

$$\Phi : \Gamma\backslash\mathcal{H} \to \mathbb{C} \quad \longrightarrow \quad \mathcal{A} : O(\Gamma_{d+k,d})\backslash G_{d+k,d} \to \mathbb{C}$$

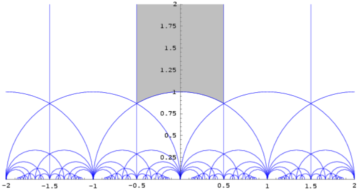
between modular forms on $\mathcal{H}$ and automorphic forms on the Grassmannian $G_{d+k,d}$, or Narain moduli space

$$G_{d+k,d} = \frac{O(d+k,d)}{O(d+k)\times O(d)} \ni (g_{ij}, B_{ij}, Y_i^a)$$

- Theta correspondences are one of the few general ways (together with Langlands-Eisenstein series) to construct automorphic forms.

Unfolding trick

- In the physics literature, the time-honored way to evaluate such integrals has been the unfolding trick or orbit method:



$$\int_{\Gamma\backslash\mathcal{H}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f|_0\gamma = \int_{\Gamma_\infty\backslash\mathcal{H}} f$$

$$f|_w\gamma(\tau) = (c\tau+d)^{-w}\, f\left(\frac{a\tau+b}{c\tau+d}\right)$$

- E.g for $d=1$, representing $\Gamma_{(1,1)} = R\sum_{m,n} e^{-\pi R^2|m-n\tau|^2/\tau_2}$,

$$\int_{\mathcal{F}} \Gamma_{1,1} = R\int_{\mathcal{F}} \mathrm{d}\mu + R\int_{\mathcal{S}} \mathrm{d}\mu \sum_{m\neq 0} e^{-\pi R^2 m^2/\tau_2}$$
$$= \frac{\pi}{3} R + \frac{\pi}{3} R^{-1}$$

Unfolding trick, revisited

- For higher dimensional lattices, the theta series $\Gamma_{d+k,d}$ involves several different orbits of $SL(2,\mathbb{Z})$. The orbit decomposition breaks manifest invariance under the automorphism group $O(\Gamma_{d+k,d})$.
- I will present an alternative method for computing such modular integrals, which keeps T-duality manifest at all stages. The method is inspired by the Rankin-Selberg method commonly used in number theory.
- The result is typically expressed as a field theory amplitude with an infinite number of BPS states running through the loops.
- The method is in principle applicable to higher genus amplitudes, though for the most part I will focus on genus one.

Rankin-Selberg method I

- Consider the completed non-holomorphic Eisenstein series

$$E^\star(\tau;s) = \zeta^\star(2s) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \tau_2^s|\gamma = \tfrac{1}{2}\,\zeta^\star(2s) \sum_{(c,d)=1} \frac{\tau_2^s}{|c\,\tau+d|^{2s}}$$

  where $\zeta^\star(s) \equiv \pi^{-s/2}\,\Gamma(s/2)\,\zeta(s) = \zeta^\star(1-s)$.
- $E^\star(\tau;s)$ is convergent for $\mathrm{Re}(s) > 1$, and has a meromorphic continuation to all s, invariant under $s \mapsto 1-s$, with simple poles at $s = 0, 1$ with constant residue:

$$E^\star(\tau;s) = \frac{1}{2(s-1)} + \tfrac{1}{2}\left(\gamma - \log(4\pi\,\tau_2\,|\eta(\tau)|^4)\right) + \mathcal{O}(s-1)\,,$$

Rankin-Selberg method (cont.)

- For any cusp form $F(\tau)$, consider the Rankin-Selberg transform

$$\mathcal{R}^\star(F,s) = \int_{\mathcal{F}} \mathrm{d}\mu\, E^\star(\tau;s)\, F(\tau)$$

- By the unfolding trick, $\mathcal{R}^\star(F,s)$ is proportional to the Mellin transform of the constant term $F_0(\tau_2) = \int_{-1/2}^{1/2} \mathrm{d}\tau_1\, F(\tau)$,

$$\begin{aligned}\mathcal{R}^\star(F;s) =& \zeta^\star(2s) \int_{\mathcal{S}} \mathrm{d}\mu\, \tau_2^s\, F(\tau) \\ =& \zeta^\star(2s) \int_0^\infty \mathrm{d}\tau_2\, \tau_2^{s-2}\, F_0(\tau_2)\,,\end{aligned}$$

Rankin-Selberg method (cont.)

- The RS transform is in fact proportional to the L-function $L(s) = \sum_n a_n n^{-s}$ associated to F.
- It inherits the meromorphicity and functional relations of $E^\star$, e.g. $\mathcal{R}^\star(F;s) = \mathcal{R}^\star(F;1-s)$.
- Since the residue of $E^\star(\tau;s)$ at $s=0,1$ is constant, the residue of $\mathcal{R}^\star(F;s)$ at $s=1$ is proportional to the modular integral of F,

$$\mathrm{Res}_{s=1}\mathcal{R}^\star(F;s) = \frac{1}{2}\int_{\mathcal{F}} \mathrm{d}\mu\, F$$

Rankin-Selberg-Zagier method I

- This was extended by Zagier to the case where $F^{(0)}$ is of power-like growth $F^{(0)}(\tau) \sim \varphi(\tau_2)$ at the cusp: the renormalized integral

$$\text{R.N.} \int_{\mathcal{F}} \mathrm{d}\mu\, F(\tau) = \lim_{\mathcal{T}\to\infty} \left[\int_{\mathcal{F}_{\mathcal{T}}} \mathrm{d}\mu\, F(\tau) - \hat{\varphi}(\mathcal{T}) \right]$$

$$\varphi(\tau_2) = \sum_{\alpha} c_\alpha \tau_2^\alpha \;, \quad \hat{\varphi}(\mathcal{T}) = \sum_{\alpha\neq 1} c_\alpha \frac{\tau_2^\alpha}{\alpha - 1} + \sum_{\alpha=1} c_\alpha \log \tau_2$$

is related to the Mellin transform of the (regularized) constant term

$$\mathcal{R}^\star(F;s) = \zeta^\star(2s) \int_0^\infty \mathrm{d}\tau_2\, \tau_2^{s-2} \left(F^{(0)} - \varphi \right) \;,$$

via

$$\text{R.N.} \int_{\mathcal{F}} \mathrm{d}\mu\, F(\tau) = 2\, \mathrm{Res}_{s=1} \mathcal{R}^\star(F;s) + \delta$$

Rankin-Selberg-Zagier method II

- The scheme dependent correction δ depends only on the leading behavior $\varphi(\tau_2)$,

$$\delta = 2\,\mathrm{Res}_{s=1}\left[\zeta^\star(2s)\,h_{\mathcal{T}}(s) + \zeta^\star(2s-1)\,h_{\mathcal{T}}(1-s)\right] - \hat{\varphi}(\mathcal{T})\,,$$

  where $h_{\mathcal{T}}(s) = \int_0^{\mathcal{T}} \mathrm{d}\tau_2\,\varphi(\tau_2)\,\tau_2^{s-2}$.
- The Rankin-Selberg transform $\mathcal{R}^\star(F;s)$ can be understood as the renormalized integral

$$\mathcal{R}^\star(F;s) = \mathrm{R.N.}\int_{\mathcal{F}} \mathrm{d}\mu\,F(\tau)\,\mathcal{E}^\star(s;\tau)$$

- According to this prescription, $\mathrm{R.N.}\int_{\mathcal{F}} \mathrm{d}\mu\,\mathcal{E}^\star(\tau;s) = 0$!

Epstein series from modular integrals

- The RSZ method applies immediately to integrals with $\Phi = 1$:

$$\mathcal{R}^\star(\Gamma_{d,d}; s) = \zeta^\star(2s) \int_0^\infty \mathrm{d}\tau_2\, \tau_2^{s+d/2-2} \sum_{p_L^2 - P_R^2 = 0} e^{-\pi\tau_2\,(p_L^2+p_R^2)}$$

$$= \zeta^\star(2s)\, \frac{\Gamma(s+\frac{d}{2}-1)}{\pi^{s+\frac{d}{2}-1}}\, \mathcal{E}_V^d(g,B; s+\tfrac{d}{2}-1)$$

where $\mathcal{E}_V^d(g,B;s)$ is the constrained Epstein series

$$\mathcal{E}_V^d(g,B;s) \equiv \sum_{\substack{(m_i,n^i)\in\mathbb{Z}^{2d}\setminus(0,0)\\ m_i n^i = 0}} \mathcal{M}^{-2s}\,, \qquad \mathcal{M}^2 = p_L^2 + p_R^2$$

Epstein series and BPS state sums I

- This is identified as a sum over all BPS states of momentum m_i and winding n^i, with mass

$$\mathcal{M}^2 = (m_i + B_{ik} n^k) g^{ij} (m_j + B_{jl} n^l) + n^i g_{ij} n^j$$

subject to the BPS condition $m_i n^i = 0$. Invariance under $O(\Gamma_{d,d})$ is manifest.

- The constrained Epstein Zeta series $\mathcal{E}^d_V(g, B; s)$ converges absolutely for $s + \frac{d}{2} - 1 > 1$. The RSZ method shows that it admits a meromorphic continuation in the s-plane satisfying

$$\mathcal{E}^{d\star}_V(s) = \pi^{-s}\,\Gamma(s)\,\zeta^\star(2s - d + 2)\,\mathcal{E}^d_V(s) = \mathcal{E}^{d\star}_V(d - 1 - s)\,,$$

with a simple pole at $s = 0, \frac{d}{2} - 1, \frac{d}{2}, 1$ (double poles if $d = 2$).

Epstein series and BPS state sums II

- The residue at $s=\frac{d}{2}$ produces the modular integral of interest:

$$\mathrm{R.N.}\int_{\mathcal{F}} \mathrm{d}\mu\, \varGamma_{d,d}(g,B) = \frac{\Gamma(d/2-1)}{\pi^{d/2-1}}\, \mathcal{E}^{d}_{V}\left(g,B;\tfrac{1}{2}\,d-1\right)$$

rigorously proving an old conjecture of Obers and myself (1999).

- For $d=2$, the BPS constraint can be solved, leading to

$$\mathcal{E}^{2\star}_{V}(T,U;s) = 2\,E^{\star}(T;s)\,E^{\star}(U;s)$$

hence to Dixon-Kaplunovsky-Louis famous result (1989)

$$\int_{\mathcal{F}} \left(\varGamma_{2,2}(T,U)-\tau_2\right)\, d\mu = -\log\left(\frac{8\pi e^{1-\gamma}}{3\sqrt{3}}\, T_2\, U_2\, |\eta(T)\,\eta(U)|^4\right)$$

Relation with other constructions

- The differential equations

$$0 = \left[\Delta_{\mathrm{SO}(d,d)} - 2\,\Delta_{\mathrm{SL}(2)} + \tfrac{1}{4}\,d(d-2)\right]\ \Gamma_{(d,d)}(g,B)$$
$$0 = \left[\Delta_{\mathrm{SL}(2)} - \tfrac{1}{2}\,s(s-1)\right]\ E^\star(\tau;s)\,,$$

  imply that $\mathcal{E}_V^{d\star}(s)$ is an eigenmode of the Laplace-Beltrami operator on the Grassmannian $G_{d,d}$ with eigenvalue $s(s-d+1)$, and more generally, of all $O(d,d)$ invariant differential operators.
- $\mathcal{E}_V^{d\star}(g,B;s)$ is proportional to the Langlands-Eisenstein series of $O(d,d)$ with infinitesimal character $\rho - 2s\alpha_1$.
- The residue at $s=\frac{d}{2}$ is the minimal theta series, attached to the minimal representation of $SO(d,d)$ (functional dimension $2d-3$).

Ginzburg Rallis Soudry; Kazhdan BP Waldron; Green Vanhove Miller

Modular integrals with unphysical tachyons I

- For many cases of interest, the integrand is NOT of moderate growth at the cusp, rather it grows exponentially, due to the heterotic unphysical tachyon.
- In mathematical terms, $\Phi(\tau) \in \mathbb{C}[\hat{E}_2, E_4, E_6, 1/\Delta]$ is a almost, weakly holomorphic modular form with weight $w = -k/2 \leq 0$.
- The RSZ method fails, however the unfolding trick could still work provided $\Phi(\tau)$ can be represented as a uniformly convergent Poincaré series with seed $f(\tau)$ is invariant under $\Gamma_\infty : \tau \to \tau + n$,

$$\Phi(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} f(\tau)|_w \gamma$$

- Convergence requires $f(\tau) \ll \tau_2^{1-\frac{w}{2}}$ as $\tau_2 \to 0$. The choice $f(\tau) = 1/q^\kappa$ works for $w > 2$ but fails for $w \leq 2$.

Various Poincaré series representations I

- One option is to insert a non-holomorphic convergence factor à la Hecke-Kronecker, i.e. choose $f(\tau) = \tau_2^{s-\frac{w}{2}}\, q^{-\kappa}$

$$E(s,\kappa,w) \equiv \frac{1}{2} \sum_{(c,d)=1} \frac{(c\tau+d)^{-w}\, \tau_2^{s-\frac{w}{2}}}{|c\tau+d|^{2s-w}}\, e^{-2\pi\mathrm{i}\kappa\, \frac{a\tau+b}{c\tau+d}}$$

Selberg;Goldfeld Sarnak; Pribitkin

- This converges absolutely for $\mathrm{Re}(s) > 1$, but the analytic continuation to $s = \frac{w}{2}$ is tricky, and leads to holomorphic anomalies.
- Moreover, $E(s,\kappa,w)$ is not an eigenmode of the Laplacian, rather

$$\left[\Delta_w + \tfrac{1}{2}\, s(1-s) + \tfrac{1}{8}\, w(w+2)\right]\, E(s,\kappa,w) = 2\pi\kappa\, (s - \tfrac{w}{2})\, E(s{+}1,\kappa,w)$$

Niebur-Poincaré series I

- We shall use another regularization which does not require analytic continuation and preserves the action of the Laplacian: the Niebur-Poincaré series

$$\mathcal{F}(s,\kappa,w) = \tfrac{1}{2} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathcal{M}_{s,w}(-\kappa\tau_2)\, e^{-2\pi i\kappa\tau_1} \,|_w\, \gamma$$

Niebur; Hejhal; Bruinier Ono Bringmann...

where $\mathcal{M}_{s,w}(y)$ is proportional to a Whittaker function, so that

$$\left[\Delta_w + \tfrac{1}{2}\, s(1-s) + \tfrac{1}{8}\, w(w+2)\right]\, \mathcal{F}(s,\kappa,w) = 0$$

- The seed $f(\tau) = \mathcal{M}_{s,w}(-\kappa\tau_2)\, e^{-2\pi i\kappa\tau_1}$ satisfies

$$f(\tau) \sim_{\tau_2\to 0} \tau_2^{\mathrm{Re}(s)-\frac{w}{2}}\, e^{-2\pi i\kappa\tau_1} \qquad f(\tau) \sim_{\tau_2\to\infty} \frac{\Gamma(2s)}{\Gamma(s+\frac{w}{2})}\, q^{-\kappa}$$

hence $\mathcal{F}(s,\kappa,w)$ converges absolutely for $\mathrm{Re}(s) > 1$.

Niebur-Poincaré series II

- Under raising and lowering operators,

$$D_w = \frac{\mathrm{i}}{\pi}\left(\partial_\tau - \frac{\mathrm{i}w}{2\tau_2}\right)\ , \qquad \bar{D}_w = -\mathrm{i}\pi\,\tau_2^2\partial_{\bar\tau}\ ,$$

  the NP series transforms as

$$\begin{aligned} D_w\cdot\mathcal{F}(s,\kappa,w) &= 2\kappa\,(s+\tfrac{w}{2})\,\mathcal{F}(s,\kappa,w+2)\,,\\ \bar{D}_w\cdot\mathcal{F}(s,\kappa,w) &= \frac{1}{8\kappa}(s-\tfrac{w}{2})\,\mathcal{F}(s,\kappa,w-2)\,. \end{aligned}$$

- Under Hecke operators,

$$H_{\kappa'}\cdot\mathcal{F}(s,\kappa,w) = \sum_{d|(\kappa,\kappa')} d^{1-w}\,\mathcal{F}(s,\kappa\kappa'/d^2,w)\ .$$

- The construction generalizes straightforwardly to congruence subgroups of $SL(2,\mathbb{Z})$: one NP series $\mathcal{F}_{\mathfrak{a}}(s,\kappa,w)$ for each cusp.

Niebur-Poincaré series III

- For $s = 1 - \frac{w}{2}$, relevant for weakly holomorphic modular forms, the seed simplifies to

$$f(\tau) = \Gamma(2-w)\left(q^{-\kappa} - \bar{q}^{\kappa}\sum_{\ell=0}^{-w}\frac{(4\pi\kappa\tau_2)^{\ell}}{\ell!}\right)$$

- For $w < 0$, the value $s = 1 - \frac{w}{2}$ lies in the convergence domain, but $\mathcal{F}(1-\frac{w}{2},\kappa,w)$ is in general NOT holomorphic, but rather a weakly harmonic Maass form,

$$\Phi = \sum_{m=-\kappa}^{\infty} a_m\, q^m + \sum_{m=1}^{\infty} m^{w-1}\,\bar{b}_m\,\Gamma(1-w,4\pi m\tau_2)\,q^{-m}$$

- For any such form, $\bar{D}\Phi = \tau_2^{2-w}\bar{\Psi}$ where $\Psi = \sum_{m\geq 1} b_m q^m$ is a holomorphic cusp form of weight $2-w$, the shadow of the Mock modular form $\Phi^{-} = \sum_{m=-\kappa}^{\infty} a_m\, q^m$.

Niebur-Poincaré series IV

- If $|w|$ is small enough, the negative frequency coefficients $\bar{b}_m$ vanish and Φ is in fact a weakly holomorphic modular form:

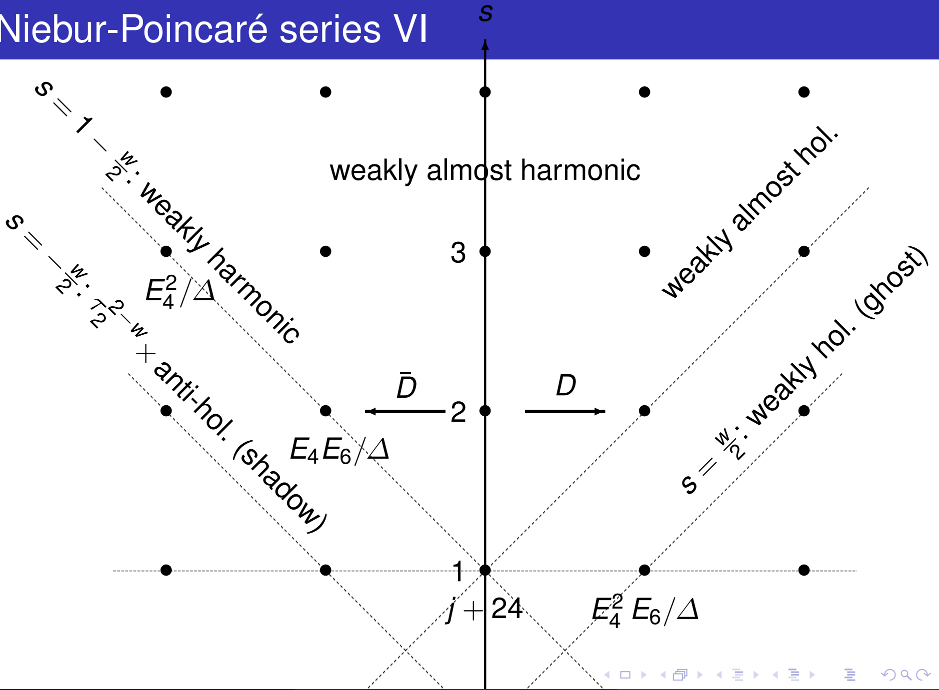
w	$\mathcal{F}(1-\frac{w}{2},1,w)$	$\mathcal{F}(1-\frac{w}{2},1,2-w)$
0	$j+24$	$E_4^2 E_6/\Delta$
-2	$3!\, E_4 E_6/\Delta$	$E_4(j-240)$
-4	$5!\, E_4^2/\Delta$	$E_6(j+204)$
-6	$7!\, E_6/\Delta$	$E_4^2(j-480)$
-8	$9!\, E_4/\Delta$	$E_4 E_6(j+264)$
-10	$11!\, \Phi_{-10}$	$(mess)$
-12	$13!/\Delta$	$E_4^2 E_6(j+24)$
-14	$15!\, \Phi_{-14}$	$(mess)$

Niebur-Poincaré series V

- Theorem (Bruinier) : any weakly holomorphic modular form of weight $w \leq 0$ with polar part $\Phi = \sum_{-\kappa \leq m < 0} a_m\, q^m + \mathcal{O}(1)$ can be represented as a linear combination of Niebur-Poincaré series

$$\Phi = \frac{1}{\Gamma(2-w)} \sum_{-\kappa \leq m < 0} a_m\, \mathcal{F}(1 - \tfrac{w}{2}, m, w) + a_0'\, \delta_{w,0}$$

(The same holds for congruence subgroups of $SL(2,\mathbb{Z})$, provided the polar parts at all cusps match)



s
weakly almost harmonic
$s = 1 - \frac{w}{2}$: weakly harmonic
$s = -\frac{w}{2}$: τ_2^{2-w} + anti-hol. (shadow)
weakly almost hol.
$s = \frac{w}{2}$: weakly hol. (ghost)
3
2
1
$\bar{D}$
D
E_4^2/Δ
$E_4 E_6/\Delta$
$j + 24$
$E_4^2\, E_6/\Delta$

Unfolding the modular integral

- Using Bruinier's thm, any modular integral can be expressed as a linear combination of

$$\mathcal{I}_{d+k,d}(s,\kappa) = \mathrm{R.N.} \int_{\mathcal{F}} \mathrm{d}\mu\, \Gamma_{d+k,d}(G,B,Y)\, \mathcal{F}(s,\kappa,-\tfrac{k}{2})$$

- Using the unfolding trick, one arrives at the BPS state sum

$$\begin{aligned}\mathcal{I}_{d+k,d}(s,\kappa) =& (4\pi\kappa)^{1-\frac{d}{2}}\, \Gamma(s+\tfrac{2d+k}{4}-1) \\ & \times \sum_{\mathrm{BPS}} {}_2F_1\left(s-\tfrac{k}{4}\,,\, s+\tfrac{2d+k}{4}-1\,;\, 2s\,;\, \tfrac{4\kappa}{p_{\mathrm{L}}^2}\right) \left(\frac{p_{\mathrm{L}}^2}{4\kappa}\right)^{1-s-\frac{2d+k}{4}}\end{aligned}$$

Bruinier; Angelantonj Florakis BP

where $\sum_{\mathrm{BPS}} \equiv \sum_p \delta(p_{\mathrm{L}}^2 - p_{\mathrm{R}}^2 - 4\kappa)$. This converges absolutely for $\mathrm{Re}(s) > \frac{2d+k}{4}$ and can be analytically continued to $\mathrm{Re}(s) > 1$ with a simple pole at $s = \frac{2d+k}{4}$.

Unfolding the modular integral

- The result is manifestly $O(\Gamma_{d+k,d})$ invariant, and requires no choice of chamber in Narain modular space. Singularities on $G_{d+k,d}$ arise when $p_L^2 = 0$ for some lattice vector.
- For the relevant values $s = 1 - \frac{w}{2} + n$, the result can be written using elementary functions, e.g.

$$\mathcal{I}_{2+k,2}(1 + \tfrac{k}{4}, \kappa) = -\,\Gamma(2 + \tfrac{k}{2}) \sum_{\mathrm{BPS}} \left[\log\left(\frac{p_{\mathrm{R}}^2}{p_{\mathrm{L}}^2}\right) + \sum_{\ell=1}^{k/2} \frac{1}{\ell} \left(\frac{p_{\mathrm{L}}^2}{4\kappa}\right)^{-\ell} \right]$$

One example

- Consider $\mathrm{Het}/T^2 \times K3$ at $\mathbb{Z}_2$ orbifold point with gauge group broken to $E_8 \times E_7 \times SU(2)$. The gauge threshold for E_7 is

$$\Delta_{\mathrm{E}_7} = -\frac{1}{12} \int_{\mathcal{F}} \mathrm{d}\mu \, \Gamma_{2,2} \, \frac{\hat{E}_2 \, E_4 \, E_6 - E_4^3}{\Delta}$$

Expressing the elliptic genus as a linear combination

$$\frac{\hat{E}_2 \, E_4 \, E_6 - E_4^3}{\Delta} = \mathcal{F}(2,1,0) - 6 \, \mathcal{F}(1,1,0) - 864$$

one arrives at

$$\Delta_{\mathrm{E}_7} = \sum_{\mathrm{BPS}} \left[1 + \frac{p_{\mathrm{R}}^2}{4} \log \left(\frac{p_{\mathrm{R}}^2}{p_{\mathrm{L}}^2} \right) \right] - 72 \log \left(4\pi \, e^{-\gamma} \, T_2 \, U_2 \, |\eta(T) \, \eta(U)|^4 \right)$$

Fourier expansion I

- The Fourier expansion in T_1 (or U_1) is obtained by solving the BPS constraint. E.g. for $\kappa = 1$, all solutions to $m_1 n^1 + m_2 n^2 = 1$ are

$$\begin{cases} m_1 = b + dM,\ n^1 = -c \\ m_2 = a + cM,\ n^2 = d \end{cases} \quad , \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash SL(2,\mathbb{Z})\ , m \in \mathbb{Z}$$

- After Poisson resumming over M, the sum over γ reproduces a sum of Niebur-Poincaré series in U,

$$\begin{aligned} \mathcal{I}^{(0)}(s,1) =& 2^{2s-1}\sqrt{4\pi}\Gamma(s-\tfrac{1}{2}) T_2^{1-s} \mathcal{E}(U;s) \\ &+ \sum_{N\neq 0} 2\sqrt{\tfrac{T_2}{|N|}}\, K_{s-\frac{1}{2}}(2\pi |N| T_2)\, e^{-2\pi \mathrm{i} N T_1} \left[\mathcal{F}(s,|N|,0;U) + \mathrm{cc}\right] \end{aligned}$$

Fourier expansion II

- For $s = 1$, relevant for weakly holomorphic modular forms, one recovers the usual Borcherds products

$$\begin{aligned}\mathcal{I}_{2,2}(1,1) =& \mathcal{I}_{2,2}^{(0)} + \sum_{N>0} \frac{q_T^N}{N} H_N^{(U)} \cdot [j(U) + 24] + \text{ccc} \\ =& \mathcal{I}_{2,2}^{(0)} - \log \left| \prod_{M,N} (1 - q_T^M q_U^N)^{c(MN)} \right|^2\end{aligned}$$

- For $s = 1 + n$, relevant for almost holomorphic modular forms,

$$\mathcal{I}_{2,2}(n+1,1) = \frac{(-D_T D_U)^n}{2^{n+1} n!} \left[\sum_{N>0} \frac{q_T^N}{N^{2n+1}} H_N^{(U)} \cdot \mathcal{F}(n+1,1,-2n;U) \right] + \dots$$

exhibiting Obers-Kiritsis generalized holomorphic prepotentials.

Rankin-Selberg method at higher genus I

- String amplitudes at genus $h \leq 3$ take the form

$$\mathcal{A}_h = \int_{\mathcal{F}_h} \mathrm{d}\mu_h \, \Gamma_{d+k,d,h} \, \Phi(\Omega) \;, \quad \mathrm{d}\mu_h = \frac{\mathrm{d}\Omega}{[\det \mathrm{Im}\Omega]^{h+1}}$$

  where $\mathcal{F}_h$ is a fundamental domain of the action of $\Gamma = Sp(2h, \mathbb{Z})$ on Siegel's upper half plane $\mathcal{H}_h = Sp(2h)/U(h)$, and $\Phi(\Omega)$ is a Siegel modular form of weight $-k/2$.
- For $h > 3$, the integral is restricted to the Schottky locus and we cannot say much.
- We would like to generalize the previous methods to the case where $\Phi(\Omega)$ is a almost holomorphic modular form with poles inside $\mathcal{F}_h$, such as $1/\chi_{10}$. As a first step, take $k = 0$, $\Phi = 1$.

Rankin-Selberg method at higher genus II

- The genus h analog of $\mathcal{E}^\star(s;\tau)$ is the non-holomorphic Siegel-Eisenstein series

$$\mathcal{E}^\star_h(s;\Omega) = \zeta^\star(2s) \prod_{j=1}^{[h/2]} \zeta^\star(4s-2j) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} |\Omega_2|^s|\gamma$$

where $\Gamma_\infty = \left\{ \begin{pmatrix} A & B \\ 0 & A^{-t} \end{pmatrix} \right\}$, $\quad |\Omega_2| = |\det \mathrm{Im}\Omega|$.

- The sum converges absolutely for $\mathrm{Re}(s) > (h+1)/2$ and can be meromorphically continued to the full s plane. The analytic continuation is invariant under $s \mapsto \frac{h+1}{2} - s$, and has a simple pole at $s = \frac{h+1}{2}$ with constant residue $r_h = \frac{1}{2} \prod_{j=1}^{[h/2]} \zeta^\star(2j+1)$

Rankin-Selberg method at higher genus III

- For any cusp form $F(\Omega)$, the Rankin-Selberg transform can be computed by unfolding the integration domain against the sum,

$$\mathcal{R}^\star_h(F;s) = \int_{\mathcal{F}_h} \mathrm{d}\mu_h\, F(\Omega)\, \mathcal{E}^\star_h(\Omega,s)$$
$$= \zeta^\star(2s) \prod_{j=1}^{[h/2]} \zeta^\star(4s-2j) \int_{GL(h,\mathbb{Z})\backslash\mathcal{P}_h} \mathrm{d}\Omega_2\, |\Omega_2|^{s-h-1}\, F_0(\Omega_2)$$

  where $\mathcal{P}_h$ is the space of positive definite real matrices, and $F_0(\Omega_2) = \int_0^1 \mathrm{d}\Omega_1 F(\Omega)$ is the constant term of F.

- The residue at $s = \frac{h+1}{2}$ is proportional to the average of F,

$$\mathrm{Res}_{s=\frac{h+1}{2}} \mathcal{R}^\star_h(F;s) = r_h \int_{\mathcal{F}_h} F\,.$$

Rankin-Selberg method at higher genus IV

- The Siegel-Narain theta series is not a cusp form, instead its zero-th Fourier mode is

$$\Gamma^{(0)}_{d,d,h}(g,B;\Omega) = |\Omega_2|^{d/2} \sum_{(m_i^\alpha, n^{i\alpha})\in\mathbb{Z}^{2d}, m_i^{(\alpha} n^{i\beta)}=0} e^{-\pi \mathrm{Tr}(M^2\Omega_2)}$$

  where

$$M^{2;\alpha\beta} = (m_i^\alpha + B_{ik} n^{k\alpha}) g^{ij} (m_j^\beta + B_{jl} n^{l\beta}) + n^{i\alpha} g_{ij} n^{j\beta}$$

  Terms with $\mathrm{Rk}(m_i^\alpha, n^{i\alpha}) < h$ do not decay rapid ly at $\Omega_2 \to \infty$.
- The Siegel-Eisenstein series $\mathcal{E}^\star_h(\Omega, s)$ similarly has non-decaying constant term of the form $e^{-\mathrm{Tr}(T\Omega_2)}$ with $\mathrm{Rk}(T) < h$.

Rankin-Selberg method at higher genus V

- The regularized Rankin-Selberg transform is obtained by subtracting non-suppressed terms, and yields a 2-loop field theory amplitude, with BPS states running in the loops,

$$\mathcal{R}_h(\Gamma_{d,d,h};s) = \int_{\mathcal{P}_h} \frac{\mathrm{d}\Omega_2}{|\Omega_2|^{h+1-s-\frac{d}{2}}} \sum_{\mathrm{BPS}} e^{-\pi \mathrm{Tr}(M^2\Omega_2)}$$

$$= \Gamma_h(s-\tfrac{h-d}{2}) \sum_{\mathrm{BPS}} \left[\det M^2\right]^{\frac{h+1-d}{2}-s}$$

$$\sum_{\mathrm{BPS}} = \sum_{\substack{(m_i^\alpha, n^{i\beta})\in\mathbb{Z}^{4d},\\ m_i^{(\alpha} n^{i\beta)}=0, \det M^2\neq 0}} , \qquad \Gamma_h(s) = \pi^{\frac{1}{4}h(h-1)} \prod_{k=1}^{h-1} \Gamma(s-\tfrac{k}{2})$$

- The modular integral of $\Gamma_{d,d,h}$ is then proportional to the residue of $\mathcal{R}_h(\Gamma_{d,d,h};s)$ at $s=(h+1)/2$, up to a scheme dependent term δ.

Rankin-Selberg method at higher genus VI

- For $h = d = 2$, either by computing the BPS sum, or by unfolding the Siegel-Narain theta series, one finds

$$\mathcal{R}_2^\star(\Gamma_{2,2}, s) = 2\zeta^\star(2s)\zeta^\star(2s-1)\zeta^\star(2s-2) \times [\mathcal{E}_1^\star(T; 2s-1) + \mathcal{E}_1^\star(U; 2s-1)]$$

hence

$$\mathcal{A}_2 = 2\zeta^\star(2)\,[\mathcal{E}_1^\star(T;2) + \mathcal{E}_1^\star(U;2)]$$

proving the conjecture by Obers and BP (1999).

- For $h = d = 3$,

$$\mathcal{R}_3^\star(\Gamma_{3,3}; s) = \zeta^\star(2s)\,\zeta^\star(2s-1)\,\zeta^\star(2s-2)\,\zeta^\star(2s-3) \left[\mathcal{E}_S^{\star,SO(3,3)}(2s-1) + \mathcal{E}_C^{\star,SO(3,3)}(2s-1)\right]$$

hence

$$\mathcal{A}_3^{d=3} = 2\zeta^\star(2)\zeta^\star(4)\,\left[\mathcal{E}_S^{\star,SO(3,3)}(3) + \mathcal{E}_C^{\star,SO(3,3)}(3)\right]$$

Conclusion - Outlook

- Modular integrals can be efficiently computed using Rankin-Selberg type methods. The result is expressed as a field theory amplitude with BPS states running in the loop.
- T-duality and singularities from enhanced gauge symmetry are manifest. Instanton expansions can be obtained in some cases by solving the BPS constraint.
- The RSZ method also works at higher genus, at least for $g = 2, 3$. For computing modular integrals with $\Phi \neq 1$ it will be important to develop Poincaré series representations for Siegel modular forms with poles at Humbert divisors, such as $1/\Phi_{10}$.
- Non-BPS amplitudes where Φ is not almost weakly holomorphic are challenging !

String Math 2016

- Mark on your calendars:

String Math 2016

Collège de France, Paris

June 27-July 2nd, 2016