Maass forms and mock modular forms in physics

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Introduction

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- Computable examples in QFT and string theory are often restricted to topological field theories, or some protected quantities in supersymmetric QFTs or string vacua. So to a large extent, Physics = Mathematics. Motivations and methods however differ.
- Modular forms usually occur when they have to, meaning when the physical set-up is invariant under $SL(2,\mathbb{Z})$, or more generally some arithmetic group $G(\mathbb{Z})$. This invariance may be manifest, e.g. when the geometry contains a T^2. If not, it may be made so by coining a new name (e.g M-theory, F-theory, (2,0) theory....).

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- Another prime example are two-dimensional CFTs on a torus, or more generally on a Riemann surface of genus h: the partition function has to be invariant under $Sp(2h, \mathbb{Z})$. For $h = 1$ the elliptic genus is usually a holomorphic Jacobi form (but sometimes it can be meromorphic, or have a holomorphic anomaly)

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- Two-dimensional CFTs of course form the basis of perturbative string theory. They also occur on the worldsheet of some more exotic strings, coming from D-branes or M5-branes wrapped on Calabi-Yau manifolds. Hence the relation to black hole counting.

- Certain two-dimensional CFTs also happen to have arithmetic symmetries, e.g. a free boson on a torus T^d is invariant under $O(\Lambda_{d,d})$, known as T-duality. String amplitudes on T^d are then automorphic wrt to $O(\Lambda_{d,d})$ at each loop order.

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- On top of this, there may be arithmetic dualities mixing different loop orders (along with non-perturbative effects), known as U-dualities. For example, type IIB string theory in $D = 10$ is invariant under $SL(2,\mathbb{Z})$. This is not manifest, so call it F-theory on T^2. This is where (non-harmonic) Maass forms made their first appearance in string theory.

Green Gutperle 1997

Outline

1. Mock modular forms in SUSY gauge theories
2. Mock modular forms in two-dimensional SCFTs
3. Mock modular forms and Gromov-Witten invariants
4. Mock modular forms in black hole counting

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Electric-magnetic duality in $D = 4$ gauge theories

- Classically, $\mathcal{N} = 4$ Super-Yang Mills is a generalization of Maxwell electromagnetism where the gauge group $U(1)$ is replaced by a semi-simple compact group G, with scalars + fermions added so as to enforce invariance under the super-Poincaré group with $\mathcal{N} = 4$ spinorial supercharges on $\mathbb{R}^{3,1}$.

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- The functional integral over $\{A_\mu, \phi_I, \psi_\alpha\}$ leads to a conformally invariant QFT with 16 supercharges, depending on $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2}$. Montonen, Olive, Witten and Sen gave compelling evidence that it is invariant under electric-magnetic duality (a.k.a. S-duality)

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} p \\ q \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}, ad - bc = 1$$

for some congruence subgroup $\Gamma \subset SL(2, \mathbb{Z})$ (depending on choice of G). Here, (p, q) are the electric and magnetic charges.

Vafa-Witten invariants and modular forms

- The quantum theory on $\mathbb{R}^{3,1}$ is defined by Wick rotation from $\mathbb{R}^4$. It can be twisted into a topological field theory, which makes sense on any 4-manifold M. and whose functional integral localizes on self-dual configurations,

$$Z_M^G = \sum_{n\geq 1} \chi(\mathcal{M}_n)\, q^{n-s} + \delta Z_M^G$$

  Here $\mathcal{M}_n$ is the moduli space of instantons of charge n on $\mathcal{M}$, and δZ_M^G is a possible contribution from the boundary of $\mathcal{M}_n$.

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- Invariance under S-duality requires that (for suitable choice of s) Z_M^G should be modular with weight $w = -\chi_M/2$ under $\Gamma_0(2N)$.

Vafa-Witten invariants and modular forms

- Example 1: (based on Göttsche)

$$Z_{K3}^{SU(2)} = \frac{1}{8} f(2\tau) + \frac{1}{4} f\left(\frac{\tau}{2}\right) + \frac{1}{4} f\left(\frac{\tau+1}{2}\right) , \quad f(\tau) = 1/\eta^{24}$$

is a weakly holomorphic modular form of weight -12 under $\Gamma_0(2)$.

Vafa-Witten invariants and modular forms

- Example 2: (based on Klyachko, Yoshioka)

$$Z_{\mathbb{P}^2}^{SU(2)} = \frac{3G_0}{\eta^6} + \delta Z_M^G$$

where $G_0 = \sum_{n\geq 0} H(4n) q^n = -\frac{1}{12} + \frac{1}{2}q + \frac{4}{3}q^2 + \dots$ is the generating function of Hurwitz class numbers.

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- As shown by Zagier (1975), G_0 transforms non-homogeneously under $\Gamma_0(4)$,

$$G_0\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{-\frac{3}{2}}\left[G_0(\tau) + \frac{i}{4\pi\sqrt{2}}\int_{-i\infty}^{-d/c} \frac{\theta_3(2u)}{[-i(\tau-u)]^{3/2}} \mathrm{d}u\right]$$

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- The modular anomaly can be cancelled by adding

$$\hat{G}_0 = G_0 + \frac{i}{4\pi\sqrt{2}}\int_{-i\infty}^{\bar{\tau}} \frac{\theta_3(2u)}{[-i(\tau-u)]^{3/2}}\mathrm{d}u \ .$$

Vafa Witten conjectured that δZ^G_M provides the missing piece.

G_0 as an Eisenstein series

- Rk1: $\hat{G}_0$ is essentially an Eisenstein series $E(s \to \frac{3}{4}, w = \frac{3}{2})$

$$E(s, w; \tau) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} \tau_2^{s - \frac{w}{2}} |_w \gamma$$

E converges for $\mathrm{Re}(s) > 1$ and satisfies

$$\begin{aligned} \left(\partial_\tau - \frac{iw}{2\tau_2}\right) E(s, w) &= -\frac{i}{2}(s + \frac{w}{2})\, E(s, w+2) \\ \tau_2^2 \partial_{\bar\tau} E(s, w) &= \frac{i}{2}(s - \frac{w}{2})\, E(s, w-2) \\ \left[\Delta_\tau - (s - \frac{w}{2})(s - 1 + \frac{w}{2})\right] E(s, w) &= 0 \end{aligned}$$

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- $E(s, w)$ is harmonic for $s = \frac{w}{2}$ or $s = 1 - \frac{w}{2}$. For $w = \frac{3}{2}$, $E(s \to \frac{3}{4})$ is a harmonic Maass form, with shadow proportional to the residue of $\mathrm{Res}_{s=\frac{w}{2}} E(s, w-2)$, consistent with $\partial_{\bar\tau} \hat{G}_0 = \tau_2^{3/2} \overline{\theta_3(2\tau)}$.

G_0 as an Appell-Lerch sum

- Rk2: G_0 is essentially an Appell-Lerch sum, or indefinite theta series of signature $(1,1)$:

$$\sum_{N=0}^{\infty} H(N)\, q^N = -\frac{1}{2\theta_3(\tau+\frac12)} \sum_{n\in\mathbb{Z}} \frac{n(-1)^n\, q^{n^2}}{1+q^{2n}} - \frac{1}{12}\theta_3^3(\tau)$$

Vafa-Witten invariants at higher rank

- Using wall-crossing/blow-ups, Manschot (2014) managed to compute Vafa-Witten invariants for $M=\mathbb{P}^2$, $G=SU(N)$ for any N in terms of generalized Appell-Lerch sums of the form

$$\sum_{k\in\mathbb{Z}^r} \frac{q^{\frac{1}{2}Q(k)}\, e^{2\pi \mathrm{i} B(k,v)}}{\prod_{j=1}^{s}(1-e^{2\pi i u_j}\, q^{B(k,m_j)})}$$

for some positive definite quadratic form $Q(k)$ and vectors $m_{j=1..s}\in\mathbb{Z}^r$. Expanding out the denominator, this is an indefinite theta series of signature (r,s) with $s\leq N-1$.

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- The modular completion of indefinite theta series of signature (r, s) involves iterated Eichler integrals of order s, leading to a new class of mock modular forms of higher depth.

Alexandrov Banerjee Manschot BP; Westerholt-Raum; Kudla; Zagier Zwegers...

More modular forms from gauge theories

- Twisted $\mathcal{N} = 2$ SYM with $G = SU(2)$ instead leads to Donaldson invariants of the 4-manifold $\mathcal{M}$. The computation involves a modular integral over the u-plane, and the result is again expressed in terms of Hurwitz class numbers.

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- Generalization to higher rank or other SUSY theories with 8 supercharges may produce new examples of mock modular forms...
- SUSY gauge theories on 3-manifolds (e.g. on knot complements) also lead to topological invariants with modular properties. Ex: $\mathcal{N} = 4$ SYM / complex Chern-Simons. Ramanujan's mock theta functions arise in the context of $\mathcal{N} = 2$ SYM.

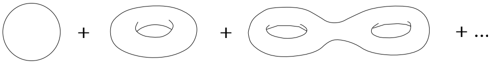
Witten 2010; Gukov Putrov Vafa 2016

Outline

Two-dimensional CFTs in perturbative string theory I

- Two-dimensional SCFTs lie at the basis of pertubative string theory:

$$\mathcal{A}(p_1, \ldots, p_n) = \sum_{h=0}^{\infty} g_s^{2h-2} \int_{\mathfrak{M}_{h,n}} \langle V_1 \ldots V_n \rangle_\Sigma + \mathcal{O}(e^{-1/g_s})$$



where $\mathfrak{M}_{h,n}$ is the moduli space of genus h super-Riemann surfaces Σ with n marked points, and $\langle V_1 \ldots V_n \rangle_\Sigma$ is a correlator in a certain superconformal field theory (SCFT) on Σ, which encodes the background in which the strings propagate.

Elliptic genera of two-dimensional SCFTs

- At each genus, the integral over the locations of the punctures and supermoduli produces a top form $\mathcal{A}_h$ on the moduli space $\mathcal{M}_h$ of genus h curves. For $h \leq 3$, $\mathcal{M}_h$ is isomorphic to a fundamental domain in Siegel's upper half plane $\mathcal{H}_h/Sp(2h,\mathbb{Z})$, and $\mathcal{A}_h$ is a (non-holomorphic) Siegel modular form.

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- The torus amplitude ($h=1$) describes one-loop corrections to classical dynamics ($h=0$). The spectrum of the SCFT is encoded in the vacuum one-loop amplitude ($h=n=0$), which must be invariant under $SL(2,\mathbb{Z})$,

$$\mathcal{A}_1 = \langle 1 \rangle_{T^2} = \mathrm{Tr}\, q^{L_0 - \frac{c_L}{24}}\, \bar{q}^{\bar{L}_0 - \frac{c_R}{24}} \ , \quad q = e^{2\pi \mathrm{i} \tau}$$

Elliptic genera of two-dimensional SCFTs

- Space-time SUSY requires one additional supersymmetry on the worldsheet. The one-loop vacuum amplitude in the (odd,odd) spin structure defines the elliptic genus

$$\mathcal{E}(\tau, z) = \mathrm{Tr}_{RR}(-1)^{(J_0+\bar{J}_0)}\, e^{2\pi i z J_0}\, q^{L_0-\frac{c_L}{24}}\, \bar{q}^{\bar{L}_0-\frac{c_R}{24}}$$

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- For a SCFT with discrete spectrum, e.g. a non-linear sigma model on a compact CY n-fold, all states with $\bar{L}_0 \neq \frac{c_R}{24}$ are paired up, hence $\mathcal{E}(\tau,z)$ is a weakly holomorphic Jacobi form of weight 0, index $n/2$. E.g. for K3,

$$\mathcal{E}_{K3}(\tau,z) = 8\left[\left(\frac{\theta_2(\tau,z)}{\theta_2(\tau,0)}\right)^2 + \left(\frac{\theta_3(\tau,z)}{\theta_3(\tau,0)}\right)^2 + \left(\frac{\theta_4(\tau,z)}{\theta_4(\tau,0)}\right)^2\right]$$

Witten 1988; Kawai Yamada Yang 1993

Elliptic genera of two-dimensional SCFTs

- When the spectrum includes a continuum part, holomorphy in τ may be lost due to a mismatch between the density of states of bosons and fermions. This is well known in the context of SUSY quantum mechanics, where the 'Witten index' $\mathrm{Tr}(-1)^F e^{-\beta H}$ is formally independent of β, but may acquire β dependence due to contributions of the continuum.

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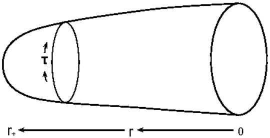
- This phenomenon was observed more recently in non-linear sigma models with non-compact or singular target space (e.g the 'cigar' $SL(2)/U(1)$, or ALE/ALF spaces, or gauged linear sigma models which flow to such spaces).

Troost Ashok, Eguchi Sugawara, Harvey Murthy, ...

Elliptic genera of two-dimensional SCFTs

- In those cases, the elliptic genus can be decomposed as $\mathcal{E} = \mathcal{E}_{\rm disc} + \mathcal{E}_{\rm cont}$, where $\mathcal{E}_{\rm disc}$ is a holomorphic but not modular, while $\mathcal{E}_{\rm cont}$ a non-holomorphic Eichler-type integral such that the sum is modular. E.g.

$$\mathcal{E}_{\frac{SL(2)_k}{U(1)}}(\tau, z) = \frac{\theta_1}{\eta^3} \sum_{m\in\mathbb{Z}} \frac{y^{2m} q^{km^2}}{1 - y^{1/k} q^m}$$



$$- \frac{\theta_1}{2\eta^3} \sum_{r,s\in\mathbb{Z}} y^{\frac{r}{k}-2s} q^{ks^2-rs} \left[\mathrm{sgn}(r+\epsilon) - \mathrm{Erf}\left(r\sqrt{\frac{\pi\tau_2}{k}} \right) \right]$$

Troost 2010

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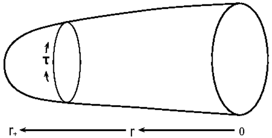

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Troost 2010

- Holomorphic/modular anomalies can also affect couplings in the low-energy effective action, obtained by integrating the elliptic genus over the Poincaré upper half plane.

Carlevaro Israel; Harvey Murthy

Elliptic genera and Mathieu moonshine

- Generalized Appell-Lerch sums also appear in characters of $N = 2$ SCFT and affine Lie superalgebras.

Eguchi Taormina 1988, Semikhatov Taormina Tipunin 2003

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- Due to this, mock modular forms can arise even for SCFTs with a discrete spectrum. E.g. upon decomposing the elliptic genus into $N = 4$ characters of $K3$, one finds

$$\chi_{K3}(\tau, z) = \frac{\theta_1^2(\tau, z)}{\eta^3(\tau)} \left[12\mu(\tau, z) + 2H^{(2)}(\tau) \right]$$

where

$$H^{(2)}(\tau) = q^{-1/8}(-1 + 45q + 231q^2 + 770q^3 + \dots)$$

is a mock modular form of weight $1/2$, with shadow proportional to η^3, whose coefficients are suggestive of an underlying M_{24} symmetry...

Eguchi Ooguri Tachikawa 2010

Outline

Mock modular forms and GW invariants I

- In topological string theory on a compact CY threefold X, the quantum intersection numbers C_{ijk} can be integrated to a holomorphic prepotential, $C_{ijk} = \partial_i \partial_j \partial_k F(t^i)$ given by

$$F(t^i) = \frac{1}{6}\kappa_{ijk} t^i t^j t^k + \frac{1}{2} A_{ij} t^i t^j + B_i t^i + C + \sum_{k>0} n_k \,\mathrm{Li}_3(e^{2\pi i k_i t^i})$$

  where n_k are the genus-zero GW (or BPS, or Gopakumar-Vafa) invariants, counting rational curves in X, while A_{ij}, B_i, C are ambiguous.

- Under monodromies around singularities in Kähler moduli space, C_{ijk} must transform covariantly, but $F(t^i)$ may pick up an additional quadratic polynomial in t^i.

Mock modular forms and GW invariants II

- When X is K3-fibered, the fiber part $F^{(1)}$ can be computed using heterotic/type II duality, in terms of a modular integral (or theta lifting)

$$\int_{SL(2,\mathbb{Z})\backslash\mathcal{H}} \frac{\mathrm{d}\tau\mathrm{d}\bar{\tau}}{\tau_2^2} Z_\Lambda \, D\Phi = \mathrm{Re}\left[\Box F^{(1)}\right]$$

  where Z_Λ is a Siegel-Narain theta series for a lattice Λ of signature $(k+2,2)$, $\Phi(\tau)$ is a weakly holomorphic modular form of weight $-\frac{k}{2}-2$, and $\Box$ is the Maass raising operator on $\frac{O(k+2,2)}{O(k+2)\times O(2)}$.

- For $k=0$, $O(\Lambda) \sim SL(2,\mathbb{Z})_T \times SL(2,\mathbb{Z})_U$, and $F^{(1)}$ transforms with modular weight $(-2,-2)$, up to a period of an Eisenstein series !

Harvey Moore 1995; Antoniadis Ferrara Gava Narain Taylor 1995

Mock modular forms and GW invariants III

- More generally, if $\Phi(\tau)$ is an almost weakly holomorphic modular form of weight $-\frac{k}{2} - 2n$,

$$\int_{SL(2,\mathbb{Z})\backslash\mathcal{H}} \frac{\mathrm{d}\tau \mathrm{d}\bar{\tau}}{\tau_2^2} Z_\Lambda \, D^n \Phi(\tau) = \mathrm{Re}\left[\Box^n F^{(n)}\right]$$

  where $F^{(n)}$ is a holomorphic mock modular form of weight $-2n$ on $O(k+2,2)/O(k+2) \times O(2)$. For $k=1$, $O(3,2) \sim Sp(4)$, so this provides examples of Siegel mock modular forms (including one which underlies the Kawazumi-Zhang invariant) !

Kiritsis Obers 2001; Angelantonj Florakis BP 2015; BP 2016

- Mock modular forms (and more exotic objects) also show up in open topological strings, e.g. on elliptic orbifolds / LG models.

Lau Zhou 2014; Bringmann Rolen Zwegers 2015

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Mock modular forms and black holes

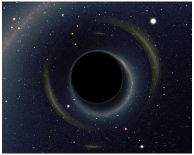
- Superconformal field theories not only arise on the worldsheet of perturbative strings, but also on solitonic strings obtained by wrapping D-branes or M5-branes on supersymmetric cycles. For example, M5/K3 is equivalent to the heterotic string !

Mock modular forms and black holes

- Superconformal field theories not only arise on the worldsheet of perturbative strings, but also on solitonic strings obtained by wrapping D-branes or M5-branes on supersymmetric cycles. For example, M5/K3 is equivalent to the heterotic string !
- Upon further compactification on a circle $S_1(R)$, the solitonic string wound around S_1 leads to a tower of solitonic particles in one dimension lower.

Mock modular forms and black holes

- At strong coupling or large charge, these BPS particles become BPS black holes, and their number is expected to match the Bekenstein-Hawking entropy, $\Omega(\gamma) \sim e^{\mathcal{A}/(4G_N)}$ where $\mathcal{A}$ is the horizon area.



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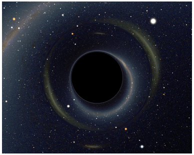
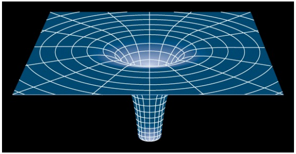


- At weak coupling, these BPS particles are counted (with sign) by the elliptic genus. By the Hardy-Ramanujan-Cardy formula, the number of such states with momentum m around the circle grows like $\Omega(\gamma) \sim e^{2\pi\sqrt{c_L m/6}}$, in agreement with BH entropy.

Strominger Vafa 1996; Maldacena Strominger Witten 1997; ...

Quantum gravity in AdS_3 and Rademacher sums

- The modular invariance underlying BPS black hole degeneracies was later traced to the AdS_3 region near the horizon: after Wick rotation, AdS_3 is a solid torus, and the partition function of gravity in AdS_3 involves a sum over all possible fillings of the torus. This leads to a Poincaré series (or Rademacher sum) representation for the elliptic genus,



$$\mathcal{E}(\tau,z) = \sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} \left[\sum_{n-\Delta<0} a_n\, q^{n-\Delta} \right] |\gamma$$

Dijkgraaf Maldacena Moore Verlinde 2000

Quantum gravity in AdS_3 and Rademacher sums

- For weight $w \leq 2$, such sums diverge and must be regularized, either by truncating the sum, keeping holomorphy (but potentially breaking modular invariance), or by modifying the seed, keeping modular invariance (but breaking holomorphy).

Manschot Moore 2007; Niebur 1973, Hejhal 1983; Bruinier-Funke 2004

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- In either case, the cancellation of modular/holomorphic anomaly puts constraints on the polar coefficients a_n. Demanding modular invariance / holomorphy may however be too strong, since the spectrum of the black string CFT may have a continuous part, especially when wall-crossing can occur...

Niebur-Poincaré series and modular integrals

- Incidentally, such Poincaré series representations are also useful for one-loop computations in perturbative string theory, allowing to compute modular integrals using unfolding methods:

$$\int_{SL(2,\mathbb{Z})\backslash\mathcal{H}} \mathrm{d}\mu_1\, Z_\Lambda \times \left[\sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} f|\gamma \right] = \int_{\Gamma_\infty\backslash\mathcal{H}} \mathrm{d}\mu_1\, Z_\Lambda \times f$$

using Brunier-Funke's seed $f = \mathcal{M}_{s,w}(-\kappa\tau_2)e^{-2\pi i\kappa\tau_1}$.

Angelantonj Florakis BP 2011-16

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- For $X = CY_3$, some progress has been made the problem is still open, both on the math and physics sides (see later).

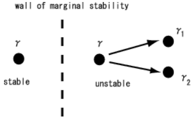
Mock modular forms and black holes

- In type II string compactified on $K3 \times T^2$, or equivalently Het/T^6, exact degeneracies of 1/4-BPS black holes are Fourier coefficients of a certain meromorphic Siegel modular form of genus two:

$$\Omega(\gamma; z) = \int_{\mathcal{C}(z)} \mathrm{d}\rho\, \mathrm{d}v\, \mathrm{d}\sigma\, \frac{e^{\mathrm{i}\pi(\rho Q^2 + \sigma P^2 + v Q\cdot P)}}{\Phi_{10}(\rho, v, \sigma)} \qquad [*]$$

The integration contour $\mathcal{C}(z)$ depends on the moduli z, such that [*] is consistent with the (primitive) wall-crossing formula

$$\Delta\Omega(\gamma_1 + \gamma_2) = (-1)^{\langle\gamma_1,\gamma_2\rangle} \langle\gamma_1, \gamma_2\rangle\, \Omega(\gamma_1)\, \Omega(\gamma_2)$$



across the wall of marginal stability for the decay into a pair of 1/2-BPS states $\gamma \to \gamma_1 \oplus \gamma_2$.

Dijkgraaf Verlinde Verlinde 1995; Cheng Verlinde 2007; Banerjee See Srivastava 2008

Siegel modular forms and $N = 4$ black holes

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- The corresponding coupling receives perturbative corrections up to two-loop on the heterotic side, and is equal non-perturbatively to a genus-two theta lifting of $1/\Phi_{10}$:

$$f_{D^2F^4} = \int_{\mathcal{F}_2} \frac{\mathrm{d}\Omega \mathrm{d}\bar{\Omega}}{|\det \mathrm{Im}\Omega|^3} \frac{Z_{\Lambda_{24,8}}[P_{abcd}]}{\Phi_{10}}$$

Gaiotto 2005; Dabholkar Gaiotto 2006; Bossard Cosnier-Horeau BP 2016

Siegel modular forms and $N = 4$ black holes

- In a suitable chamber, the integral over σ picks up Fourier-Jacobi coefficients

$$\frac{1}{\Phi_{10}(\rho, v, \sigma)} = \sum_{m=-1}^{\infty} \psi_m(\rho, v)\, e^{2\pi i m\sigma}$$

  where $\psi_m(\rho, v)$ is a meromorphic Jacobi form of weight -10, index m, with a double pole at $v = 0$: $\psi_{-1} = 1/(\eta^{18}\theta_1^2), \ldots$

Siegel modular forms and $N = 4$ black holes

- Each ψ_m can be decomposed into $\psi = \psi_m^F + \psi_m^P$ where ψ_m^F is holomorphic in v (hence has the usual theta series decomposition), while ψ_m^P is proportional to an Appell-Lerch sum,

$$\psi_m^P(\rho, v) = \frac{p_{24}(m+1)}{\Delta(\rho)} \sum_{s\in\mathbb{Z}} \frac{q^{ms^2+s}\, e^{2\pi\mathrm{i}(2ms+1)v}}{(1-q^s e^{2\pi\mathrm{i}v})^2}$$

Physically, ψ_m^F counts single-centered black holes, which exist for all values of the moduli, while ψ_m^P counts bound states of two 1/2-BPS black holes.

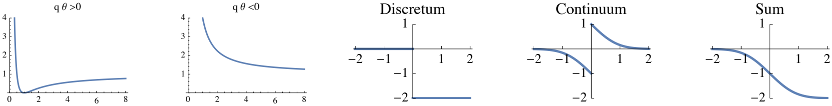
Dabholkar Murthy Zagier 2012

Meromorphic Jacobi forms and mock modular forms

- ψ^F_m and ψ^P_m are not separately modular invariant, but each of them has a non-holomorphic completion. E.g.

$$\widehat{\psi}^P_m(\rho, v) = \frac{p_{24}(m+1)}{\Delta(\rho)} \sum_{s,\ell\in\mathbb{Z}} \left[\ell \left(\operatorname{sgn} s + \operatorname{sgn} \ell\right) + F\left(\ell \frac{\pi\tau_2}{m}\right)\right] \times\ q^{ms^2+\ell s} e^{2\pi\mathrm{i}(2ms+\ell)v}$$

where $F(x) = \operatorname{erfc}(x) + \frac{e^{-x^2}}{\sqrt{\pi}x}$. The non-holomorphic term can be understood as coming from the continuum of scattering states.



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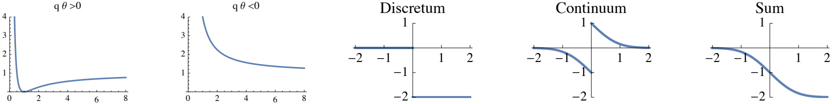

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- The precise physical interpretation of the non-holomorphic completion of the elliptic genus ψ^P_m remains unclear.

Mock modular forms and black holes

- In type II string theory compactified on a genuine CY threefold X, the knowledge about exact BPS indices is much more sparse, in part because the moduli space itself is much more complicated (e.g. it involves Gromov-Witten invariants of X), and because bound states with an arbitrary number of constituents can form.

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- In the sector with vanishing D6-brane charge, the elliptic genus of one M5-brane wrapped on a very ample divisor P is governed by a vector-valued holomorphic modular form $h_{P,\mu}$ of weight $-\frac{b_2+2}{2}$. From the knowledge of BPS indices of a few low-lying states, one can derive an infinite number of DT invariants.

Maldacena Strominger Witten 1997; ...; Gaiotto Strominger Yin 2006

Mock modular forms and black holes

- If $P = \sum_{i=1}^{n} P_i$ is reducible, the SCFT has a continuous spectrum, and $h_{P,\mu}$ no longer need to be modular. When X is elliptically fibered and $P = nB$, the BPS indices are related to $SU(n)$ Vafa-Witten invariants, so mock modularity is expected.

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- Since $IIA/X \times S^1 = M/X \times T^2 = IIB/X \times \widehat{S}_1$, $\mathcal{M}$ must have an isometric action of $SL(2,\mathbb{Z})$. In the one-instanton approximation, valid for very ample divisor, this is tantamount to the modular invariance of the M5-brane elliptic genus !

Mock modular forms and black holes

- At the multi-instanton level, one finds that $h_{\sum_{i=1}^n m_i P_i,\mu}$ is a mock modular form of weight $-\frac{b_2+2}{2}$ and depth $n-1$, with shadow expressed in terms of $h_{\sum_{i=1}^n m_i' P_i,\mu}$ with $m_i' \leq m_i$.

Alexandrov Manschot BP, 2012; A Banerjee M BP 2016-17

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- This relies on a twistorial construction of the QK metric on $\mathcal{M}$, and new insights about indefinite theta series, which were in fact gained by thinking about this physical problem. It would be very useful to have a manifestly modular invariant construction of $\mathcal{M}$, including D6 and KK monopole corrections (or D5-NS5 in the type IIB set-up).

Alexandrov Persson BP; Alexandrov Banerjee

Conclusion and outlook

- Physics provides many examples of almost modular holomorphic quantities. The lack of modular invariance/covariance may be due to the fact that this quantity is only part of the full physical observable (e.g. discrete spectrum contribution to the elliptic genus), while the full observable is modular (but not holomorphic).

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- Or it could be that the physical quantity is intrisically ambiguous, and the lack of modular covariance can be absorbed by a "gauge transformation" (e.g. prepotential in GW theory, or Darboux coordinates in QK geometry).
- In many cases, the modular anomaly can be traced to an indefinite theta series. Ordinary mock modular forms occur when the signature is Lorentzian, but arbitrary signature can occur, leading to mock modular forms of higher 'depth'.

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- Holomorphy or quasi-holomorphy is only expected for quantities protected by supersymmetry. More general observables will involve real-analytic modular forms (e.g. modular graph functions in one-loop string amplitudes), which may still be significant for number theory.

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- $SL(2,\mathbb{Z})$ and its congruence subgroups are the simplest in a long series of arithmetic symmetries occurring in string theory: $Sp(2g,\mathbb{Z})$, automorphism groups of lattices, monodromy groups of CY families, arithmetic subgroups of reductive groups, etc.
- Irrespective from string theory, modular forms and iterated Eichler integrals also seem to be relevant for amplitudes in QFT...

Broadhurst; Brown; Bloch Vanhove; Adams Bogner Weinzerl; ...

Back-up: indefinite theta series of general signature I

Let $Q(k) = B(k,k)$ an (even) integer quadratic form of signature (r,s) on $\Lambda = \mathbb{Z}^n$. For $\Phi : \mathbb{R}^n \to \mathbb{C} \in L_1$, consider the theta series (setting characteristics to zero for simplicity)

$$\Theta_\mu[\Phi](\tau) = \sum_{k \in \Lambda + \mu} \Phi\left(\sqrt{2\tau_2}k\right) q^{\frac{1}{2}Q(k)}$$

- For suitable choices of (C_i, C_i'), $i = 1 \ldots s$, the theta series $\Theta_\mu[\Phi_s]$ with locally constant kernel

$$\Phi(x) = \frac{1}{2^s} \prod_{i=1}^{s} \left(\mathrm{sgn} B(C_i, x) - \mathrm{sgn} B(C_i', x)\right)$$

  converges to a holomorphic function of τ;

Back-up: indefinite theta series of general signature II

- Its modular completion is obtained by expanding out the product, and replacing each product $\prod_{i=1}^{s} \mathrm{sgn} B(C_i, x)$ by the generalized error function

$$E_s(C_1, \ldots, C_s; x) = \int_{\langle C_1, \ldots, C_s \rangle} \mathrm{d}^s y \, e^{-\pi Q(y - x_+)} \prod_{i=1}^{s} \mathrm{sgn} B(C_i, y)$$

  where x_+ is the orthogonal projection on the plane $\langle C_1, \ldots, C_s \rangle$.
- The shadow is an indefinite theta series of signature $(r, s-1)$.
- For $s = 1$, this reduces to Zwegers' trick $\mathrm{sgn}(x) \to \mathrm{erf}(x\sqrt{\pi})$.

Alexandrov Banerjee Manschot BP 2016