Attractor flow trees and scattering diagrams

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with Pierrick Bousseau, Pierre Descombes and Bruno Le Floch

Introduction

- In type IIA string theory compactified on a Calabi-Yau threefold X, the BPS spectrum consists of bound states of D6-D4-D2-D0 branes, with charge $\gamma \in H_{\rm even}(X, \mathbb{Q})$.
- BPS states saturate the bound $M(\gamma) \geq |Z(\gamma)|$, where the central charge $Z \in \mathrm{Hom}(\Gamma, \mathbb{C})$ depends on the complexified Kähler moduli.
- The index $\Omega_z(\gamma)$ counting BPS states is robust under complex structure deformations, but in general depends on $z \in \mathcal{M}_K$.
- Mathematically, the Donaldson-Thomas invariant $\Omega_z(\gamma)$ counts stable objects with ch $E = \gamma$ in the derived category of coherent sheaves $\mathcal{C} = D^b\mathrm{Coh}(X)$.

Introduction

- $\Omega_z(\gamma)$ is locally constant on $\mathcal{M}_K$, but can jump across real codimension one walls of marginal stability $\mathcal{W}(\gamma_L, \gamma_R) \subset \mathcal{M}_K$, where the phases of the central charges $Z(\gamma_L)$ and $Z(\gamma_R)$ with $\gamma = m_L \gamma_L + m_R \gamma_R$ become aligned *[Kontsevich Soibelman'08, Joyce Song'08]*
- Physically, multi-centered black hole solutions with constituent charges $\gamma_i = m_{L,i} \gamma_L + m_{R,i} \gamma_R$ (dis)appear across the wall.

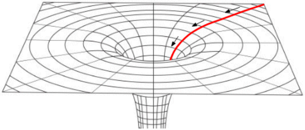
$$\frac{\langle \gamma_L, \gamma_R \rangle}{r} = \frac{2\,\mathrm{Im}[\bar{Z}(\gamma_L)\, Z(\gamma_R)]}{|Z(\gamma_L + \gamma_R)|}, \quad \Delta\Omega(\gamma) = \pm |\langle \gamma_L, \gamma_R \rangle|\, \Omega(\gamma_L) \Omega(\gamma_R)$$

Denef'02, Denef Moore '07, ..., Manschot BP Sen '11

Introduction

- These multi-centered bound states are expected to decay away as one follows the attractor flow equations *[Ferrara Kallosh Strominger'95]*

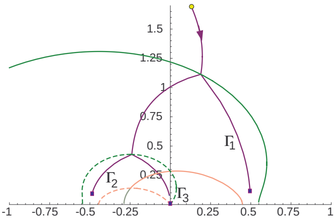
$$\mathrm{AF}_\gamma : \quad r^2 \frac{\mathrm{d}z^a}{\mathrm{d}r} = -g^{a\bar b}\partial_{\bar b}|Z_z(\gamma)|^2$$



- $|Z_z(\gamma)|$ decreases along the flow until it reaches a local minimum at the attractor point $z_\star(\gamma)$, independent of moduli at infinity. We define the attractor invariant as $\Omega_\star(\gamma) = \Omega_{z_\star(\gamma)}(\gamma)$.
- $z_\star(\gamma)$ may be a regular attractor point, corresponding to a spherically symmetric black hole, or a conifold point where $Z_{z_\star(\gamma)}(\gamma) = 0$. For non-compact CY3, only the second option is allowed.

The Split Attractor Flow Conjecture

- Starting from $z \in \mathcal{M}_K$, following AF_γ and recursively applying the WCF formula at whenever the flow crosses a wall of marginal stability, one can in principle express $\Omega_z(\gamma)$ in terms of attractor invariants.



Denef Moore'07

The Split Attractor Flow Conjecture (SFAC)

- In terms of the rational DT invariants

$$\bar{\Omega}_z(\gamma) := \sum_{k|\gamma} \frac{1}{k^2} \Omega_z(\gamma/k)$$

  the result takes the form

$$\bar{\Omega}_z(\gamma) = \sum_{\gamma=\sum \gamma_i} \frac{g_z(\{\gamma_i\})}{\mathrm{Aut}(\{\gamma_i\})} \prod_i \bar{\Omega}_\star(\gamma_i)$$

  where $g_z(\{\gamma_i\})$ is a sum over attractor flow trees.

- The Split Attractor Flow Conjecture *[Denef'00, Denef Moore'07]* is the statement that only a finite number of decompositions $\gamma = \sum \gamma_i$ contribute to the index $\bar{\Omega}_z(\gamma)$.

The Split Attractor Flow Conjecture

- Unfortunately it is not known a priori which constituents γ_i can contribute, except for the obvious constraints

$$\sum_i \gamma_i = \gamma \,, \quad \sum_i |Z_{z_\star(\gamma_i)}(\gamma_i)| < |Z_z(\gamma)|$$

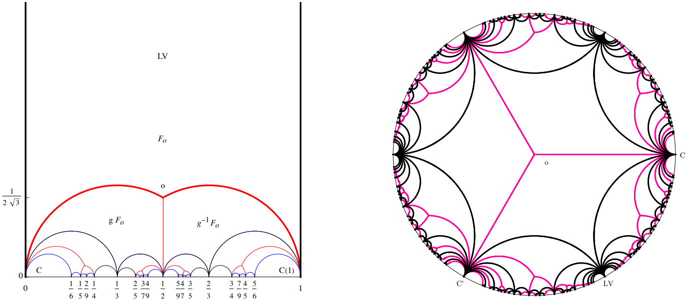
- In particular, there can be cancellations between D-branes and anti-D-branes, and contributions from conifold states which are massless at their attractor point are difficult to bound.
- Even if SAFC holds, one still has to compute the attractor indices $\Omega_\star(\gamma)$, a tall order for regular attractor points.
- Our aim is to investigate the SAFC for one of the simplest examples of CY threefolds, $X = K_{\mathbb{P}^2} = \widetilde{\mathbb{C}^3/\mathbb{Z}_3}$, revisiting the analysis of *[Douglas Fiol Romelsberger'00]*.

Summary

- We show that the only possible constituents are the D4-brane $\mathcal{O}_{\mathbb{P}^2}$, the anti-D4-brane $\mathcal{O}_{\mathbb{P}^2}[1]$, and their images thereof under $\Gamma_1(3)$, each carrying attractor index $\Omega_\star(\gamma) = 1$.
- In the vicinity of the orbifold point, the only populated states are bound states of the fractional branes $\mathcal{O}[-1], \Omega(1), \mathcal{O}(-1)[1]$.
- Instead, the full BPS spectrum at large volume arises as bound states of fluxed D4 and anti-D4-branes $\mathcal{O}(m), \mathcal{O}(m)[1]$, with effective bounds on the number and flux of the constituents.
- A key role is played by scattering diagrams, which provide the correct mathematical framework for the SAFC, at least for local CY threefolds.

Kähler moduli space

- The Kähler moduli space of $X = K_{\mathbb{P}^2}$ is the modular curve $X_1(3) = \mathbb{H}/\Gamma_1(3)$. It admits two cusps LV, C and one orbifold point o of order 3.



- A BPS state on X is a stable object E in the bounded derived category $\mathcal{C}$ of compactly supported sheaves on X, with charge $\gamma(E) = [r, d, \mathrm{ch}_2] \sim [D4, D2, D0]$

Central charge as Eichler integral

- The central charge $Z_\tau(\gamma)$ is a linear combination

$$Z_\tau(\gamma) = -rT_D(\tau) + dT(\tau) - 1 \cdot \mathrm{ch}_2$$

where T_D, T are multi-valued holomorphic functions on $\mathcal{M}_K$, single valued on the universal cover $\mathbb{H}$, satisfying a third order Picard-Fuchs equation.

- While T, T_D can be expressed in terms of Meijer G-functions, it is more efficient to represent them as Eichler-type integrals,

$$\boxed{\begin{pmatrix} T \\ T_D \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/3 \end{pmatrix} + \int_{\tau_o}^{\tau} \begin{pmatrix} 1 \\ \rho \end{pmatrix} C(\rho)\, \mathrm{d}\rho}$$

where $C(\tau) = \frac{\eta(\tau)^9}{\eta(3\tau)^3} = 1 - 9q + 27q^2 + \dots$ is a weight 3 Eisenstein series for $\Gamma_1(3)$.

Central charge as Eichler integral

- This provides an computationally efficient analytic continuation of Z_τ throughout $\mathbb{H}$, and gives access to monodromies:

$$\tau \mapsto \frac{a\tau + b}{c\tau + d} \quad \begin{pmatrix} 1 \\ T \\ T_D \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ m & d & c \\ m_D & b & a \end{pmatrix} \cdot \begin{pmatrix} 1 \\ T \\ T_D \end{pmatrix}$$

  where (m, m_D) are period integrals of C from τ_o to $\frac{d\tau_o - b}{a - c\tau_o}$.
- At large volume $\tau \to \mathrm{i}\infty$, using $C = 1 + \mathcal{O}(q)$ one finds

$$T = \tau + \mathcal{O}(q), \quad T_D = \frac{1}{2}\tau^2 + \frac{1}{8} + \mathcal{O}(q)$$

  in agreement with $Z_\tau(\gamma) \sim -\int_S e^{-\tau H} \sqrt{\mathrm{Td}(S)}\, \mathrm{ch}(E)$.

Space of Bridgeland stability conditions

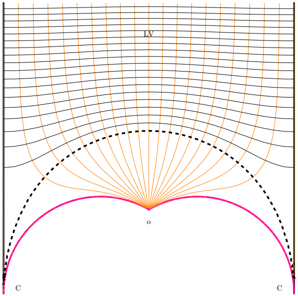
- Donaldson-Thomas invariants are defined in the larger space of Bridgeland stability conditions $\mathrm{Stab}\,\mathcal{C} = \{\sigma = (Z, \mathcal{A})\}$, where $Z : \Gamma \to \mathbb{C}$ is a linear map and $\mathcal{A} \subset \mathcal{C}$ an Abelian sub category locally determined by Z. In particular, $\dim_{\mathbb{C}} \mathrm{Stab}\,\mathcal{C} = \dim \Gamma = 3$.
- $G = \widetilde{GL(2,\mathbb{R})}^+$ acts on $\mathrm{Stab}\,\mathcal{C}$ by $\begin{pmatrix}\mathrm{Re}Z\\ \mathrm{Im}Z\end{pmatrix} \mapsto \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}\mathrm{Re}Z\\ \mathrm{Im}Z\end{pmatrix}$, leaving $\Omega_\sigma(\gamma)$ invariant. Using $\mathbb{C}^\times \subset G$, one can always set $Z([D0]) = -1$.
- The physical moduli space is a particular one-dimensional slice $(Z_\tau, \mathcal{A}_\tau)$ inside $\mathrm{Stab}\,\mathcal{C}$, known as Π-stability. Another natural slice is the large volume slice with central charge

$$Z^{LV}_\rho(\gamma) = -r\frac{\rho^2}{2} + d\rho - \mathrm{ch}_2 \,, \quad \rho = s + \mathrm{i}t$$

- For $\mathrm{Im}\tau$ large enough, the physical and large volume slices are related by the action of G.

Space of Bridgeland stability conditions

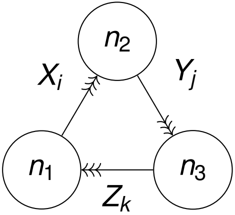
- Specifically, this holds in the region $w > \frac{1}{2}s^2$ where $(s, w) := \left(\frac{\mathrm{Im}\, T_D}{\mathrm{Im}\, T}, -\frac{\mathrm{Im}(T \bar{T}_D)}{\mathrm{Im}\, T}\right) \simeq (\tau_1, \frac{1}{2}|\tau|^2)$ above dashed line:



- The large volume slice does not cover the region around the orbifold point, and covers only part of the conifold point.

BPS Spectrum around the orbifold point

- The category $D^b\,\mathrm{Coh}_c(K_{\mathbb{P}^2})$ is isomorphic to the category of representations of a quiver with potential (Q, W), whose nodes correspond to fractional branes on $\mathbb{C}^3/\mathbb{Z}_3$ *[Douglas Fiol Romelsberger'00]*



$$W = \sum \epsilon_{ijk} X_i Y_j Z_k$$

$$\begin{aligned} E_1 &= \mathcal{O}[-1], & \gamma_1 &= [-1, 0, 0] \\ E_2 &= \Omega(1), & \gamma_2 &= [2, -1, -\tfrac{1}{2}] \\ E_3 &= \mathcal{O}(-1)[1], & \gamma_3 &= [-1, 1, -\tfrac{1}{2}] \end{aligned}$$

$$\begin{aligned} r &= 2n_2 - n_1 - n_3 \\ d &= n_3 - n_2 \\ \mathrm{ch}_2 &= -\tfrac{1}{2}(n_2 + n_3) \end{aligned}$$

- The quiver description is valid in a region where the central charges $Z(E_i)$ lie in a common half-plane. This includes the vicinity of the orbifold point, where $Z_{\tau_o}(\gamma_i) = 1/3$ for $i = 1, 2, 3$.

Attractor indices for quivers

- In that region, $\Omega_\tau(\gamma)$ coincides with the quiver index $\Omega_\theta(\gamma)$ counting θ-semi-stable representations of dimension vector γ, for suitable FI parameters $\theta(\tau) \in \mathbb{R}^{Q_0}$.
- Recall that a representation of dimension vector γ is θ-semi-stable iff $(\theta, \gamma') \leq (\theta, \gamma)$ for any subrepresentation. Specifically,

$$\theta_i = -\mathrm{Re}(e^{-\mathrm{i}\psi} Z(\gamma_i)) \quad \text{with} \quad \mathrm{Im}(e^{-\mathrm{i}\psi} Z(\gamma_i)) > 0 \;\forall i$$

- In the quiver context, the attractor point (aka self-stability condition) is $\theta_\star(\gamma)$ such that *[Manschot BP Sen'13; Bridgeland'16]*

$$\forall \gamma', \quad (\theta_\star(\gamma), \gamma') = \langle \gamma', \gamma \rangle := \sum_{a:i\to j} (n'_i n_j - n'_j n_i)$$

and the (quiver) attractor invariant is defined as $\Omega_\star(\gamma) := \Omega_{\theta_\star(\gamma)}(\gamma)$

The Flow Tree formula for quivers

- In *[Alexandrov BP'18]*, we conjectured a precise version of SAFC which expresses $\bar{\Omega}_\theta(\gamma)$ in terms of the attractor invariants:

$$\bar{\Omega}_\theta(\gamma) = \sum_{\gamma=\sum \gamma_i} \frac{g_\theta(\{\gamma_i\})}{\mathrm{Aut}(\{\gamma_i\})} \prod_i \bar{\Omega}_\star(\gamma_i)$$

  The coefficients $g_\theta(\{\gamma_i\})$ involve a sum over rooted binary trees, whose edges are embedded in FI-space along straight lines $\theta_0 + \lambda\theta_\star(\gamma_e)$, which are the analogue of attractor flows.
- The sum is manifestly finite, since γ_i lie in the positive cone $\mathbb{N}^{Q_0}$.
- The formula was proven mathematically in *[Argüz Bousseau'21]* using the formalism of scattering diagrams. See also Mozgovoy's proof using operads.

Scattering diagrams in a nutshell

- For any quiver with potential (Q, W), the scattering diagram $\mathcal{D}_Q$ is the set of real codimension-one rays $\{\mathcal{R}(\gamma), \gamma \in \mathbb{Z}^{Q_0}\}$ defined by

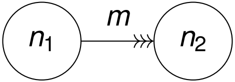
$$\mathcal{R}(\gamma) = \{\theta \in \mathbb{R}^{Q_0} : (\theta, \gamma) = 0,\ \bar{\Omega}_\theta(\gamma) \neq 0\}$$

- Each point along $\mathcal{R}(\gamma)$ is endowed with an automorphism of the quantum torus algebra generated by $\mathcal{X}_\gamma \mathcal{X}_{\gamma'} = (-y)^{\langle \gamma, \gamma' \rangle} \mathcal{X}_{\gamma+\gamma'}$,

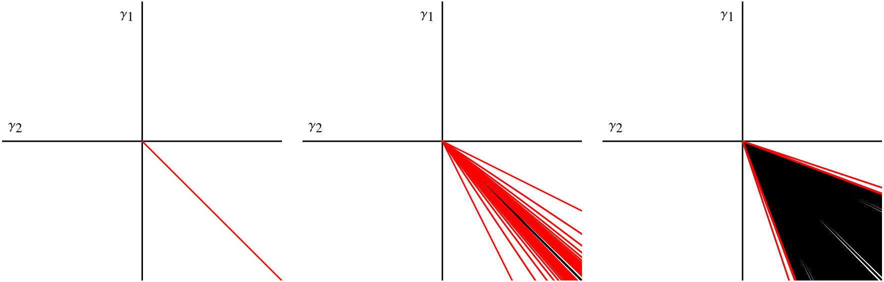
$$\mathcal{U}_\theta(\gamma) = \exp\left(\frac{\bar{\Omega}_\theta(\gamma)}{y^{-1} - y} \mathcal{X}_\gamma\right) = \mathrm{Exp}\left(\frac{\Omega_\theta(\gamma)}{y^{-1} - y} \mathcal{X}_\gamma\right)$$

- The WCF ensures that the diagram is consistent: for any generic closed path $\mathcal{P} : t \in [0, 1] \in \mathbb{R}^{Q_0}$, $\prod_i \mathcal{U}_{\theta(t_i)}(\gamma_i)^{\epsilon_i} = 1$ *[Bridgeland'16]*

Scattering diagram for Kronecker quiver



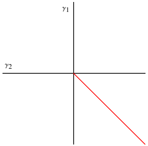
$$\theta_1 > 0, \theta_2 < 0 : \quad \dim \mathcal{M}_\theta(\gamma) = m n_1 n_2 - n_1^2 - n_2^2 + 1$$



m=1 m=2 m=3

Consistent scattering diagrams

- At each intersection, outgoing rays (and corresponding DT invariants) are determined from incoming rays by the consistency condition. E.g. for $K_1 = A_2$, this is the famous five-term relation

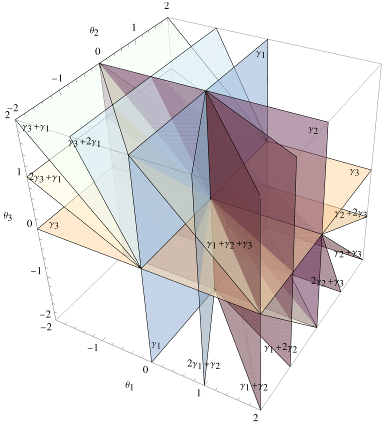


$$\mathcal{U}_{\gamma_1}\mathcal{U}_{\gamma_2} = \mathcal{U}_{\gamma_2}\mathcal{U}_{\gamma_1+\gamma_2}\mathcal{U}_{\gamma_1}$$

- A consistent scattering diagram is uniquely determined from the initial rays $\mathcal{R}_\star(\gamma)$, defined as those which contain $\theta_\star(\gamma)$.
- The Flow Tree Formula of *[Alexandrov BP'18]* determines the indices of outgoing rays produced by scattering initial rays *[Argüz Bousseau '20]* (see also the operadic approach of *[Mozgovoy'19]*)

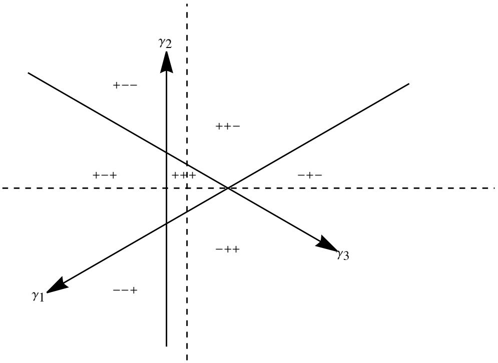
Attractor invariants for $K_{\mathbb{P}^2}$

- In *[Beaujard BP Manschot'20]*, we conjectured that the attractor indices $\Omega_\star(\gamma)$ vanish except for $\gamma = \gamma_i$ or $\gamma = k(\gamma_1 + \gamma_2 + \gamma_3) = k[D0]$. This is now a theorem *[Bousseau Descombes Le Floch BP'22]*.



A 2D slice of the orbifold scattering diagram

Let $\mathcal{D}_o$ be the restriction of $\mathcal{D}_Q$ to the hyperplane $\theta_1 + \theta_2 + \theta_3 = 1$:



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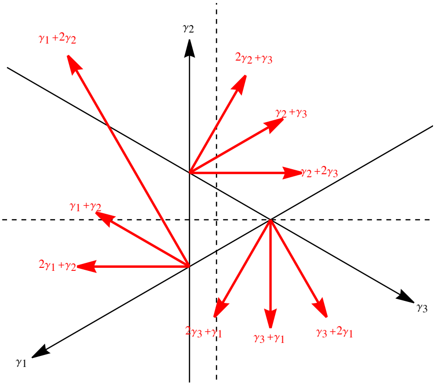

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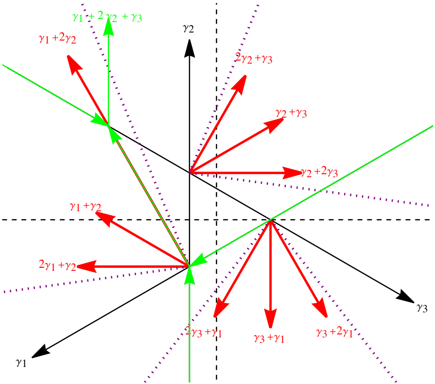
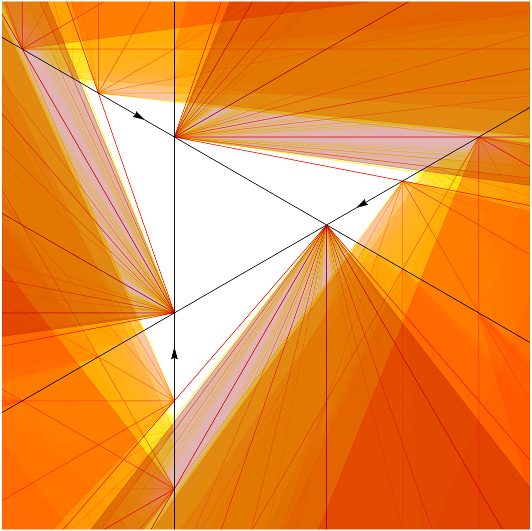


A 2D slice of the orbifold scattering diagram

The full scattering diagram $\mathcal{D}_Q$ includes regions with dense set of rays:



Scattering diagrams on triangulated categories

- For a general triangulated category $\mathcal{C}$, define the scattering diagram $\mathcal{D}_\psi(\mathcal{C})$ as the set of codimension-one loci in Stab $\mathcal{C}$,

$$\mathcal{R}_\psi(\gamma) = \{\sigma : \arg Z(\gamma) = \psi + \tfrac{\pi}{2},\ \bar{\Omega}_Z(\gamma) \neq 0\}$$

equipped with (a suitable regularization of) the automorphism

$$\mathcal{U}_\sigma(\gamma) = \exp\left(\frac{\bar{\Omega}_\sigma(\gamma)}{y^{-1}-y}\mathcal{X}_\gamma\right) = \mathrm{Exp}\left(\frac{\Omega_\sigma(\gamma)}{y^{-1}-y}\mathcal{X}_\gamma\right)$$

- The WCF ensures that the diagram $\mathcal{D}_\psi$ is still locally consistent at each codimension-two intersection.

Flow trees from scattering diagrams

- To see the relation to SAFC, note that for any local CY threefold, the central charge $Z_z(\gamma)$ is holomorphic in z^a, hence its phase is constant along the flow $\frac{\mathrm{d}z^a}{\mathrm{d}\mu} = -g^{a\bar{b}}\partial_{\bar{b}}|Z_z(\gamma)|^2$:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\mu}\log\frac{Z(\gamma)}{\bar{Z}(\gamma)} = -\frac{1}{2}\partial_a Z(\gamma) g^{a\bar{b}}\partial_{\bar{b}}\bar{Z}(\gamma) + \frac{1}{2}\partial_a Z(\gamma) g^{a\bar{b}}\partial_{\bar{b}}\bar{Z}(\gamma) = 0$$

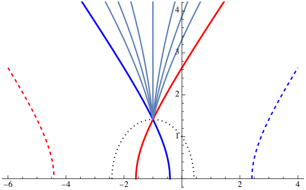
  thus the attractor flow takes place along the ray $\mathcal{R}_\psi(\gamma)$, and can only split when $\mathcal{R}(\gamma_L)$ and $\mathcal{R}(\gamma_R)$ intersect.
- Moreover, by holomorphy $|Z_z(\gamma)|^2$ has no local minima so the only attractor points are conifold points with $Z_z(\gamma_i) = 0$.
- In complex dimension one, attractor flow lines $\simeq$ scattering rays ! Attractor flow trees are subsets of $\mathcal{D}_\psi$ which produce an outgoing ray $\mathcal{R}_\psi(\gamma)$ passing through the desired point z.

Large volume scattering diagram

- The scattering diagram $\mathcal{D}_\psi^{\rm LV}$ along the large volume slice

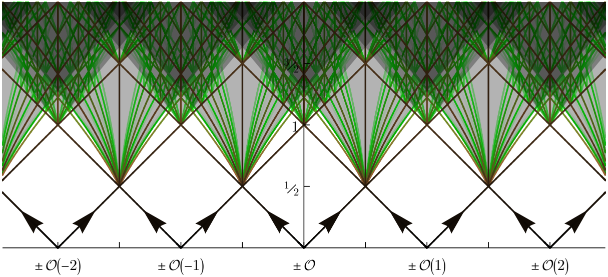
$$Z_\rho^{LV}(\gamma) = -\frac{1}{2} r\rho^2 + d\rho - \mathrm{ch}_2 \ , \quad \rho = s + \mathrm{i} t$$

  was determined for $\psi = 0$ in *[Bousseau'19]*. Other values of ψ are reached by mapping $(s,t) \mapsto (s - t\tan\psi, t/\cos\psi)$.
- Each ray $\mathcal{R}_0(\gamma)$ is a branch of hyperbola asymptoting to $t = \pm(s - \frac{d}{r})$ for $r \neq 0$, or a vertical line when $r = 0$. Walls of marginal stability $\mathcal{W}(\gamma,\gamma')$ are half-circles centered on real axis.



Large volume scattering diagram

- Initial rays correspond to $\mathcal{O}(m)$ and $\mathcal{O}(m)[1]$, with charge $\gamma_m = \pm[1, m, \frac{1}{2}m^2]$, emanating from $(s,t) = (m,0)$ on the boundary where $Z^{LV}_\rho(\gamma_m) = 0$ *[Bousseau'19]*



- Physically, the BPS spectrum along the large volume slice originates from bound states of fluxed D4-branes and anti-D4 branes.

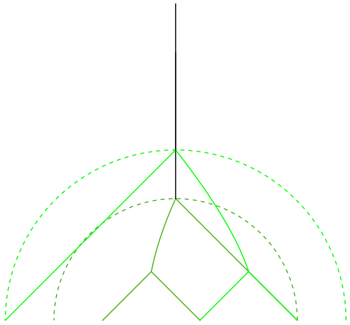
SAFC holds along large volume slice

- Rays stay inside the 'forward light-cone', and $\varphi_s(\gamma) = 2(d - sr) = 2\mathrm{Im}Z_\gamma/t$ increases along the ray.
- The first scatterings occur after a time $t \geq \frac{1}{2}$, after each constituent $k_i\mathcal{O}(m_i)$ has moved by $|\Delta s| \geq \frac{1}{2}$, by which time $\varphi_s(\gamma_i) \geq |k_i|$.
- Since $\varphi_s(\gamma)$ is additive at each vertex, this gives a bound on the number and charges of constituents contributing to $\Omega_{(s,t)}(\gamma)$:

$$\sum_i k_i[1, m_i, \frac{1}{2}m_i^2] = \gamma\ , \quad s - t \leq m_i \leq s + t, \quad \sum |k_i| \leq \varphi_s(\gamma)$$

- Thus, SAFC holds along the large volume slice !

Example: Flow trees for $\gamma = [0, 4, 1)$



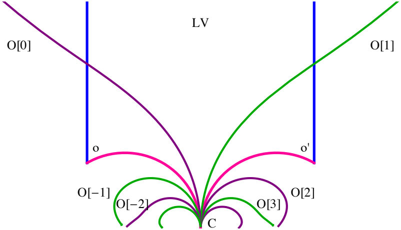
- $\{\{-3\mathcal{O}(-2), 2\mathcal{O}(-1)\}, \mathcal{O}\}$:
  $3\mathcal{O}(-2) \to 2\mathcal{O}(-1) \oplus \mathcal{O} \to E$
  $\Omega_1 = K_3(2,3) K_{12}(1,1) \to -156$

- $\{-\mathcal{O}(-3), \{-\mathcal{O}(-1), 2\mathcal{O}\}\}$:
  $\mathcal{O}(-3) \oplus \mathcal{O}(-1) \to 2\mathcal{O} \to E$
  $\Omega_2 = K_3(1,2) K_{12}(1,1) \to -36$

Total: $\Omega_\infty(\gamma) = -192 = GV_4^{(0)}$

Exact scattering diagram

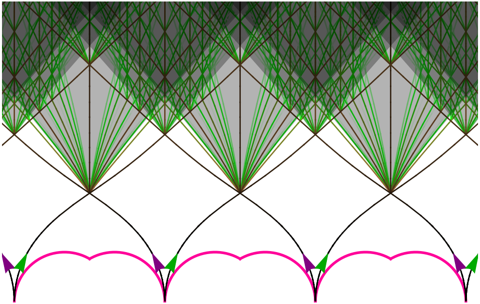
- The scattering diagram $\mathcal{D}^{\Pi}_{\psi}$ along the physical slice should interpolate between $\mathcal{D}^{\rm LV}_{\psi}$ and $\mathcal{D}_o$, and be invariant under $\Gamma_1(3)$.
- Under $\tau \mapsto \frac{\tau}{3n\tau+1}$ with $n \in \mathbb{Z}$, $\mathcal{O} \mapsto \mathcal{O}[n]$. Hence there is a doubly infinite family of initial rays emitted at $\tau = 0$, associated to $\mathcal{O}[n]$:



- Similarly, there must be an infinite family of rays emitted from $\tau = \frac{p}{q}$ with $q \neq 0 \mod 3$, corresponding to $\Gamma_1(3)$-images of $\mathcal{O}$.

Exact scattering diagram for small ψ

- For $|\psi|$ small enough, the only rays which reach the large volume region are those associated to $\mathcal{O}(m)$ and $\mathcal{O}(m)[1]$. Thus, the scattering diagram $\mathcal{D}_\psi^\Pi$ is isomorphic to $\mathcal{D}_0^{\rm LV}$ inside $\cup_n \mathcal{F}(n)$:



Scattering diagram in affine coordinates

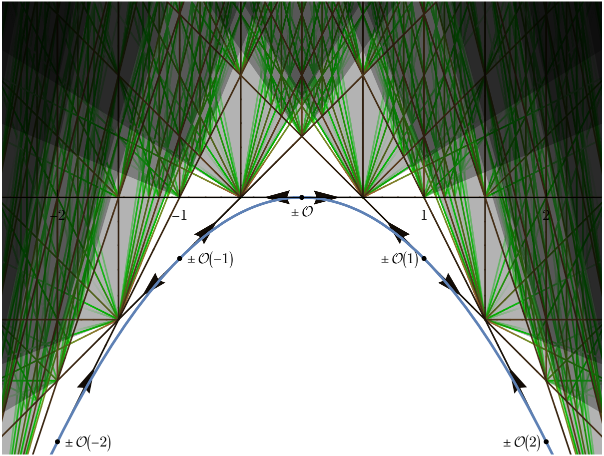
- To see this, one can map both of them to the (x, y)-plane

$$x = \frac{\mathrm{Re}\left(e^{-\mathrm{i}\psi} T\right)}{\cos\psi}\,, \quad y = -\frac{\mathrm{Re}\left(e^{-\mathrm{i}\psi} T_D\right)}{\cos\psi}$$

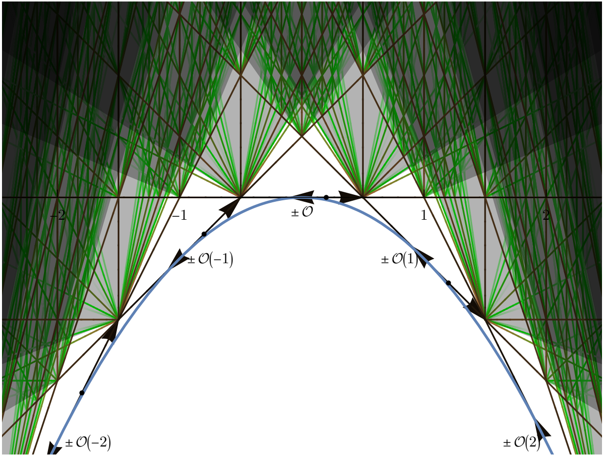
  such that $\mathcal{R}_\psi(\gamma)$ becomes a line segment $rx + dy - \mathrm{ch}_2 = 0$.
- The initial rays $\mathcal{R}_\psi(\mathcal{O}(m))$ are tangent to the parabola $y = -\frac{1}{2}x^2$ at $x = m$, but the origin of each ray is shifted to $x = m + \mathcal{V}\tan\psi$ where $\mathcal{V}$ is the quantum volume

$$\mathcal{V} = \mathrm{Im}\, T(0) = \frac{27}{4\pi^2}\mathrm{Im}\left[\mathrm{Li}_2(e^{2\pi \mathrm{i}/3})\right] \simeq 0.463$$

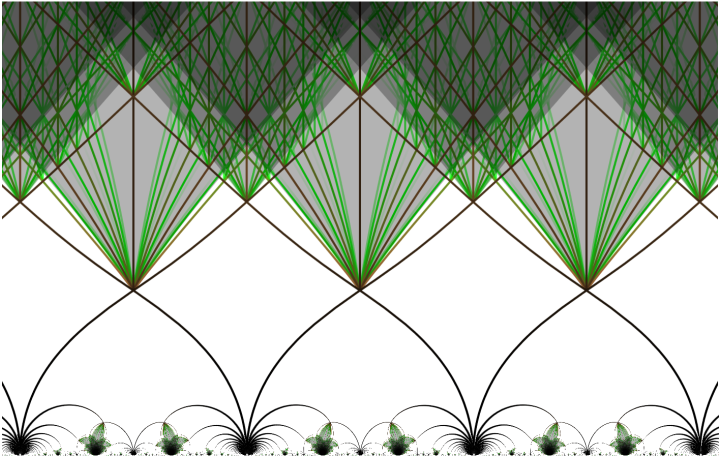
Affine scattering diagram, $\psi = 0$



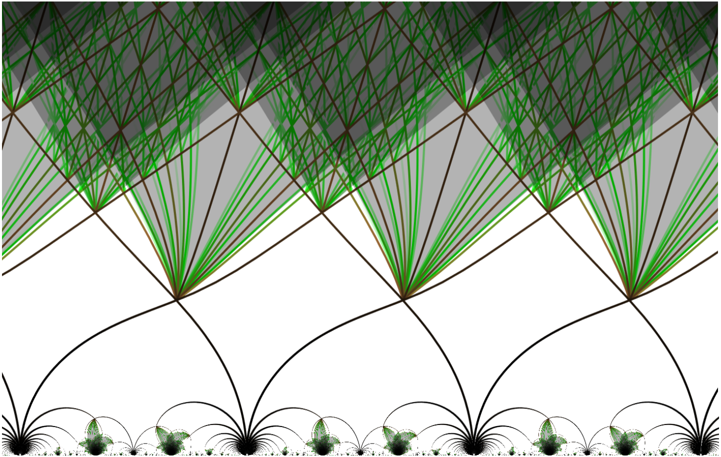
Affine scattering diagram, $|\mathcal{V} \tan \psi| < 1/2$



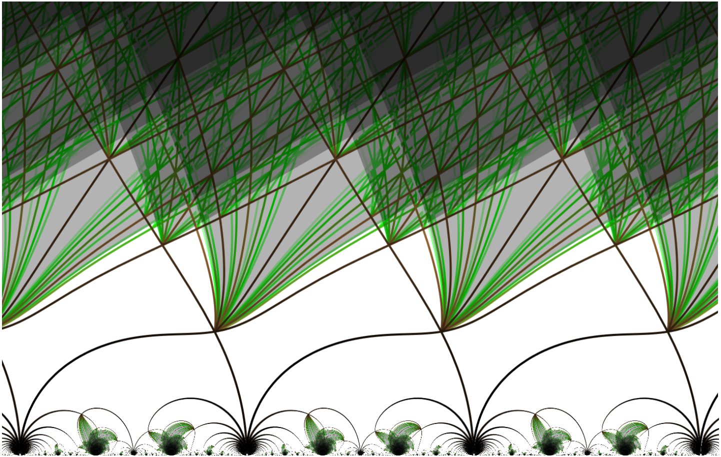
Exact scattering diagram, $\psi = 0$



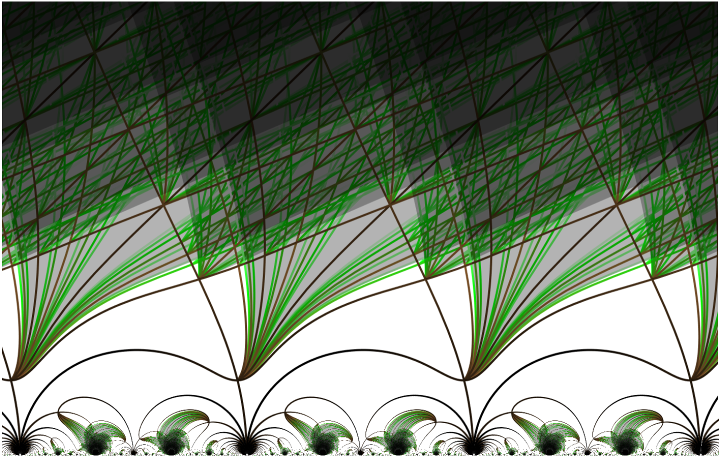
Exact scattering diagram, $\psi = 0.3$



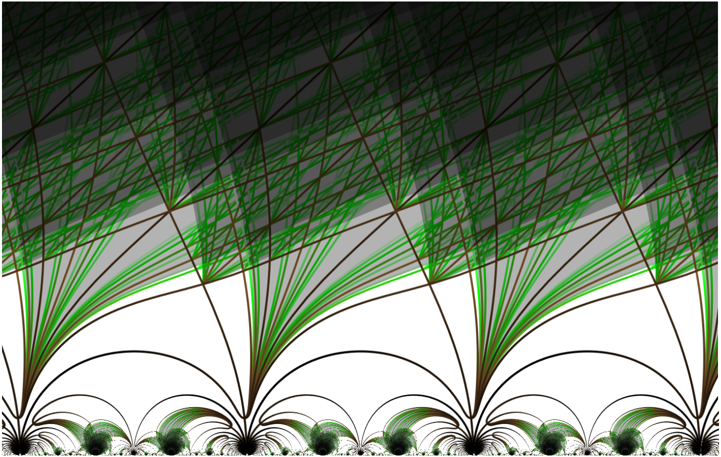
Exact scattering diagram, $\psi = 0.6$



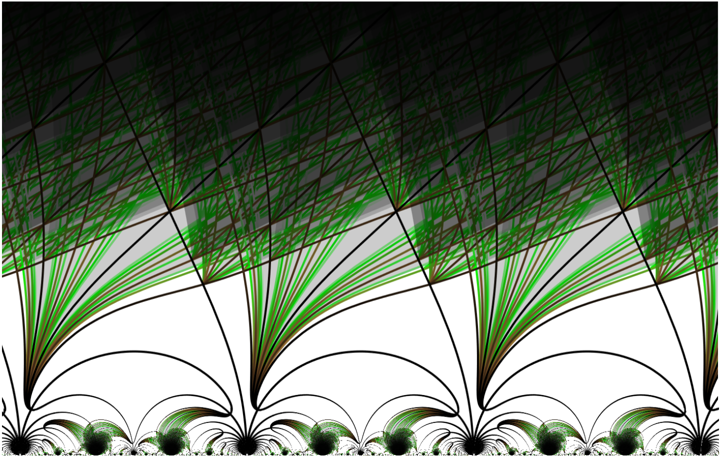
Exact scattering diagram, $\psi = 0.8$



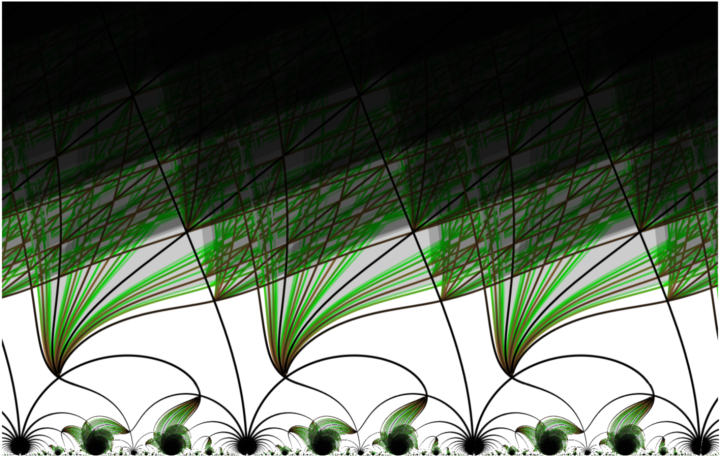
Exact scattering diagram, $\psi = 0.824$



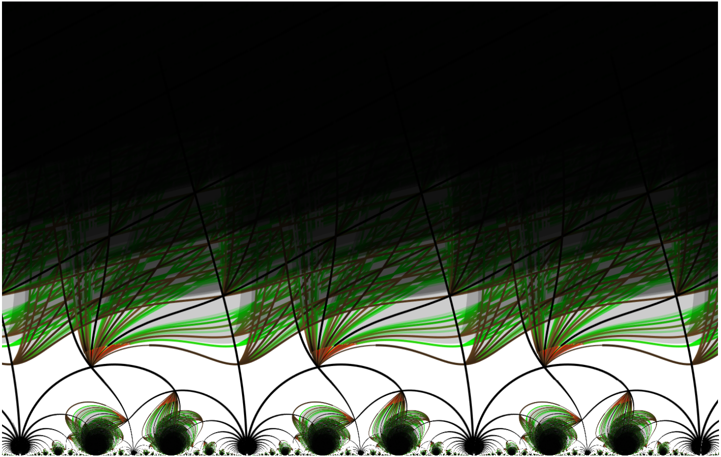
Exact scattering diagram, $\psi = 0.825$



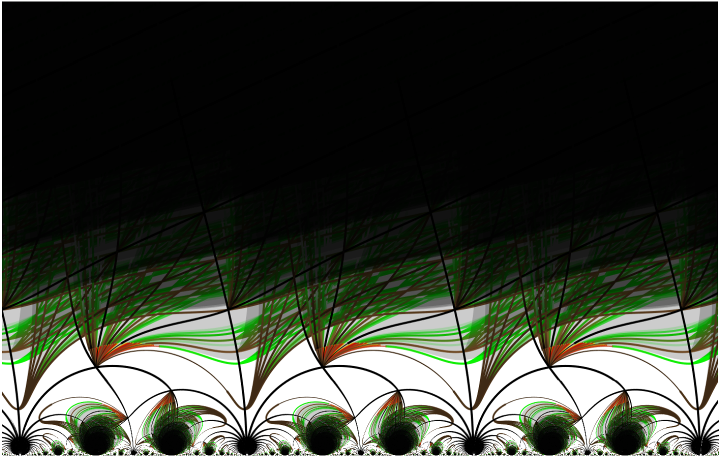
Exact scattering diagram, $\psi = 0.9$



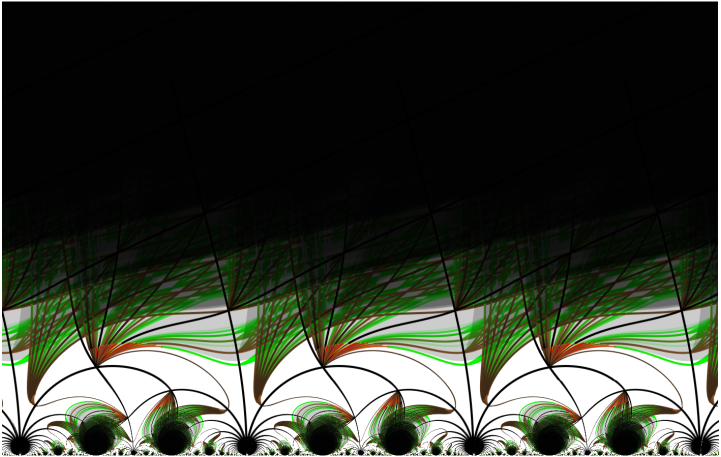
Exact scattering diagram, $\psi = 1.1$



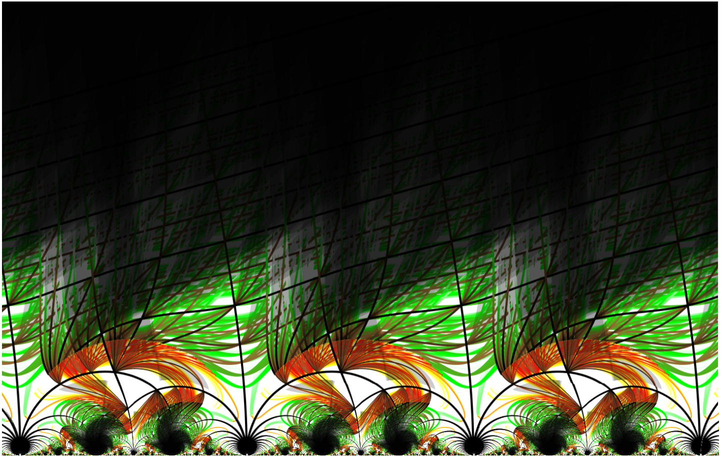
Exact scattering diagram, $\psi = 1.137$



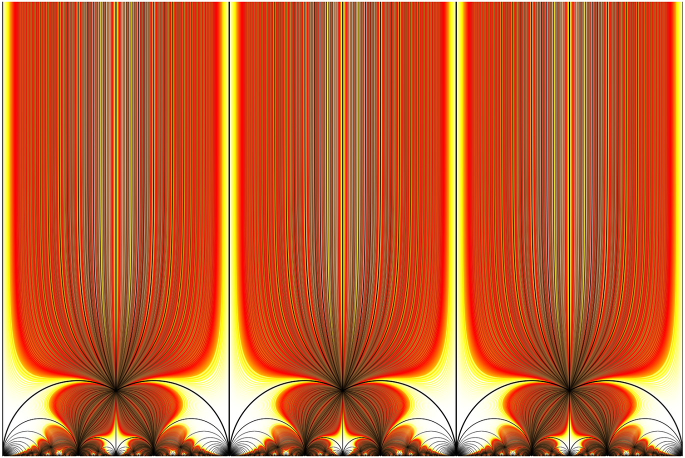
Exact scattering diagram, $\psi = 1.139$



Exact scattering diagram, $\psi = 1.3$

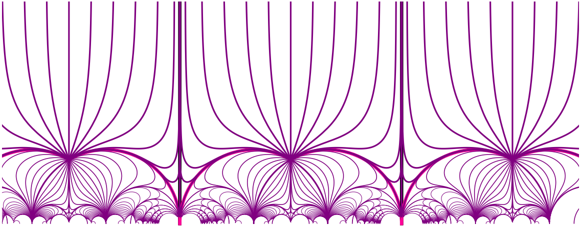


Exact scattering diagram, $\psi = \pi/2$



Exact scattering diagram for $\psi = \pm\frac{\pi}{2}$

- For $\psi = \pm\frac{\pi}{2}$, the geometric rays $\{\mathrm{Im} Z_\tau(\gamma) = 0\}$ coincide with lines of constant $s = \frac{\mathrm{Im}\, T_D}{\mathrm{Im}\, T} = \frac{d}{r}$, independent of ch_2:



- Hence, there is no wall-crossing between τ_o and $\tau = \mathrm{i}\infty$ when $-1 \leq \frac{d}{r} \leq 0$, explaining why the Gieseker index $\Omega_\infty(\gamma)$ agrees with the quiver index $\Omega_c(\gamma)$ in the anti-attractor chamber.

Douglas Fiol Romelsberger'00, Beaujard BP Manschot'20

Conclusion - outlook

- Scattering diagrams are the appropriate mathematical framework for attractor flow trees in the case of local CY3. This is because $Z(\gamma)$ is holomorphic on $\mathcal{M}_K$, so the gradient flow preserves the phase of $Z(\gamma)$.
- This provides an effective way of computing BPS invariants in any chamber, and a natural decomposition into elementary constituents. Does it help e.g. in understanding modularity ?
- It will be interesting to extend this description to other toric CY3, such as local del Pezzo surfaces. *[Le Floch BP Schimannek, in progress]*
- For compact CY3, $Z(\gamma) = e^{K/2} Z_{\rm hol}(\gamma)$ is not longer holomorphic, so $\arg Z(\gamma)$ is not constant along the flow. Can one still use scattering diagrams to construct the BPS spectrum ?

Thanks for your attention !