BPS Dendroscopy on Local Projective Plane

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"Intersections of Strings and QFT IV"
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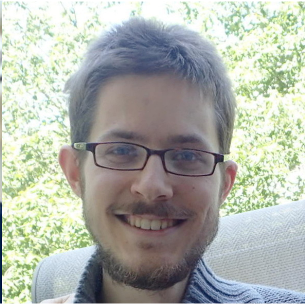
My amazing co-authors



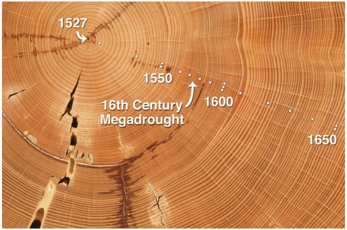


Based on 'BPS Dendroscopy on Local $\mathbb{P}^2$' [2210.10712]

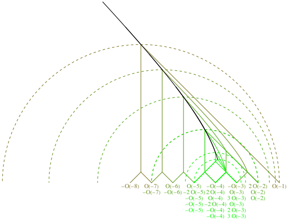
with Pierrick Bousseau, Pierre Descombes and Bruno Le Floch

δενδρον= tree

Dentrology



Dendrochronology



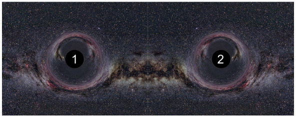
Dendroscopy

Introduction

- In type IIA string theory compactified on a Calabi-Yau threefold X, the BPS spectrum consists of bound states of D6-D4-D2-D0 branes, with charge $\gamma \in H_{\rm even}(X, \mathbb{Q})$.
- BPS states saturate the bound $M(\gamma) \geq |Z(\gamma)|$, where the central charge $Z \in \mathrm{Hom}(\Gamma, \mathbb{C})$ depends on the complexified Kähler moduli.
- The index $\Omega_z(\gamma)$ counting BPS states is robust under complex structure deformations, but in general depends on $z \in \mathcal{M}_K$.
- Mathematically, the Donaldson-Thomas invariant $\Omega_z(\gamma)$ counts stable objects with $\mathrm{ch}\, E = \gamma$ in the derived category of coherent sheaves $\mathcal{C} = D^b\mathrm{Coh}(X)$.

Introduction

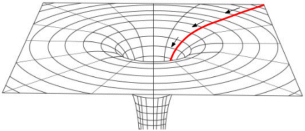
- $\Omega_z(\gamma)$ is locally constant on $\mathcal{M}_K$, but can jump across real codimension one walls of marginal stability $\mathcal{W}(\gamma_L, \gamma_R) \subset \mathcal{M}_K$, where the phases of the central charges $Z(\gamma_L)$ and $Z(\gamma_R)$ with $\gamma = m_L \gamma_L + m_R \gamma_R$ become aligned *[Kontsevich Soibelman'08, Joyce Song'08]*
- Physically, multi-centered black hole solutions with charges $\gamma_i = m_{L,i}\gamma_L + m_{R,i}\gamma_R$ (dis)appear across the wall *[Denef'02, Denef Moore '07, ..., Manschot BP Sen '11]*.



$$\frac{\langle \gamma_L, \gamma_R \rangle}{r} = \frac{2\,\mathrm{Im}[\bar{Z}(\gamma_L)\, Z(\gamma_R)]}{|Z(\gamma_L + \gamma_R)|}, \qquad \Delta\Omega(\gamma) = \pm |\langle \gamma_L, \gamma_R \rangle|\, \Omega(\gamma_L)\Omega(\gamma_R)$$

- These multi-centered bound states are expected to decay away as one follows the attractor flow equations *[Ferrara Kallosh Strominger'95]*

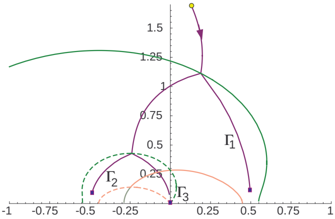
$$\mathrm{AF}_\gamma : \quad r^2 \frac{\mathrm{d}z^a}{\mathrm{d}r} = -g^{a\bar{b}} \partial_{\bar{b}} |Z_z(\gamma)|^2$$



- Let $z_\star(\gamma)$ be the endpoint of the flow, or attractor point. Since $Z_z(\gamma)$ decreases along the flow, $z_\star(\gamma)$ can either be a regular local minimum of $|Z_z(\gamma)|$ with $|Z_{z_\star(\gamma)}(\gamma)| > 0$, or a conifold point if $Z_{z_\star(\gamma)}(\gamma) = 0$.
- We define the attractor invariant as $\Omega_\star(\gamma) = \Omega_{z_\star(\gamma)}(\gamma)$.

The Split Attractor Flow Conjecture

- Starting from $z \in \mathcal{M}_K$, following AF_γ and recursively applying the WCF formula at whenever the flow crosses a wall of marginal stability, one can in principle express $\Omega_z(\gamma)$ in terms of attractor invariants.



Denef Moore'07

The Split Attractor Flow Conjecture (SFAC)

- In terms of the rational DT invariants

$$\bar\Omega_z(\gamma) := \sum_{k|\gamma} \frac{y-1/y}{k(y^k-y^{-k})}\Omega_z(\gamma/k)_{y\to y^k}$$

the result takes the form

$$\bar\Omega_z(\gamma) = \sum_{\gamma=\sum\gamma_i} \frac{g_z(\{\gamma_i\})}{\mathrm{Aut}(\{\gamma_i\})} \prod_i \bar\Omega_\star(\gamma_i)$$

where $g_z(\{\gamma_i\})$ is a sum over attractor flow trees.

- The Split Attractor Flow Conjecture *[Denef'00, Denef Moore'07]* is the statement that only a finite number of decompositions $\gamma = \sum\gamma_i$ contribute to the index $\bar\Omega_z(\gamma)$.

The Split Attractor Flow Conjecture

- Unfortunately one does not know a priori which constituents γ_i can contribute, except for the obvious constraints

$$\sum_i \gamma_i = \gamma \; , \quad \sum_i |Z_{z_\star(\gamma_i)}(\gamma_i)| < |Z_z(\gamma)|$$

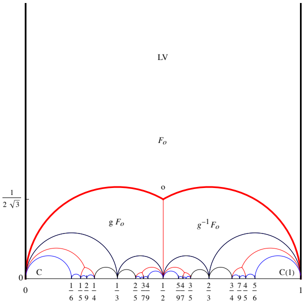
- In particular, there can be cancellations between D-branes and anti-D-branes, and contributions from conifold states which are massless at their attractor point are difficult to bound.
- Even if SAFC holds, one still has to compute the attractor indices $\Omega_\star(\gamma)$, a tall order for regular attractor points.

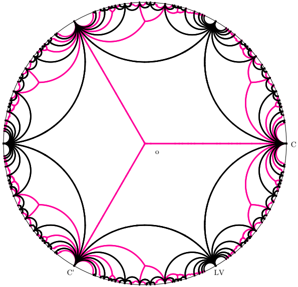
Summary

- Our aim is to investigate the Split Attractor Flow Conjecture for one of the simplest examples of CY threefolds, namely $X = K_{\mathbb{P}^2} = \widetilde{\mathbb{C}^3/\mathbb{Z}_3}$ *[Douglas Fiol Romelsberger'00]*.
- We show that the only possible constituents are the D4-brane $\mathcal{O}_{\mathbb{P}^2}$ and anti-D4-brane $\mathcal{O}_{\mathbb{P}^2}[1]$, and images thereof under $\Gamma_1(3)$, each carrying attractor index $\Omega_\star(\gamma) = 1$.
- In particular, in the large volume region the full BPS spectrum arises as bound states of fluxed D4 and anti-D4-brane, with effective bounds on the number and flux of the constituents.
- A key role is played by scattering diagrams, which provide the correct mathematical framework for the SAFC, at least for local CY threefolds.

Kähler moduli space

- By local mirror symmetry, the Kähler moduli space of $X = K_{\mathbb{P}^2}$ is the quotient $X_1(3) = \mathbb{H}/\Gamma_1(3)$. It admits two cusps LV, C and one orbifold point o of order 3.





- A BPS state on X is a stable object E in the bounded derived category $\mathcal{C}$ of compactly supported sheaves on X, with charge $\gamma(E) = \mathrm{ch}(\pi_*(E)) = [r, d, \mathrm{ch}_2] \sim [D4, D2, D0]$

Central charge as Eichler integral

- The central charge $Z_\tau(\gamma)$ is a linear combination

$$Z_\tau(\gamma) = -rT_D(\tau) + dT(\tau) - \mathrm{ch}_2$$

where T_D, T are single-valued functions on $\mathbb{H}$ (but not on $\mathcal{M}_K$). They are periods of a one-form λ with logarithmic singularities on the mirror curve, satisfying a Picard-Fuchs equation of degree 3.

- It turns out that $\partial_\tau \lambda$ is holomorphic, so its periods are proportional to $(1, \tau)$. Integrating along a path from o to τ, one can establish the Eichler-type integral representation

$$\boxed{\begin{pmatrix} T \\ T_D \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/3 \end{pmatrix} + \int_{\tau_o}^{\tau} \begin{pmatrix} 1 \\ u \end{pmatrix} C(u)\, \mathrm{d}u}$$

where $C(\tau) = \frac{\eta(\tau)^9}{\eta(3\tau)^3} = 1 - 9q + \dots$ is a weight 3 Eisenstein series for $\Gamma_1(3)$.

Central charge as Eichler integral

- This provides an computationally efficient analytic continuation of Z_τ throughout $\mathbb{H}$, and gives access to monodromies:

$$\tau \mapsto \frac{a\tau+b}{c\tau+d} \qquad \begin{pmatrix} 1 \\ T \\ T_D \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ m & d & c \\ m_D & b & a \end{pmatrix} \cdot \begin{pmatrix} 1 \\ T \\ T_D \end{pmatrix}$$

  where (m, m_D) are period integrals of C from τ_o to $\frac{d\tau_o-b}{a-c\tau_o}$.

- At large volume $\tau \to \mathrm{i}\infty$, using $C = 1 + \mathcal{O}(q)$ one finds

$$T = \tau + \mathcal{O}(q), \quad T_D = \frac{1}{2}\tau^2 + \frac{1}{8} + \mathcal{O}(q)$$

  in agreement with $Z_\tau(\gamma) \sim -\int_S e^{-\tau H} \sqrt{\mathrm{Td}(S)}\, \mathrm{ch}(E)$.

Space of Bridgeland stability conditions

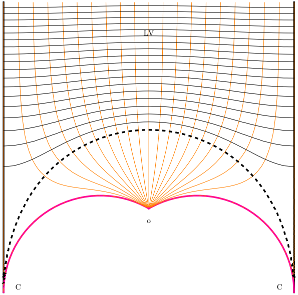
- Donaldson-Thomas invariants are defined in the larger space of Bridgeland stability conditions $\mathrm{Stab}\,\mathcal{C} = \{\sigma = (Z, \mathcal{A})\}$, where $Z : \Gamma \to \mathbb{C}$ is a linear map and $\mathcal{A} \subset \mathcal{C}$ an Abelian category (heart of t-structure) satisfying various axioms, e.g. $\mathrm{Im} Z(\gamma(E)) \geq 0 \ \forall E \in \mathcal{A}$.
- The group $\widetilde{GL(2,\mathbb{R})}^+$ acts on $\mathrm{Stab}\,\mathcal{C}$ by linear transformations of $(\mathrm{Re} Z, \mathrm{Im} Z)$ with positive determinant, leaving $\Omega_\sigma(\gamma)$ invariant.
- For τ_2 large enough, one can use $\widetilde{GL(2,\mathbb{R})}^+$ to absorb the $1/8$ and $\mathcal{O}(q)$ corrections to $Z_\tau(\gamma)$ and reach the large volume slice

$$Z^{LV}_{(s,t)}(\gamma) = -\frac{r}{2}(s + \mathrm{i}t)^2 + d(s + \mathrm{i}t) - \mathrm{ch}_2 \ ,$$

with $\tau \simeq s + \mathrm{i}t$.

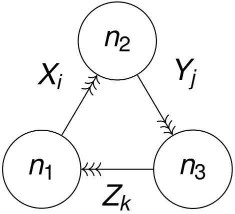
Space of Bridgeland stability conditions

- Specifically, this holds in the region $w > \frac{1}{2}s^2$ where $(s,w) := \left(\frac{\operatorname{Im} T_D}{\operatorname{Im} T}, -\frac{\operatorname{Im}(T\bar{T}_D)}{\operatorname{Im} T}\right)$. In $(s, t = \sqrt{2w - s^2})$ plane:



- The large volume slice does not cover the region around the orbifold point, and covers only part of the conifold point.

- The category $D^b\,\mathrm{Coh}_c(K_{\mathbb{P}^2})$ is isomorphic to the category of representations a quiver with potential (Q,W), whose nodes correspond to fractional branes on $\mathbb{C}^3/\mathbb{Z}_3$:



$$W = \sum \epsilon_{ijk} X_i Y_j Z_k$$

$$\begin{array}{lcll}
E_1 & = & \mathcal{O}[-1] & \gamma_1 = [-1,0,0] \\
E_2 & = & \Omega(1), & \gamma_2 = [2,-1,-\frac{1}{2}] \\
E_3 & = & \mathcal{O}(-1)[1] & \gamma_3 = [-1,1,-\frac{1}{2}] \\
 & & r = & 2n_2 - n_1 - n_3 \\
 & & d = & n_3 - n_2 \\
 & & \mathrm{ch}_2 = & -\frac{1}{2}(n_2+n_3)
\end{array}$$

- The quiver description is valid in a region where the central charges $Z(E_i)$ lie in a common half-plane, which includes the orbifold point $\tau_o = -\frac{1}{2} + \frac{\mathrm{i}}{2\sqrt{3}}$, where $Z_{\tau_o}(\gamma_i) = 1/3$ for $i = 1,2,3$.

Attractor flow tree formula for quivers

- In that region, $\Omega_\tau(\gamma)$ coincides with the quiver index $\Omega_\theta(\gamma)$ counting θ-semi-stable representations of dimension vector γ, upon setting $\theta_i = -\mathrm{Re}(e^{-\mathrm{i}\psi} Z_\tau(\gamma_i))$ with ψ s.t. $\mathrm{Im}(e^{-\mathrm{i}\psi} Z_\tau(\gamma_i)) > 0$.
- For fixed FI parameters $\theta \in \mathbb{R}^{Q_0}$, a representation of dim γ is θ-semi-stable iff $(\theta, \gamma') \leq (\theta, \gamma)$ for any subrepresentation.
- In the quiver context, there is a notion of attractor stability condition (aka self-stability condition) *[Manschot BP Sen'13; Bridgeland'16]*

$$(\theta_\star(\gamma), \gamma') = \langle \gamma', \gamma \rangle := \sum_{a:i\to j} (n'_i n_j - n'_j n_i)$$

The (quiver) attractor invariant is defined as $\Omega_\star(\gamma) := \Omega_{\theta_\star(\gamma)}(\gamma)$

The Flow Tree formula for quivers

- In *[Alexandrov BP'18]*, we conjectured a precise version of SAFC which expresses $\bar{\Omega}_\theta(\gamma)$ in terms of the attractor invariants:

$$\bar{\Omega}_\theta(\gamma) = \sum_{\gamma=\sum \gamma_i} \frac{g_\theta(\{\gamma_i\})}{\mathrm{Aut}(\{\gamma_i\})} \prod_i \bar{\Omega}_\star(\gamma_i)$$

  The coefficients $g_\theta(\{\gamma_i\})$ involve a sum over rooted binary trees, whose edges are embedded in $\mathbb{R}^{Q_0}$ along straight lines $\theta_0 + \lambda\theta_\star(\gamma_e)$, which are the analogue of attractor flows.

- The sum is manifestly finite, since γ_i lie in the positive cone $\mathbb{Z}_+^{Q_0}$.
- The formula was proven mathematically in *[Argüz Bousseau'21]* using the formalism of scattering diagrams.

Flow tree formula from scattering diagrams

- For any quiver with potential (Q, W), the scattering diagram $\mathcal{D}_Q$ is the set of real codimension-one rays $\{\mathcal{R}(\gamma), \gamma \in \mathbb{Z}^{Q_0}\}$ defined by

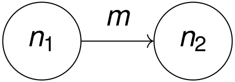
$$\mathcal{R}(\gamma) = \{\theta \in \mathbb{R}^{Q_0} : (\theta, \gamma) = 0,\ \bar{\Omega}_\theta(\gamma) \neq 0\}$$

- Each point along $\mathcal{R}(\gamma)$ is endowed with an automorphism of the quantum torus algebra,

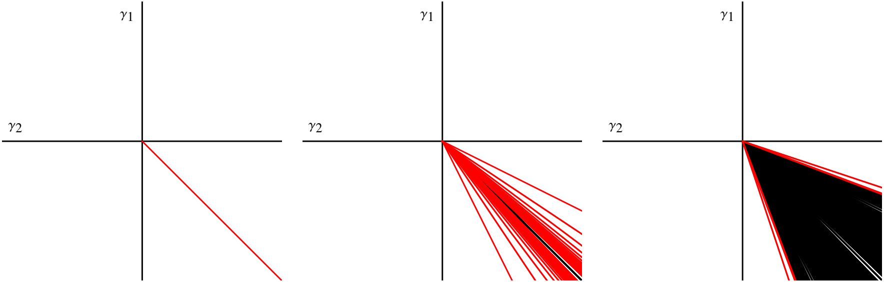
$$\mathcal{U}_\theta(\gamma) = \exp\left(\frac{\bar{\Omega}_\theta(\gamma)}{y^{-1} - y}\mathcal{X}_\gamma\right) \ , \quad \mathcal{X}_\gamma \mathcal{X}_{\gamma'} = (-y)^{\langle\gamma,\gamma'\rangle}\mathcal{X}_{\gamma+\gamma'}$$

- The WCF ensures that the diagram is consistent: for any generic closed path $\mathcal{P} : t \in [0, 1] \in \mathbb{R}^{Q_0}$, $\prod_i \mathcal{U}_{\theta(t_i)}(\gamma_i)^{\epsilon_i} = 1$ *[Bridgeland'16]*
- A consistent scattering diagram is uniquely determined from the initial rays $\mathcal{R}_\star(\gamma)$, defined as those which contain $\theta_\star(\gamma)$.
- The Flow Tree Formula of *[Alexandrov BP'18]* determines the indices of outgoing rays produced by scattering initial rays *[Argüz Bousseau '20]*.

Scattering diagram for Kronecker quiver



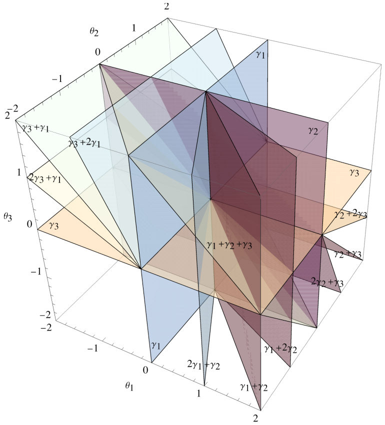
$\theta_1 > 0, \theta_2 < 0$: $\quad \dim \mathcal{M}_\theta(\gamma) = m n_1 n_2 - n_1^2 - n_2^2 + 1$



m=1 m=2 m=3

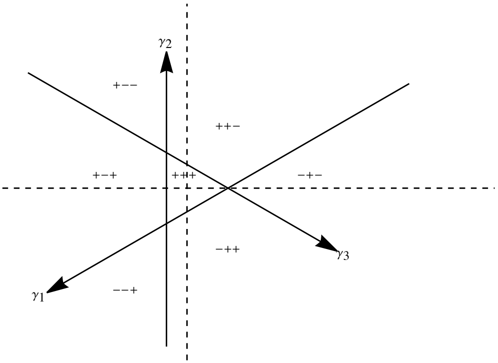
Attractor invariants for $K_{\mathbb{P}^2}$

- By studying expected dimension of the moduli space of semi-stable representations $\mathcal{M}_\theta(\gamma)$, *[Beaujard BP Manschot'20]* conjectured that the attractor index $\Omega_\star(\gamma)$ vanishes unless for $\gamma = \gamma_i$ or $\gamma = k(\gamma_1 + \gamma_2 + \gamma_3)$. This is now a theorem *[Bousseau Descombes Le Floch BP'22]*.



A 2D slice of the orbifold scattering diagram

Let $\mathcal{D}_o$ be the restriction of $\mathcal{D}_Q$ to the hyperplane $\theta_1 + \theta_2 + \theta_3 = 1$:



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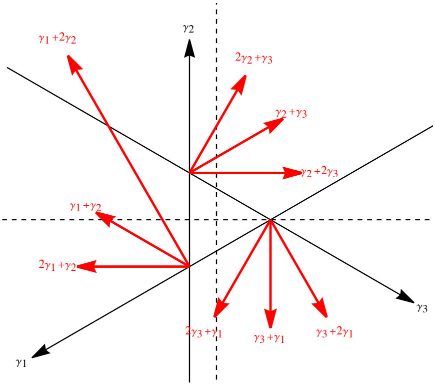

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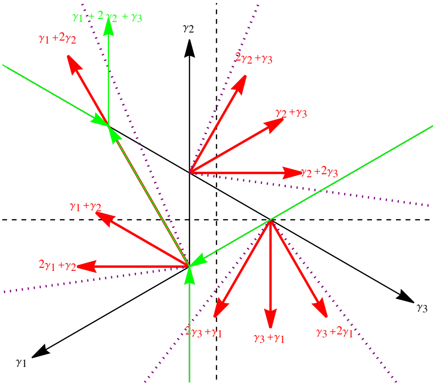
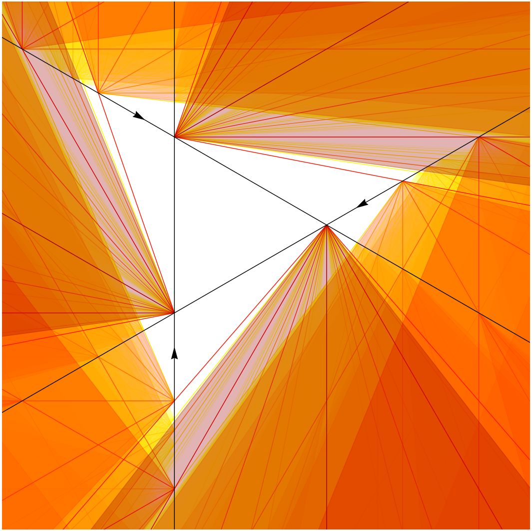


A 2D slice of the orbifold scattering diagram

The full scattering diagram $\mathcal{D}_Q$ includes regions with dense set of rays:



Scattering diagrams on triangulated categories

- For a general triangulated category $\mathcal{C}$, define the scattering diagram $\mathcal{D}_\psi(\mathcal{C})$ as the set of codimension-one loci in $\mathrm{Stab}\,\mathcal{C}$,

$$\mathcal{R}_\psi(\gamma) = \{\sigma : \arg Z(\gamma) = \psi + \tfrac{\pi}{2},\ \bar\Omega_Z(\gamma) \neq 0\}$$

  equipped with (a suitable regularization of) the automorphism

$$\mathcal{U}_\sigma(\gamma) = \exp\left(\frac{\bar\Omega_\sigma(\gamma)}{y^{-1}-y}\mathcal{X}_\gamma\right) = \mathrm{Exp}\left(\frac{\Omega_\sigma(\gamma)}{y^{-1}-y}\mathcal{X}_\gamma\right)$$

- The WCF ensures that the diagram $\mathcal{D}_\psi$ is still locally consistent at each codimension-two intersection.

Flow trees from scattering diagrams

- To see the relation to SAFC, note that for any local CY threefold, the central charge $Z_z(\gamma)$ is holomorphic in z^a, hence its phase is constant along the flow $\frac{\mathrm{d}z^a}{\mathrm{d}\mu} = -g^{a\bar{b}}\partial_{\bar{b}}|Z_z(\gamma)|^2$:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\mu}\log\frac{Z(\gamma)}{\bar{Z}(\gamma)} = -\frac{1}{2}\partial_a Z(\gamma) g^{a\bar{b}}\partial_{\bar{b}}\bar{Z}(\gamma) + \frac{1}{2}\partial_a Z(\gamma) g^{a\bar{b}}\partial_{\bar{b}}\bar{Z}(\gamma) = 0$$

  Moreover, $|Z_z(\gamma)|^2$ has no local minima so the only attractor points are conifold points with $Z_z(\gamma_i) = 0$.
- Thus, the restriction of $\mathcal{R}_\psi(\gamma)$ to the physical slice is preserved by the attractor flow. Moreover, the flow can only split when $\mathcal{R}(\gamma_L)$ and $\mathcal{R}(\gamma_R)$ intersect, and end on an initial ray $\mathcal{R}_\psi(\gamma_i)$.
- In complex dimension one, attractor flow lines and scattering rays coincide. Attractor flow trees are subsets of $\mathcal{D}_\psi$ which produce an outgoing ray $\mathcal{R}_\psi(\gamma)$ with desired charge γ, passing through the desired point z.

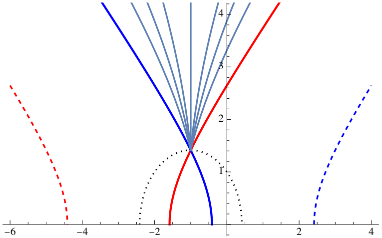
Large volume scattering diagram

- The scattering diagram $\mathcal{D}_\psi^{\rm LV}$ along the large volume slice

$$Z^{LV}_{(s,t)} = -\frac{1}{2} r(s+\mathrm{i}t)^2 + d(s+\mathrm{i}t) - \mathrm{ch}_2$$

  was determined for $\psi=0$ in *[Bousseau'19]*, using a different set of coordinates. The construction extends to any ψ by just mapping $(s,t)\mapsto(s-t\tan\psi, t/\cos\psi)$.
- Since $\mathrm{Re}Z(\gamma) = \frac12 r(t^2-s^2)+ds-\mathrm{ch}_2$, each ray $\mathcal{R}_0(\gamma)$ is contained in a branch of hyperbola asymptoting to $t=\pm(s-\frac{d}{r})$ for $r\neq 0$, or vertical a line when $r=0$. Walls of marginal stability $\mathcal{W}(\gamma,\gamma')$ are half-circles centered on real axis.

Large volume scattering diagram

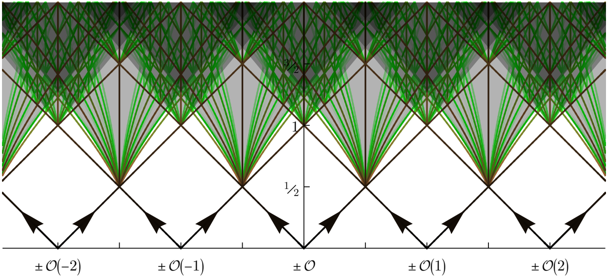


It is useful to think of $\mathcal{R}(\gamma)$ as the worldline of a fictitious particle of charge r, mass $M^2 = \frac{1}{2}d^2 - r\,\mathrm{ch}_2$ moving in a constant electric field:

- The particle travels inside the forward light-cone
- the 'electric potential' $\varphi_s(\gamma) = 2(d - sr) = 2\mathrm{Im}Z_\gamma/t$ increases along the flow.

Large volume scattering diagram

- Initial rays correspond to $\mathcal{O}(m)$ and $\mathcal{O}(m)[1]$, with charge $\gamma_m = \pm[1, m, \frac{1}{2}m^2]$, emanating from $(s,t) = (m,0)$ on the boundary where $Z^{LV}_{(s,t)}(\gamma_m) = 0$. *[Bousseau'19]*



- Physically, the BPS spectrum along the large volume slice originates from bound states of fluxed D4-branes and anti-D4 branes.

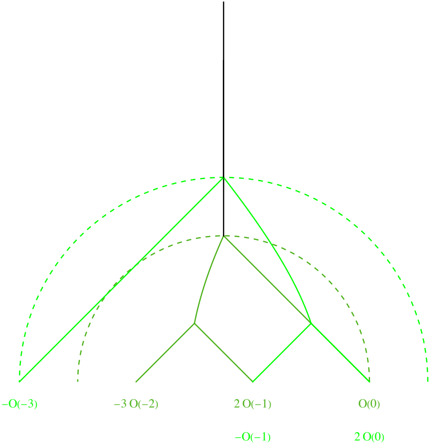
SAFC holds along large volume slice

- The first scatterings occur after a time $t \geq \frac{1}{2}$, after each constituent $k_i \mathcal{O}(m_i)$ has moved by $|\Delta s| \geq \frac{1}{2}$, by which time $\varphi_s(\gamma_i) \geq |k_i|$.
- Since $\varphi_s(\gamma)$ is additive at each vertex, this gives a bound on the number and charges of constituents contributing to $\Omega_{(s,t)}(\gamma)$:

$$\sum_i k_i [1, m_i, \frac{1}{2} m_i^2] = \gamma \ , \quad s - t \leq m_i \leq s + t, \quad \sum |k_i| \leq \varphi_s(\gamma)$$

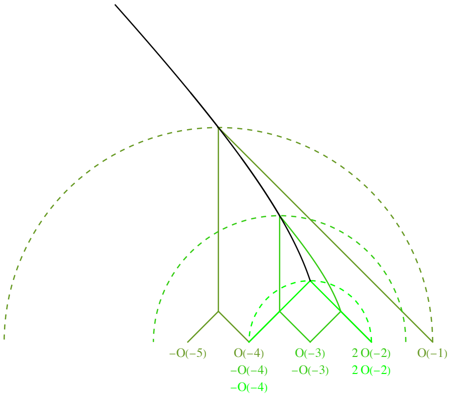
- Thus, SAFC holds along the large volume slice !

Flow trees for $\gamma = [0, 4, 1)$



- $\{\{-3\mathcal{O}(-2), 2\mathcal{O}(-1)\}, \mathcal{O}\}$:
  $3\mathcal{O}(-2) \to 2\mathcal{O}(-1) \oplus \mathcal{O} \to E$
  $K_3(2,3)K_{12}(1,1) \to -156$
- $\{-\mathcal{O}(-3), \{-\mathcal{O}(-1), 2\mathcal{O}\}\}$:
  $\mathcal{O}(-3) \oplus \mathcal{O}(-1) \to 2\mathcal{O} \to E$
  $K_3(1,2)K_{12}(1,1) \to -36$

Total: $\Omega_\infty(\gamma) = -192 = GV_4^{(0)}$

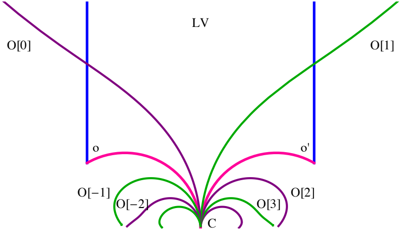


- $\{\{-\mathcal{O}(-5), \mathcal{O}(-4)\}, \mathcal{O}(-1)\}$
  $\mathcal{O}(-5) \to \mathcal{O}(-4) \oplus \mathcal{O}(-1) \to E$
  $K_3(1,1)^2 \to 9$
- $\{\{-\mathcal{O}(-4), \mathcal{O}(-3)\},$
  $\{-\mathcal{O}(-3), 2\mathcal{O}(-2)\}\}$
  $\mathcal{O}(-4) \oplus \mathcal{O}(-3) \to$
  $\mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \to E$
  $K_3(1,1)^2 K_3(1,2) \to 27$
- $\{-\mathcal{O}(-4), 2\mathcal{O}(-2)\}$
  $\mathcal{O}(-4) \to 2\mathcal{O}(-2) \to E$
  $K_6(1,2) \to 15$

Total: $\Omega_\infty(\gamma) = 51 = \chi(\mathrm{Hilb}_4 \mathbb{P}^2)$

Exact scattering diagram

- The scattering diagram $\mathcal{D}^{\Pi}_{\psi}$ along the physical slice should interpolate between $\mathcal{D}^{\rm LV}_{\psi}$ around $\tau = \mathrm{i}\infty$ and $\mathcal{D}_o$ around $\tau = \tau_o$, and be invariant under the action of $\Gamma_1(3)$.
- Under $\tau \mapsto \frac{\tau}{3n\tau+1}$ with $n \in \mathbb{Z}$, $\mathcal{O} \mapsto \mathcal{O}[n]$. Hence there is a doubly infinite family of initial rays emitted at $\tau = 0$, associated to $\mathcal{O}[n]$.



- Similarly, there must be an infinite family of initial rays coming from $\tau = \frac{p}{q}$ with $q \neq 0 \mod 3$, corresponding to $\Gamma_1(3)$-images of $\mathcal{O}$, where an object denoted by $\mathcal{O}_{p/q}$ becomes massless.

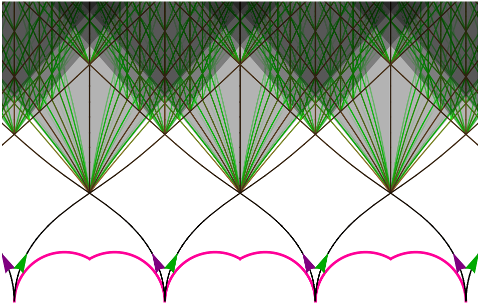
Massless objects at conifold points

τ	g	γ_C	$\Delta(\gamma_C)$	$\mathcal{O}_{p/q}$
0	1	$[1,0,1)$	0	$\mathcal{O}$
$1/5$	U^2T^{-1}	$-[5,1,6)$	$3/25$	$E \to \Omega(2)[-1] \to \mathcal{O}^{\oplus 3}[2]$
$1/4$	UT	$[4,1,6)$	$-3/32$	$E \to \mathcal{O}(1) \to \mathcal{O}^{\oplus 3}[3]$
$2/5$	UT^{-2}	$-[5,2,6)$	$12/25$	$E \to \mathcal{O}(-2) \to \mathcal{O}^{\oplus 6}$
$3/7$	$UT^{-1}VT$	$[7,3,10)$	$15/49$	$E \to \Omega(0)[1] \to \mathcal{O}^{\oplus 9}[1]$
$1/2$	TVT	$-[2,1,3)$	$3/8$	$\Omega(2)[1]$
$4/7$	$TVTUT^{-1}$	$[7,4,12)$	$15/49$	$\mathcal{O}(1)^{\oplus 9}[-1] \to \Omega(4)[-1] \to E$
$3/5$	TVT^2	$-[5,3,8)$	$12/25$	$\mathcal{O}(1)^{\oplus 6} \to \mathcal{O}(3) \to E$
$3/4$	TVT^{-1}	$[4,3,10)$	$-3/32$	$\mathcal{O}(1)^{\oplus 3}[-3] \to \mathcal{O}(0) \to E$
$4/5$	TV^2T	$-[5,4,12)$	$3/25$	$\mathcal{O}(1)^{\oplus 3}[-2] \to \Omega(2)[1] \to E$
1	T	$[1,1,3)$	0	$\mathcal{O}(1)$

$$T : \tau \mapsto \tau + 1; \quad U : \tau \mapsto 1/(3\tau + 1); \quad V = U^{-1}$$

Exact scattering diagram for small ψ

- For $|\psi|$ small enough, the only rays which reach the large volume region are those associated to $\mathcal{O}(m)$ and $\mathcal{O}(m)[1]$. Thus, the scattering diagram $\mathcal{D}_\psi^\Pi$ is isomorphic to $\mathcal{D}_0^{\mathrm{LV}}$ inside $\mathcal{F}$ and its translates:



Scattering diagram in affine coordinates

- To see this, one can map both of them to the plane

Bousseau'19

$$x = \frac{\mathrm{Re}\left(e^{-\mathrm{i}\psi} T\right)}{\cos\psi}\ , \quad y = -\frac{\mathrm{Re}\left(e^{-\mathrm{i}\psi} T_D\right)}{\cos\psi}\ ,$$

such that $\mathcal{R}_\psi(\gamma)$ becomes a line segment $rx + dy - \mathrm{ch}_2 = 0$.

- The initial rays $\mathcal{R}_{\mathcal{O}(m)}$ are tangent to the parabola $y = -\frac{1}{2}x^2$ at $x = m$, but the origin of each ray is shifted to $x = m + \mathcal{V}\tan\psi$ where $\mathcal{V}$ is the quantum volume

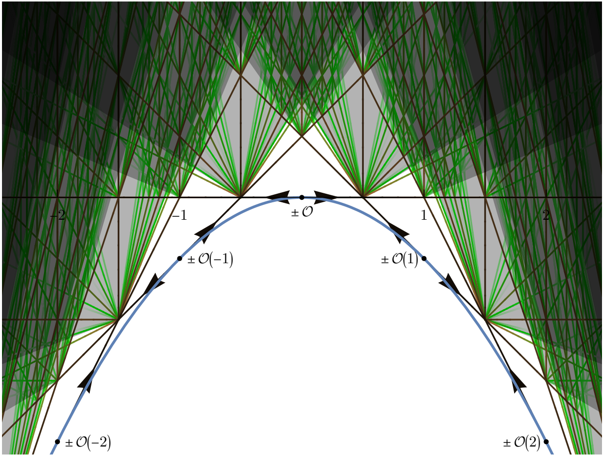
$$\mathcal{V} = \mathrm{Im}\, T(0) = \frac{27}{4\pi^2}\mathrm{Im}\left[\mathrm{Li}_2(e^{2\pi\mathrm{i}/3})\right] \simeq 0.463$$

- The topology of $\mathcal{D}_\psi^\Pi$ jumps at a discrete set of rational values

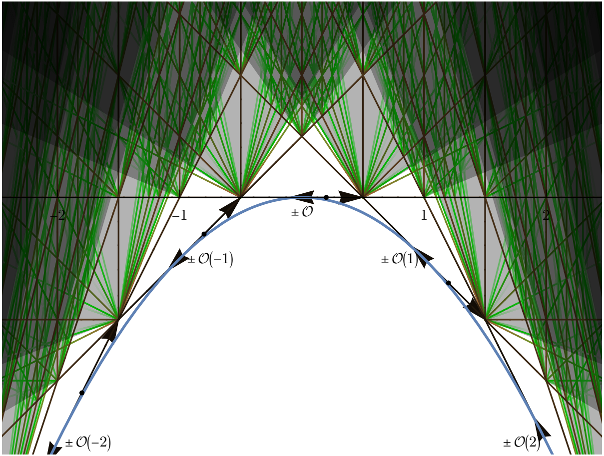
$$\mathcal{V}\tan\psi \in \{\tfrac{F_{2k}+F_{2k+2}}{2F_{2k+1}}, k \geq 0\} = \{\tfrac{1}{2}, 1, \tfrac{11}{10}, \tfrac{29}{26}, \tfrac{19}{17}, \dots\}$$

and a dense set of values in $[\frac{\sqrt{5}}{2}, +\infty)$ where secondary rays pass through a conifold point.

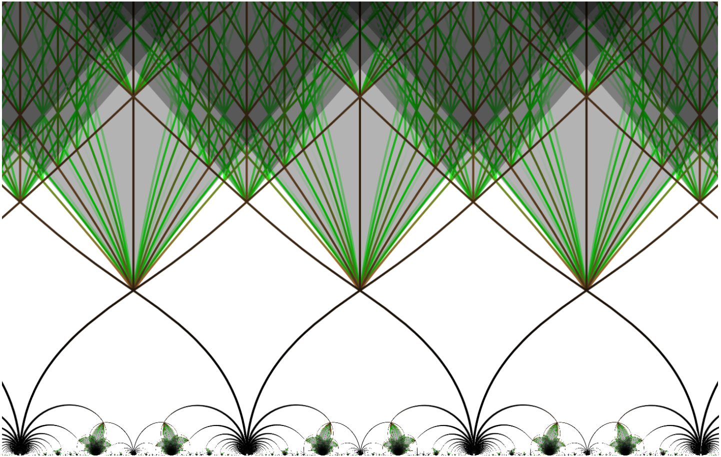
Affine scattering diagram, $\psi = 0$



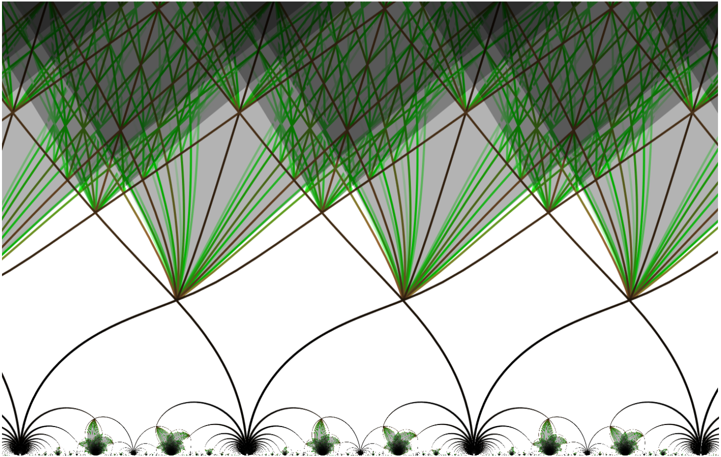
Affine scattering diagram, $|\mathcal{V}\tan\psi| < 1/2$



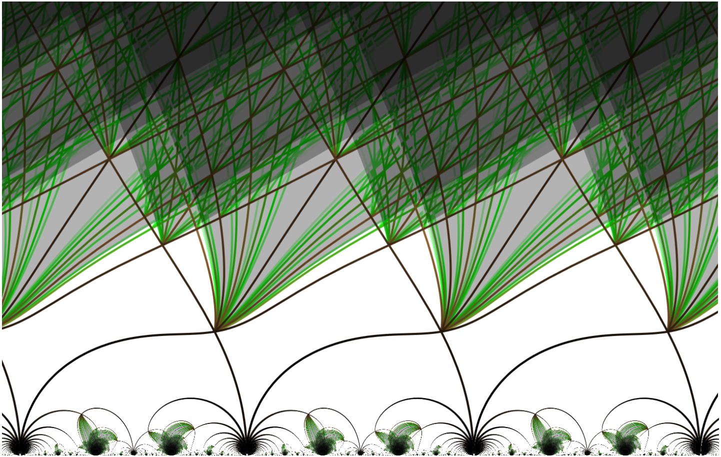
Exact scattering diagram, $\psi = 0$



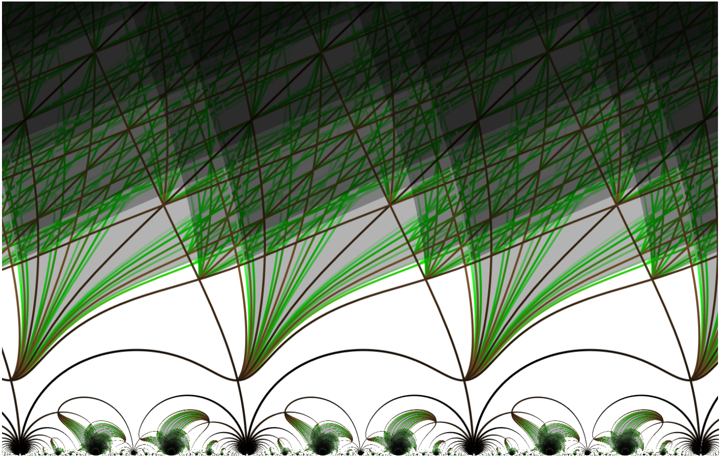
Exact scattering diagram, $\psi = 0.3$



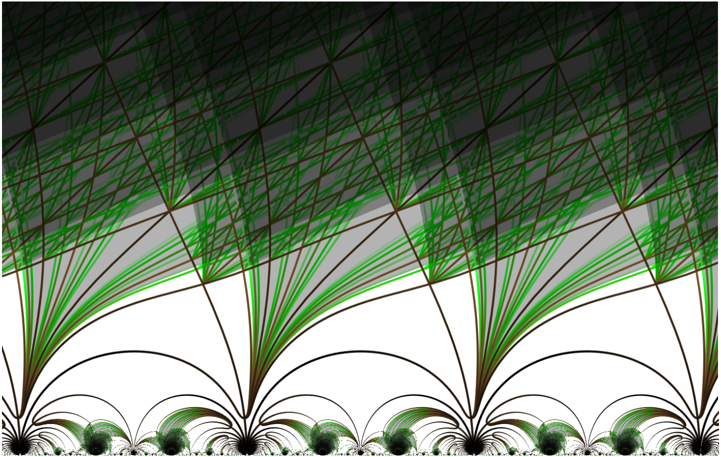
Exact scattering diagram, $\psi = 0.6$



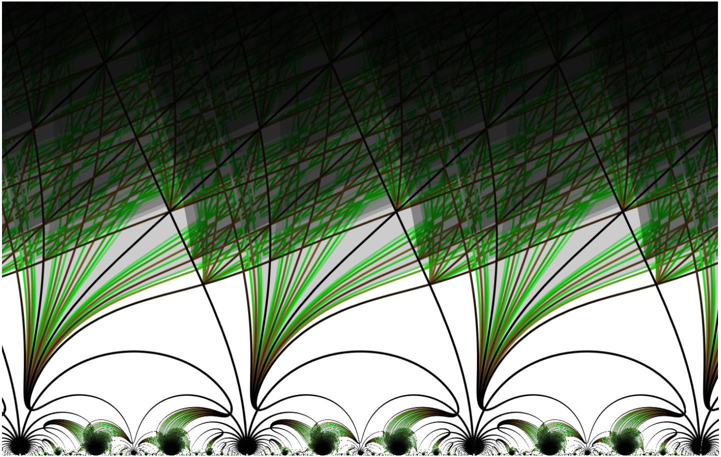
Exact scattering diagram, $\psi = 0.8$



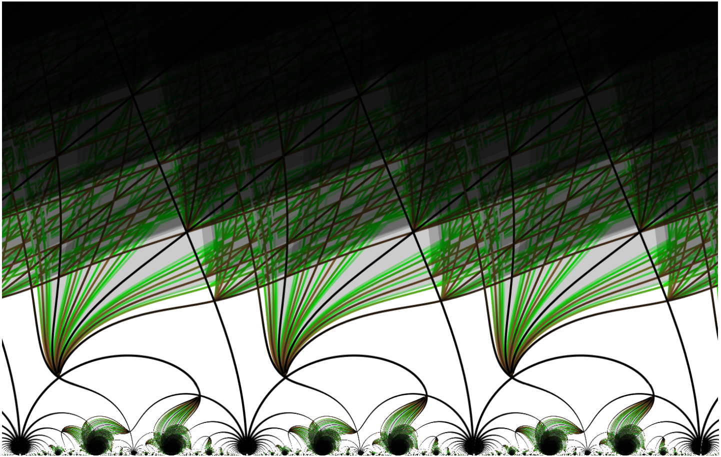
Exact scattering diagram, $\psi = 0.824$



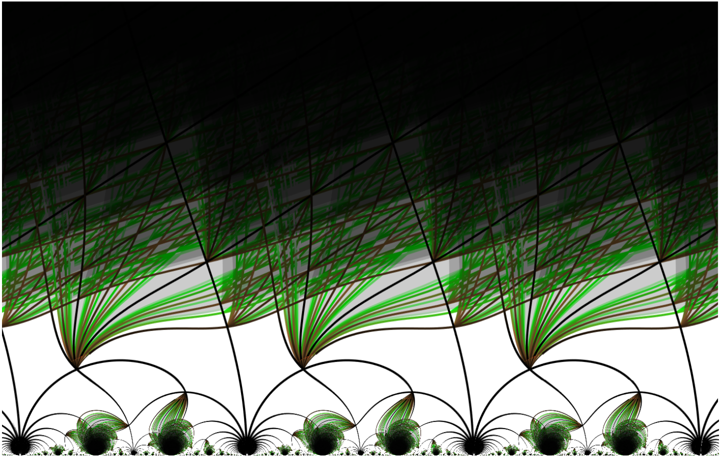
Exact scattering diagram, $\psi = 0.825$



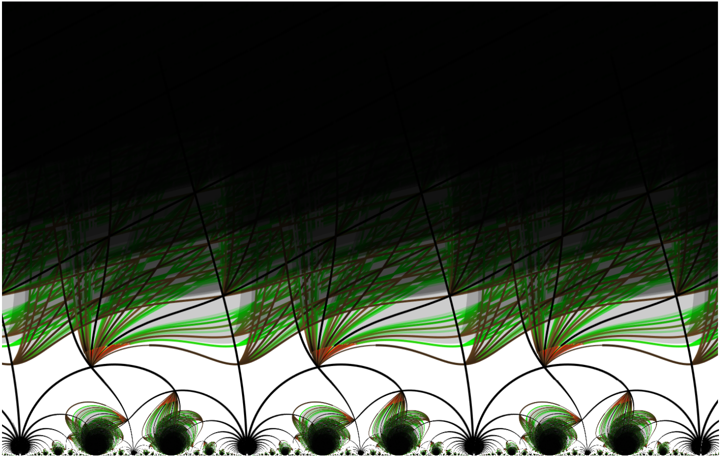
Exact scattering diagram, $\psi = 0.9$



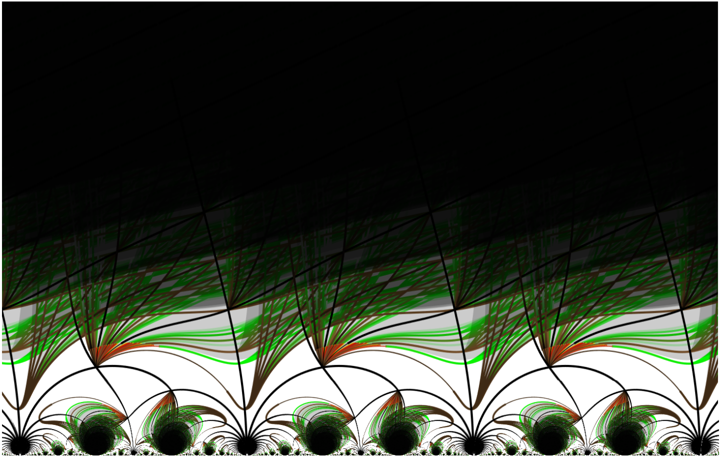
Exact scattering diagram, $\psi = 1$



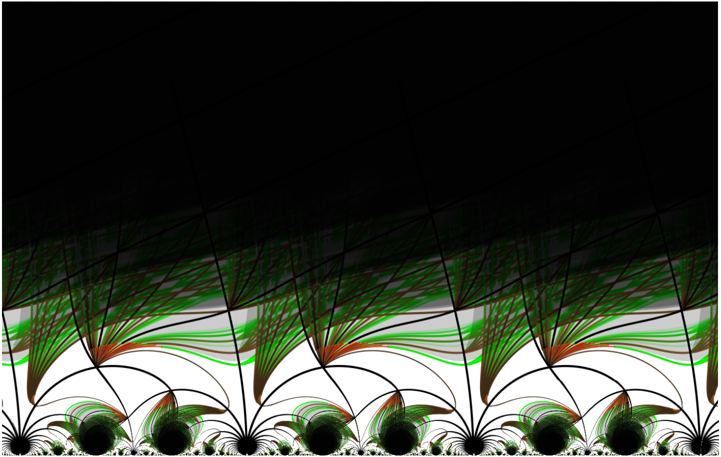
Exact scattering diagram, $\psi = 1.1$



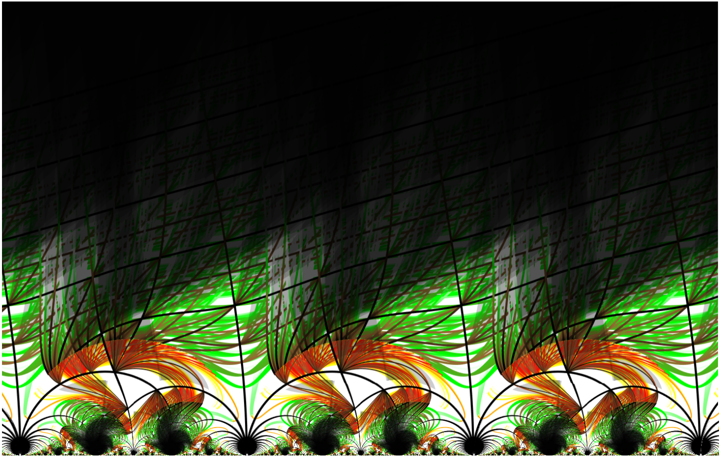
Exact scattering diagram, $\psi = 1.137$



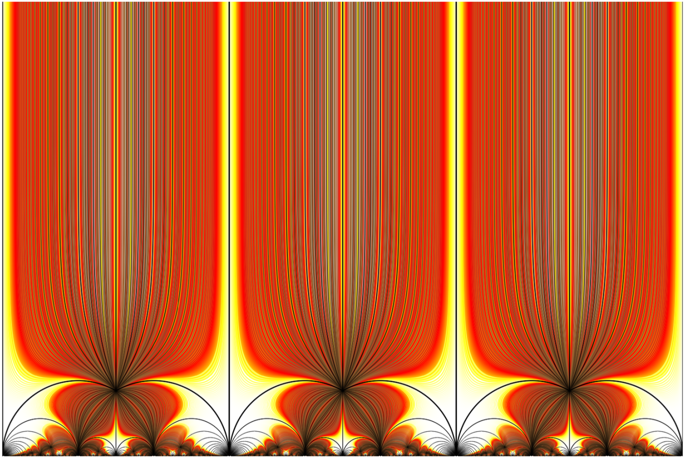
Exact scattering diagram, $\psi = 1.139$



Exact scattering diagram, $\psi = 1.3$

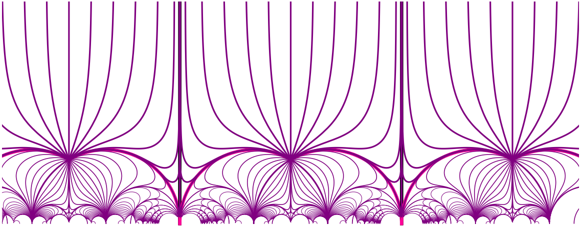


Exact scattering diagram, $\psi = \pi/2$



Exact scattering diagram for $\psi = \pm\frac{\pi}{2}$

- For $\psi = \pm\frac{\pi}{2}$, the geometric rays $\{\mathrm{Im} Z_\tau(\gamma) = 0\}$ coincide with lines of constant $s = \frac{\mathrm{Im}\, T_D}{\mathrm{Im}\, T} = \frac{d}{r}$, independent of ch_2:

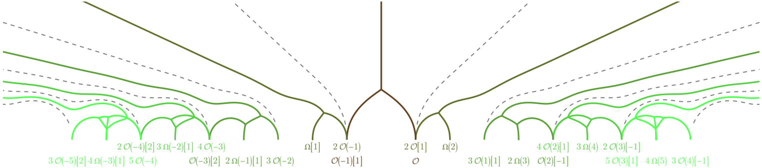


- Hence, there is no wall-crossing between τ_o and $\tau = \mathrm{i}\infty$ when $-1 \leq \frac{d}{r} \leq 0$, explaining why the Gieseker index $\Omega_\infty(\gamma)$ agrees with the quiver index $\Omega_c(\gamma)$ in the anti-attractor chamber.

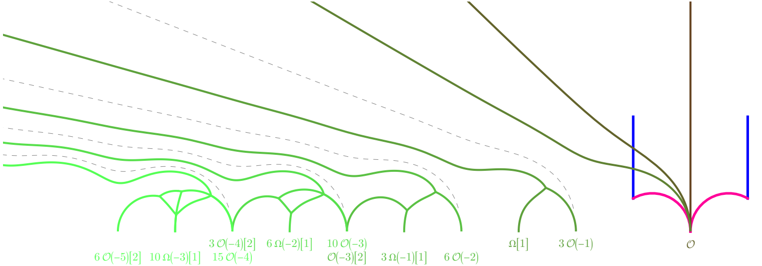
Douglas Fiol Romelsberger'00, Beaujard BP Manschot'20

Case studies

$\gamma = [0, 1, 1) = \mathrm{ch}\,\mathcal{O}_C$: $\Omega_{t \gg 1} = K_3(1, 2) K_3(1, 3)^{n-1} = y^2 + 1 + 1/y^2$

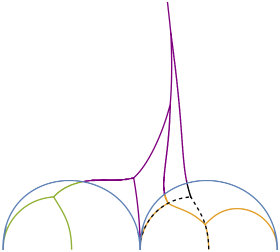


$\gamma = [1, 0, 1) = \mathrm{ch}\,\mathcal{O}$: $\Omega_{t \gg 1} = K_3(1, 3) \dots K_3(1, 3n) = 1$



SAFC along the physical slice

- In general, trees reaching the large volume region have a two-stage structure, with initial rays from a finite set of exceptional collections $\{E_1(m_i), E_2(m_i), E_3(m_i)\}$, which scatter in the vicinity of orbifold points $\tau = \tau_o + m_i$, and then further interact in the large volume region.



- The SFAC is proved by analyzing the possible leaves, defining a monotonic cost function φ and classifying allowed trees...

Conclusion - outlook

- The scattering diagram is the proper mathematical framework for the attractor flow tree formula in the case of local CY3. This is because $Z(\gamma)$ is holomorphic on $\mathcal{M}_K$, so the gradient flow preserves the phase $\arg Z(\gamma)$.
- This provides an effective way of computing (unframed) BPS invariants in any chamber, and a natural decomposition into elementary constituents. Is this mathematically meaningful ? Does it help e.g. in understanding modularity ?
- It would be interesting to extend this description to other toric CY3, such as local del Pezzo surfaces.
- For compact CY3, $Z(\gamma) = e^{K/2} Z_{\rm hol}(\gamma)$ is not longer holomorphic, so $\arg Z(\gamma)$ is not constant along the flow. Can one establish the Split Attractor Flow Tree formula in such context ?

Thanks for your attention !