Counting Calabi-Yau black holes with (mock) modular forms

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References

- *"Black holes and higher depth mock modular forms"*, with S. Alexandrov, Comm.Math.Phys. 374 (2019) 549 [arXiv:1808.08479]
- *"S-duality and refined BPS indices"*, with S. Alexandrov and J. Manschot, Comm.Math.Phys. 380 (2020) 755 [arXiv:1910.03098]
- *"Modular bootstrap for D4-D2-D0 indices on compact Calabi-Yau threefolds"*, with S. Alexandrov, N. Gaddam, J. Manschot [arXiv:2204.02207], to appear in Adv. Th. Math. Phys.
- *"Quantum geometry, stability and modularity"*, with S. Alexandrov, S. Feyzbakhsh, A. Klemm, T. Schimannek [arXiv:2301.08066]+ work in progress

- A driving force in high energy theory has been the quest for a microscopic explanation of the Bekenstein-Hawking entropy of black holes.
- String Theory provides a quantitative answer to this question in the context of BPS black holes in vacua with extended SUSY: at weak string coupling, black hole micro-states arise as bound states of D-branes (along with F-strings and NS5-branes) wrapped on the internal manifold, and can (sometimes) be counted efficiently.
- Besides confirming the consistency of string theory as a theory of quantum gravity, this has opened up many fruitful connections with mathematics.

BPS indices and Donaldson-Thomas invariants

- In the context of type IIA strings compactified on a Calabi-Yau three-fold X, BPS states are described mathematically by stable objects in the derived category of coherent sheaves $\mathcal{C} = D^b\mathrm{Coh}X$. The Chern character $\gamma = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2, \mathrm{ch}_3)$ is identified as the electromagnetic charge, or D6-D4-D2-D0-brane charge.
- The problem becomes a question in enumerative geometry: for fixed $\gamma \in K(X)$, compute the Donaldson-Thomas invariant $\Omega_z(\gamma)$ counting (semi)stable objects of class γ for a Bridgeland stability condition $z \in \mathrm{Stab}\,\mathcal{C}$, and determine its growth as $|\gamma| \to \infty$.
- Physical arguments predict that suitable generating series of rank 0 DT invariants (counting D4-D2-D0 bound states) should have specific modular properties. This gives very good control on their asymptotic growth, and allows to check whether $\Omega_z(\gamma) \simeq e^{S_{BH}(\gamma)}$.

Simplest example: Abelian three-fold

- For $X = T^6$, $\Omega_z(\gamma)$ depends only on a certain quartic polynomial $I_4(\gamma)$ in the charges, and is moduli independent. It is given by the Fourier coefficient $c(I_4(\gamma)+1)$ of a weak modular form,

$$\frac{\theta_3(2\tau)}{\eta^6(4\tau)} = \sum_{n\geq 0} c(n)\, \mathrm{q}^{n-1} = \frac{1}{\mathrm{q}} + 2 + 8\mathrm{q}^3 + 12\mathrm{q}^4 + 39\mathrm{q}^7 + 56\mathrm{q}^8 + \dots$$

Moore Maldacena Strominger 1999, BP 2005, Shih Strominger Yin 2005

Bryan Oberdieck Pandharipande Yin'15

- Recall that $f(\tau) := \sum_{n\geq 0} c(n)\mathrm{q}^{n-\Delta}$ (with $\mathrm{q} = e^{2\pi\mathrm{i}\tau}$, $\mathrm{Im}\tau > 0$) is a modular form of weight w if $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subset SL(2,\mathbb{Z})$,

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^w f(\tau) \quad \Rightarrow \quad c(n) \overset{n\to\infty}{\sim} \exp\left(4\pi\sqrt{\Delta(n-\Delta)}\right)$$

in agreement with $S_{BH}(\gamma) = \frac{1}{4}A(\gamma)$.

Wall-crossing and mock modularity

- For a general CY3, the story is more involved and interesting. First, $\Omega_z(\gamma)$ depends on the Kähler parameters z (more generally, on the stability condition), with a complicated chamber structure.
- Second, the generating series of BPS indices in the attractor chamber, denoted by $\Omega_\star(\gamma)$, are generally not modular but rather mock modular. *[Dabholkar Murthy Zagier 2012]*
- A (depth one) mock modular form of weight w transforms inhomogeneously under $\Gamma \subset SL(2,\mathbb{Z})$ (or $Mp(2,\mathbb{Z})$ if $w \in \mathbb{Z}+\frac{1}{2}$)

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^w \left[f(\tau) - \int_{-d/c}^{\mathrm{i}\infty} \overline{g(-\bar{\rho})} (\tau+\rho)^{-w} \mathrm{d}\rho \right]$$

where $g(\tau)$ is an ordinary modular form of weight $2-w$, known as the shadow.

Wall-crossing and mock modularity

- Equivalently, the non-holomorphic completion

$$\widehat{f}(\tau,\bar\tau) := f(\tau) + \int_{-\bar\tau}^{i\infty} \overline{g(-\bar\rho)}(\tau+\rho)^{-w}\mathrm{d}\rho$$

transforms like a modular form of weight w. Fourier coefficients still grow as $c(n) \sim \exp\left(4\pi\sqrt{\Delta(n-\Delta)}\right)$ but subleading corrections are markedly different.

- The Ramanujan's mock θ-functions belong to this class, along with indefinite theta series of Lorentzian signature $(1, n-1)$ *[Zwegers'02]*
- For $X = K3 \times T^2$ (or similar vacua with $\mathcal{N} = 4$ SUSY), generating series of $1/4$-BPS indices fall in this class, with shadow determined by $1/2$-BPS indices. This is because wall-crossing only involves bound states of two 1/2-BPS dyons.

Wall-crossing and mock modularity

- For a generic CY3, BPS bound states can in principle arise from an arbitrary number of constituents, hence generating series of BPS indices are expected to be higher depth mock modular forms, with specific modular anomaly *[Alexandrov Manschot BP'18-19]*
- Our goal is to exploit these (mock) modular properties to determine D4-D2-D0 indices (or rank 0 Donaldson-Thomas invariants) for compact Calabi-Yau manifolds such $\mathbb{P}^4[5]$ and other one-parameter models, revisiting the analysis in *[Gaiotto Strominger Yin '06-07]* and many subsequent works.
- A key tool will be recent results in mathematical literature *[Feyzbakhsh and Thomas'21-22]*, which relate D4-D2-D0 indices to Gopakumar-Vafa invariants.

Outline

1. Review some mathematical background on Bridgeland stability conditions on $\mathcal{C} = D^b\mathrm{Coh}X$
2. Spell out the modularity properties of rank 0 DT invariants on a general compact CY threefold
3. Test modularity for compact CY threefolds with $b_2(X) = 1$, using recent results of S. Feyzbakhsh and R. Thomas
4. Obtain new constraints on higher genus GW/GV invariants from modularity of rank 0 DT invariants

Mathematical preliminaries

- Let X a compact CY threefold, and $\mathcal{C} = D^b\mathrm{Coh}X$ the bounded derived category of coherent sheaves. Objects $E \in \mathcal{C}$ are bounded complexes of coherent sheaves $\mathcal{E}^k$ on X,

$$E = (\cdots \stackrel{d^{-2}}{\to} \mathcal{E}^{-1} \stackrel{d^{-1}}{\to} \mathcal{E}^0 \stackrel{d^0}{\to} \mathcal{E}^1 \stackrel{d^1}{\to} \dots)\,,$$

  with morphisms $d^k : \mathcal{E}^k \to \mathcal{E}^{k+1}$ such that $d^{k+1}d^k = 0$. Physically, $\mathcal{E}^k$ describe D6-branes for k even, or anti D6-branes for k odd, and d^k are open strings .
- $\mathcal{C}$ is graded by the Grothendieck group $K(\mathcal{C})$. Let $\Gamma \subset H^{\rm even}(X,\mathbb{Q})$ be the image of $K(\mathcal{C})$ under $E \mapsto \mathrm{ch}\, E = \sum_k (-1)^k \mathrm{ch}\, \mathcal{E}_k$. The lattice of electromagnetic charges Γ is equipped with the skew-symmetric (Dirac-Schwinger-Zwanziger) pairing

$$\langle E, E' \rangle = \chi(E', E) = \int_X (\mathrm{ch}\, E')^\vee\, \mathrm{ch}(E)\, \mathrm{Td}(TX) \in \mathbb{Z}$$

Bridgeland stability conditions

- Stability conditions are pairs $\sigma = (Z, \mathcal{A})$, where $Z : \Gamma \to \mathbb{C}$ is a linear map (the central charge) and $\mathcal{A} \subset \mathcal{C}$ is an Abelian subcategory (heart of bounded *t*-structure), subject to certain compatibility conditions. In particular, $\mathrm{Im} Z(E) \geq 0 \; \forall E \in \mathcal{A}$.
- Let $\mathcal{S} = \mathrm{Stab}(\mathcal{C})$ be the space of of stability conditions. If not empty, then it is a complex manifold of dimension $\mathrm{rk}\,\Gamma = b_{\mathrm{even}}(X)$, locally parametrized by $Z(\gamma_i)$ with γ_i a basis of Γ.
- Stability conditions are known to exist only for a handful of CY threefolds, including the quintic in $\mathbb{P}^4$ *[Li'18]*. Their construction depends on the conjectural Bayer-Macrì-Toda (BMT) inequality. Weak stability conditions are much easier to construct.

Physical stability conditions

- Physics/Mirror symmetry conjecturally selects a subspace $\Pi \subset \mathrm{Stab}\,\mathcal{C}$, known as 'physical slice' or slice of Π-stability conditions, parametrized by complexified Kähler structure of X, or complex structure of $\widehat{X}$. Hence $\dim_{\mathbb{C}} \Pi = b_2(X) + 1 = b_3(\widehat{X})$.
- Along this slice, the central charge is given by the period

$$Z(\gamma) = \int_{\hat{\gamma}} \Omega_{3,0}$$

  of the holomorphic 3-form on $\widehat{X}$ on a dual 3-cycle $\hat{\gamma} \in H_3(\hat{X}, \mathbb{Z})$.
- Near the large volume point in $\mathcal{M}_K(X)$, or MUM point in $\mathcal{M}_{cx}(\widehat{X})$,

$$Z(E) \sim -\int_X e^{-z^a H_a} \sqrt{Td(TX)}\, \mathrm{ch}(E)$$

  where H_a is a basis of $H^2(X, \mathbb{Z})$, and $z^a = b^a + \mathrm{i} t^a$ are the complexified Kähler moduli.

Generalized Donaldson-Thomas invariants

- Given a (weak) stability condition $\sigma = (Z, \mathcal{A})$, an object $F \in \mathcal{A}$ is called σ-semi-stable if $\arg Z(F') \leq \arg Z(F)$ for every non-zero subobject $F' \subset F$ (where $0 < \arg Z \leq \pi$).
- Let $\mathcal{M}_\sigma(\gamma)$ be the moduli stack of σ-semi-stable objects of class γ in $\mathcal{A}$. Following *[Joyce-Song'08]* one can associate the DT invariant $\bar\Omega_\sigma(\gamma) \in \mathbb{Q}$. When $\mathcal{M}_\sigma(\gamma)$ is a smooth projective variety, then $\bar\Omega_\sigma(\gamma) = (-1)^{\dim_\mathbb{C} \mathcal{M}_\sigma(\gamma)} \chi(\mathcal{M}_\sigma(\gamma))$ is integer.
- Conjecturally, the invariants $\Omega_\sigma(\gamma) := \sum_{m|\gamma} \mu(m) \frac{\bar\Omega_\sigma(\gamma/m)}{m^2}$ are integer, and coincide with the physical BPS indices.
- Examples:
  1. $\Omega_\sigma(k[pt]) = -\chi_X$ for all $k \geq 1$ throughout the space of geometric stability conditions.
  2. For any $\beta \in H_2(X, \mathbb{Z})$, $\Omega_\sigma([\beta] + k[pt]) = GV_\beta^{(0)}$ for all $k \geq 0$ in the large volume limit.

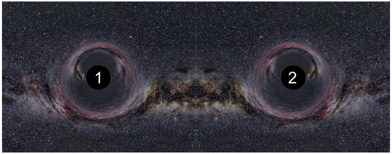
Wall-crossing

- The invariants $\bar{\Omega}_\sigma(\gamma)$ are locally constant on $\mathcal{S}$, but jump across walls of instability (or marginal stability), where the central charge $Z(\gamma)$ aligns with $Z(\gamma')$ where $\gamma' = \mathrm{ch}\, E'$ for a subobject $E' \subset E$. The jump is governed by a universal wall-crossing formula.

Joyce Song'08; Kontsevich Soibelman'08

- Physically, the jump corresponds to the (dis)appearance of multi-centered black hole bound states. In the simplest case,

$$\Delta\bar{\Omega}(\gamma_1+\gamma_2) = (-1)^{\langle\gamma_1,\gamma_2\rangle+1} |\langle\gamma_1,\gamma_2\rangle|\, \bar{\Omega}(\gamma_1)\, \bar{\Omega}(\gamma_2)$$



GV/DT/PT relation

- For a single D6-brane, the DT-invariant $DT(q,n) = \Omega(1,0,q,n)$ at large volume can be computed via the GV/DT relation

$$\sum_{Q,n} DT(Q,n)\,(-\mathrm{q})^n v^Q = M(\mathrm{q})^{\chi_X} \prod_{Q,g,\ell} \left(1-\mathrm{q}^{g-\ell-1} v^Q\right)^{(-1)^{g+\ell}\binom{2g-2}{\ell}\,\mathrm{GV}_Q^{(g)}}$$

  where $M(\mathrm{q}) = \prod_{n\geq 1}(1-\mathrm{q}^n)^{-n}$ is the Mac-Mahon function.

  Maulik Nekrasov Okounkov Pandharipande'06

- The topological string partition function is given by

$$\Psi_{\mathrm{top}}(z,\lambda) = M(\mathrm{q})^{-\chi_X/2} Z_{DT}\left(\mathrm{q}=-e^{\mathrm{i}\lambda}, v=e^{2\pi\mathrm{i}z/\lambda}\right)$$

  can be computed by the direct integration method, assuming conifold gap conditions and Castelnuovo-type bounds $g \leq g_{\max}(Q)$ *[BCOV 93, Huang Klemm Quackenbush'06]*.

Rank 0 DT invariants from GV invariants

- Thm [Feyzbakhsh Thomas'20-22]: *Let* (X, H) *be any polarized CY3 satisfying the BMT conjecture (see below). Then rank* r *DT invariants for any* $r \geq 0$ *are determined by rank 1 DT invariants, hence by GV invariants.*
- This relies on wall-crossing in a family of weak stability conditions parametrized by $(b, t) \in \mathbb{R} \times \mathbb{R}^+$, with degenerate central charge

$$Z^{\rm tilt}_{b,t}(E) = \frac{\rm i}{6} t^3 \operatorname{ch}_0 - \frac{1}{2} t^2 \operatorname{ch}^b_1 - {\rm i} t \operatorname{ch}^b_2 + 0 \operatorname{ch}^b_3$$

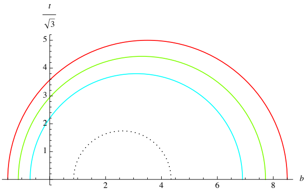
where $\operatorname{ch}^b_k = \int_X H^{3-k} e^{-bH} \operatorname{ch}(E)$. The BMT conjecture states that tilt-semistable objects exist only when $C_k := \operatorname{ch}^0_k$ satisfy

$$(C_1^2 - 2C_0 C_2)(\tfrac{1}{2} b^2 + \tfrac{1}{6} t^2) + (3C_0 C_3 - C_1 C_2) b + (2C_2^2 - 3C_1 C_3) \geq 0$$

Bayer Macri Toda'11; Bayer Macri Stellari'16

Rank 0 DT invariants from GV invariants

- Walls for tilt stability are nested half-circles in the Poincaré upper half-plane spanned by $z = b + \mathrm{i}\frac{t}{\sqrt{3}}$.



- The BMT inequality provides an empty chamber whenever the discriminant at $t = 0$ is positive:

$$8C_0C_2^3 + 6C_1^3C_3 + 9C_0^2C_3^2 - 3C_1^2C_2^2 - 18C_0C_1C_2C_3 \geq 0$$
$$\Updownarrow$$
$$\frac{8}{9\kappa}p^0(q_1)^3 - \frac{2}{3}\kappa q_0(p^1)^3 - (p^0q_0)^2 + \frac{1}{3}(p^1q_1)^2 - 2p^0p^1q_0q_1 \leq 0$$

hence when single centered black hole solutions are ruled out !

S-duality constraints on D4-D2-D0 indices

- For classes supported on an irreducible divisor $\mathcal{D}$ of class $p^a\gamma_a \in \Lambda = H_4(X,\mathbb{Z})$, the generating series of rank 0 DT invariants

$$h_{p^a,q_a}(\tau) = \sum_n \bar\Omega_\star(0,p^a,q_a,n)\, \mathrm{q}^{n+\frac12 q_a\kappa^{ab}q_b+\frac12 p^a q_a-\frac{\chi(\mathcal{D})}{24}}$$

  should be a vector-valued, weakly holomorphic modular form of weight $w=-\frac12 b_2(X)-1$ and prescribed multiplier system.
- Here, $\bar\Omega_\star(0,p^a,q_a,n)$ is the index in the large volume attractor chamber

$$\bar\Omega_\star(\gamma) = \lim_{\lambda\to+\infty} \bar\Omega_{-\kappa^{ab}q_b+\mathrm{i}\lambda p^a}(\gamma)$$

  where κ^{ab} is the inverse of the quadratic form $\kappa_{ab}=\kappa_{abc}p^c$ with Lorentzian signature $(1,b_2(X)-1)$.

S-duality constraints on D4-D2-D0 indices

- By construction, $\Omega_\star(0, p^a, q_a, n)$ is invariant under tensoring with a line bundle $\mathcal{O}(m^a H_a)$ (aka spectral flow)

$$q_a \to q_a - \kappa_{ab} m^b \,, \quad n \mapsto n - m^a q_a + \tfrac{1}{2}\kappa_{ab} m^a m^b$$

Thus, the D2-brane charge q_a can be restricted to the finite set Λ^*/Λ, of cardinal $|\det(\kappa_{ab})|$.

- h_{p^a,q_a} transforms under the Weil representation of $\mathrm{Mp}(2,\mathbb{Z})$ associated to the lattice Λ, e.g.

$$h_{p^a,q_a}(-1/\tau) = \sum_{q'_a \in \Lambda^*/\Lambda} \frac{e^{-2\pi \mathrm{i} \kappa^{ab} q_a q'_b + \frac{\mathrm{i}\pi}{4}(b_2(X) + 2\chi(\mathcal{O}_\mathcal{D}) - 2)}}{\tau^{1+\frac{1}{2}b_2(X)} \sqrt{|\det(\kappa_{ab})|}} h_{p^a,q'_a}(\tau)$$

- Equivalently, $Z_p(\tau, v) = \sum_{q \in \Lambda^*/\Lambda} h_{p,q}(\tau) \Theta_q(\tau, v)$, where $\Theta_q(\tau, v)$ is the Siegel theta series for the indefinite lattice (Λ, κ_{ab}), transforms as a (non-holomorphic) Jacobi form – the five-brane elliptic genus.

Maldacena Strominger Witten'98, Cheng de Boer Dijkgraaf Manschot Verlinde'06

Mock modularity constraints on D4-D2-D0 indices

- For γ supported on a reducible divisor class $\mathcal{D} = \sum_{i=1}^{n\geq 2} \mathcal{D}_i$, the generating series h_p (omitting q index for brevity) should be a vector-valued mock modular form of depth $n-1$.

Alexandrov Banerjee Manschot BP '16-19

- There exists explicit non-holomorphic theta series such that

$$\widehat{h}_p(\tau,\bar\tau) = h_p(\tau) + \sum_{p=\sum_{i=1}^{n\geq 2} p_i} \Theta_n(\{p_i\},\tau,\bar\tau) \prod_{i=1}^{n} h_{p_i}(\tau)$$

transforms as a modular form of weight $-\frac{1}{2}b_2(X)-1$. The completion satisfies an explicit holomorphic anomaly equation,

$$\partial_{\bar\tau}\widehat{h}_p(\tau,\bar\tau) = \sum_{p=\sum_{i=1}^{n\geq 2} p_i} \widehat{\Theta}_n(\{p_i\},\tau,\bar\tau) \prod_{i=1}^{n} \widehat{h}_{p_i}(\tau,\bar\tau)$$

Indefinite theta series

- Θ_n and $\widehat{\Theta}_n$ belongs to the class of indefinite theta series

$$\vartheta_{\Phi,q}(\tau,\bar\tau) = \sum_{k\in\Lambda+q} \Phi\left(\sqrt{2\tau_2}k\right)\, e^{-\mathrm{i}\pi\tau Q(k)}$$

  where (Λ, Q) is an even lattice of signature $(r, d-r)$, $q\in\Lambda^*/\Lambda$. Conditions for modularity were spelled out in *[Vignéras'78]*

- The relevant lattice for Θ_n and $\widehat{\Theta}_n$ is $\Lambda = H^2(X,\mathbb{Z})^{\oplus(n-1)}$, with signature $(r, d-r) = (n-1)(1, b_2(X)-1)$. The relevant Φ is a linear combination of generalized error functions $\mathcal{E}_{n-1}(\{C_i\},x) := e^{\pi Q(x_+)} \star \prod_{i=1}^{n-1} \mathrm{sgn}(C_i,x)$ where $\star$ is the convolution product.

Alexandrov Banerjee Manschot BP 2016; Nazaroglu 2016

Modularity for one-modulus compact CY

- Our aim is now to exploit these (mock) modularity constraints to determine D4-D2-D0 indices for simple compact CY threefolds with one Kähler modulus, revisiting and extending the analysis of *[Gaiotto Strominger Yin'06]*
- For $N = 1$, the generating series

$$h_{1,q} = \sum_{n\in\mathbb{Z}} \Omega_\star(0,1,q,n)\, \mathrm{q}^{n+\frac{q^2}{2\kappa}+\frac{q}{2}-\frac{\chi(\mathcal{D})}{24}} \;, \quad q \in \mathbb{Z}/\kappa\mathbb{Z}$$

should transform as a vector-valued modular form of weight $-\frac{3}{2}$ in the Weil representation of $\mathbb{Z}[\kappa]$ with $\kappa = H^3$.

Hypergeometric CY threefolds

X	χ_X	κ	$c_2(TX)$	$\chi(\mathcal{O}_\mathcal{D})$	n_1	C_1
$X_5(1^5)$	-200	5	50	5	7	0
$X_6(1^4,2)$	-204	3	42	4	4	0
$X_8(1^4,4)$	-296	2	44	4	4	0
$X_{10}(1^3,2,5)$	-288	1	34	3	2	0
$X_{4,3}(1^5,2)$	-156	6	48	5	9	0
$X_{4,4}(1^4,2^2)$	-144	4	40	4	6	1
$X_{6,2}(1^5,3)$	-256	4	52	5	7	0
$X_{6,4}(1^3,2^2,3)$	-156	2	32	3	3	0
$X_{6,6}(1^2,2^2,3^2)$	-120	1	22	2	1	0
$X_{3,3}(1^6)$	-144	9	54	6	14	1
$X_{4,2}(1^6)$	-176	8	56	6	15	1
$X_{3,2,2}(1^7)$	-144	12	60	7	21	1
$X_{2,2,2,2}(1^8)$	-128	16	64	8	33	3

Abelian D4-D2-D0 invariants

- The space of vector-valued modular form of weight $-\frac{3}{2}$ has dimension $n_1 - C_1$, where n_1 is the number of polar terms, and C_1 is the dimension of the space of cusp forms in dual weight $2 + \frac{3}{2}$.

Bantay Gannon'07, Manschot Moore'07, Manschot'08

- An overcomplete basis is given for κ even by

$$\frac{E_4^a E_6^b}{\eta^{4\kappa+c_2}} D^\ell(\vartheta_q^{(\kappa)}) \quad \text{with} \quad \vartheta_q^{(\kappa)} = \sum_{k \in \mathbb{Z}+\frac{q}{\kappa}} \mathrm{q}^{\frac{1}{2}\kappa k^2}$$

where $D = \mathrm{q}\partial_\mathrm{q} - \frac{w}{12} E_2$, is the Serre derivative and $4a + 6b + 2\ell - 2\kappa - \frac{c_2}{2} + \frac{1}{2} = -\frac{3}{2}$.

- For κ odd, the same works with $\vartheta_q^{(\kappa)} = \sum_{k \in \mathbb{Z}+\frac{q}{\kappa}+\frac{1}{2}} (-1)^{\kappa k}\, k \mathrm{q}^{\frac{1}{2}\kappa k^2}$.

Rank 0 DT invariants from GV invariants

- For a D4-D2-D0 charge $\gamma = (0, r, q, n)$ close enough to the (usual) Bogomolov-Gieseker bound, *[Toda'13, Feyzbakhsh'22]*

$$\bar{\Omega}_{r,q}(n) = \sum_{r_i, Q_i, n_i} \langle \gamma_1, \gamma_2 \rangle \, \mathrm{DT}(Q_1, n_1) \, \mathrm{PT}(Q_2, n_2)$$

  where $\mathrm{DT}(Q_1, n_1), \mathrm{PT}(Q_2, n_2)$ counts BPS states with charge $\gamma_1 = (1, 0, -Q_1, -n_1), \gamma_2 = (-1, 0, Q_2, -n_2)$, respectively

- Alternatively, one can study wall crossing for $\gamma = (-1, 0, q, n)$. For (q, n) large enough, there is an empty chamber and a single wall corresponding to $\overline{D6} \to \overline{D6} + D4$ contributes to $PT(q, n)$:

$$PT(q, n) = \langle \overline{D6(1)}, \gamma_{D4} \rangle \, \bar{\Omega}(\gamma_{D4})$$

  with $\overline{D6(1)} := \mathcal{O}_X(H)[1]$ and $\gamma_{D4} = (0, 1, q, n)$ *[Feyzbakhsh'22]*.

Modular predictions for D4-D2-D0

- Using this idea, we can compute all polar terms and many non-polar ones, and verify modular invariance. E.g. for X_5:

$$h_{1,0} = \mathrm{q}^{-\frac{55}{24}} \Big(\underline{5 - 800\mathrm{q} + 58500\mathrm{q}^2} + 5817125\mathrm{q}^3 + 75474060100\mathrm{q}^4 + 28096675153255\mathrm{q}^5 + 3756542229485475\mathrm{q}^6 + 277591744202815875\mathrm{q}^7 + 13610985014709888750\mathrm{q}^8 + \dots \Big),$$

$$h_{1,\pm 1} = \mathrm{q}^{-\frac{55}{24}+\frac{3}{5}} \Big(\underline{0 + 8625\mathrm{q}} - 1138500\mathrm{q}^2 + 3777474000\mathrm{q}^3 + 3102750380125\mathrm{q}^4 + 577727215123000\mathrm{q}^5 + \dots \Big)$$

$$h_{1,\pm 2} = \mathrm{q}^{-\frac{55}{24}+\frac{2}{5}} \Big(\underline{0 + 0\mathrm{q}} - 1218500\mathrm{q}^2 + 441969250\mathrm{q}^3 + 953712511250\mathrm{q}^4 + 217571250023750\mathrm{q}^5 + 22258695264509625\mathrm{q}^6 + \dots \Big)$$

Alexandrov, Feyzbakhsh, Klemm, BP, Schimannek'23

Mock modularity for non-Abelian D4-D2-D0 indices

- For D4-D2-D0 indices with $N = 2$ units of D4-brane charge, $\{h_{2,q}, q \in \mathbb{Z}/(2\kappa\mathbb{Z})\}$ should transform as a vv mock modular form with modular completion

$$\widehat{h}_{2,q}(\tau,\bar\tau) = h_{2,q}(\tau) + \sum_{q_1,q_2=0}^{\kappa-1} \delta^{(\kappa)}_{q_1+q_2-q}\, \Theta^{(\kappa)}_{q_2-q_1+\kappa}\, h_{1,q_1}\, h_{1,q_2}$$

where

$$\Theta^{(\kappa)}_q(\tau,\bar\tau) = \frac{(-1)^q}{8\pi} \sum_{k\in 2\kappa\mathbb{Z}+q} |k|\, \beta\left(\frac{\tau_2 k^2}{\kappa}\right) e^{-\frac{\pi i \tau}{2\kappa} k^2},$$

and $\beta(x) = 2|x|^{-1/2} e^{-\pi x} - 2\pi \mathrm{Erfc}(\sqrt{\pi|x|})$.

- The series $\Theta^{(\kappa)}_q$ is convergent but not modular invariant.

Mock modularity for non-Abelian D4-D2-D0 indices

- Suppose there exists a holomorphic function $g_q^{(\kappa)}$ such that $\Theta_q^{(\kappa)} + g_q^{(\kappa)}$ transforms as a vv modular form. Then

$$\widetilde{h}_{2,q}(\tau,\bar{\tau}) = h_{2,q}(\tau) - \sum_{q_1,q_2=0}^{\kappa-1} \delta^{(\kappa)}_{q_1+q_2-q}\, g^{(\kappa)}_{q_2-q_1+\kappa}\, h_{1,q_1}\, h_{1,q_2}$$

will be an ordinary weak holomorphic vv modular form, hence uniquely determined by its polar part.

- For $\kappa = 1$, the series $\Theta_q^{(1)}$ is the one appearing in the modular completion of the generating series of Hurwitz class numbers *[Hirzebruch Zagier 1973]* (or rank 2 Vafa-Witten invariants on $\mathbb{P}^2$)

$$\begin{aligned} H_0(\tau) &= -\tfrac{1}{12} + \tfrac{1}{2}\mathrm{q} + \mathrm{q}^2 + \tfrac{4}{3}\mathrm{q}^3 + \tfrac{3}{2}\mathrm{q}^4 + \dots \\ H_1(\tau) &= \mathrm{q}^{\frac{3}{4}}\left(\tfrac{1}{3} + \mathrm{q} + \mathrm{q}^2 + 2\mathrm{q}^3 + \mathrm{q}^4 + \dots\right) \end{aligned}$$

Thus we can choose $g_q^{(1)} = H_q(\tau)$.

Mock modularity for non-Abelian D4-D2-D0 indices

X	χ_X	κ	c_2	$\chi(\mathcal{O}_{2\mathcal{D}})$	n_2	C_2
$X_5(1^5)$	-200	5	50	15	36	1
$X_6(1^4,2)$	-204	3	42	11	19	1
$X_8(1^4,4)$	-296	2	44	10	14	1
$X_{10}(1^3,2,5)$	-288	1	34	7	7	0
$X_{4,3}(1^5,2)$	-156	6	48	16	42	0
$X_{4,4}(1^4,2^2)$	-144	4	40	12	25	1
$X_{6,2}(1^5,3)$	-256	4	52	14	30	1
$X_{6,4}(1^3,2^2,3)$	-156	2	32	8	11	1
$X_{6,6}(1^2,2^2,3^2)$	-120	1	5	2	5	0
$X_{3,3}(1^6)$	-144	9	54	21	78	3
$X_{4,2}(1^6)$	-176	8	56	20	69	3
$X_{3,2,2}(1^7)$	-144	12	60	26	117	0
$X_{2,2,2,2}(1^8)$	-128	16	64	32	185	4

Mock modularity for non-Abelian D4-D2-D0 indices

- For X_{10}, we computed the 7 polar terms + 4 non-polar terms and found a unique mock modular form reproducing this data:

$$h_{2,\mu} = \frac{5397523E_4^{12}+70149738E_4^9E_6^2-12112656E_4^6E_6^4-61127530E_4^3E_6^6-2307075E_6^8}{46438023168\eta^{100}}\vartheta_\mu^{(1,2)} + \frac{-10826123E_4^{10}E_6-14574207E_4^7E_6^3+20196255E_4^4E_6^5+5204075E_4E_6^7}{1934917632\eta^{100}}D\vartheta_\mu^{(1,2)} + (-1)^{\mu+1}H_{\mu+1}(\tau)\,h_1(\tau)^2$$

with $h_1 = \frac{203E_4^4+445E_4E_6^2}{216\,\eta^{35}} = \mathrm{q}^{-\frac{35}{24}}(\underline{3-575\mathrm{q}}+\dots)$, leading to integer DT invariants

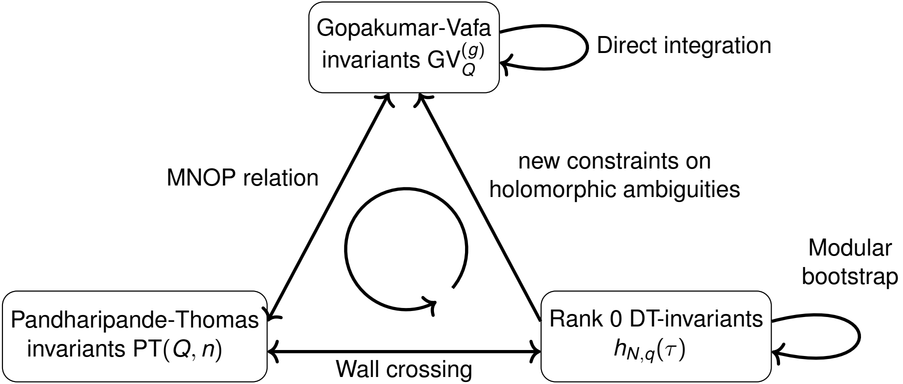
$$h_{2,0}^{(\mathrm{int})} = \mathrm{q}^{-\frac{19}{6}}\left(\underline{7-1728\mathrm{q}+203778\mathrm{q}^2-13717632\mathrm{q}^3}-23922034036\mathrm{q}^4+\dots\right)$$

$$h_{2,1}^{(\mathrm{int})} = \mathrm{q}^{-\frac{35}{12}}\left(\underline{-6+1430\mathrm{q}-1086092\mathrm{q}^2}+208065204\mathrm{q}^3+\dots\right)$$

- The extension to other one-parameter models is in progress.

Quantum geometry from stability and modularity

Conversely, we can use our knowledge of Abelian D4-D2-D0 invariants to compute GV invariants and push the direct integration method to higher genus !



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Quantum geometry from stability and modularity

X	χ_X	κ	type	$g_{\rm integ}$	$g_{\rm mod}$	$g_{\rm avail}$
$X_5(1^5)$	-200	5	F	53	69	64
$X_6(1^4,2)$	-204	3	F	48	63	48
$X_8(1^4,4)$	-296	2	F	60	80	60
$X_{10}(1^3,2,5)$	-288	1	F	50	91	65
$X_{4,3}(1^5,2)$	-156	6	F	20	24	24
$X_{6,4}(1^3,2^2,3)$	-156	2	F	14	17	17
$X_{6,6}(1^2,2^2,3^2)$	-120	1	K	18	21	21
$X_{4,4}(1^4,2^2)$	-144	4	K	26	34	34
$X_{3,3}(1^6)$	-144	9	K	29	33	33
$X_{4,2}(1^6)$	-176	8	C	50	64	50
$X_{6,2}(1^5,3)$	-256	4	C	63	78	42

Conclusion

- Wall-crossing in the full space of Bridgeland stability conditions provides a powerful tool for computing BPS indices, even though its physical interpretation away from Π-stability remains obscure.
- While modular properties of D4-D2-D0 indices are clear physically, their mathematical origin is mysterious in general, except for two special cases:
  1. For vertical D4-branes in torus-fibered CY3, it follows from the modularity of topological strings by Fourier-Mukai duality *[Klemm Manschot Wotschke'12, Oberdieck Pixton'17]*
  2. For vertical D4-branes in K3-fibered CY3, it follows from Noether-Lefschetz theory and results of Kudla-Millson and Borcherds *[Bouchard Creuztig Diaconescu Doran Quigley Sheshmani'16]*.
- Optimistically, mock modularity of D4-D2-D0 indices for arbitrary D4-brane charge might give enough constraints to fix the topological string amplitude to arbitrary genus...