BPS black holes, wall-crossing and mock modular forms

Boris Pioline

LPTHE
LABORATOIRE DE PHYSIQUE THEORIQUE ET HAUTES ENERGIES

CNRS

SORBONNE UNIVERSITÉ

Maths HEP Lunchtime Seminar
Durham, 14/12/2018

based on arXiv:1804.06928,1808.08479 with Sergei Alexandrov,

and earlier works 1605.05945,1702.05497 with S. Banerjee and J. Manschot

Precision counting of BPS black holes I

- String theory famously provides a statistical explanation of the Bekenstein-Hawking entropy of large supersymmetric black holes in terms of D-brane bound states. Those are described at weak coupling by a superconformal field theory, with entropy

$$S_{\rm CFT} = \lim_{L_0 \gg c} \log \Omega = 2\pi \sqrt{\frac{c}{6} L_0} = \frac{\mathcal{A}}{4 G_N} = S_{\rm BH}$$

Strominger Vafa 1996

- One obvious direction is to relax supersymmetry. Another is to include finite size effects, both on macroscopic side (including higher-derivative curvatures to the area law) and on microscopic side (considering moderate or small charges).

Precision counting of BPS black holes II

- For BPS black holes in $\mathcal{N}=8$ or $\mathcal{N}=4$ string vacua, the exact number of BPS black hole microstates $\Omega(\gamma, u)$ with charges $\gamma=(Q,P)$ is known to be given by a Fourier coefficient of a suitable meromorphic Siegel modular form,

$$\Omega(\gamma,u)=\oint_{\mathcal{C}(\gamma,u)} \frac{e^{2\pi \mathrm{i} \mathrm{Tr}(\tau\cdot\Gamma)}}{\Phi(\tau)}\,, \qquad \Gamma=\begin{pmatrix} Q^2 & Q\cdot P \\ Q\cdot P & P^2 \end{pmatrix}$$

Dijkgraaf Verlinde Verlinde '96; David Jatkar Sen '05-06; ...

- For large charges, $\log \Omega(Q,P)$ matches the BH-Wald entropy taking into account $\mathcal{R}^2$ and one-loop corrections to the entropy function. *[Cardoso de Wit Kappeli Mohaupt 2004; Banerjee Gutpa Mandal Sen 2011]*
- The result depends on the choice of contour $\mathcal{C}(\gamma,u)$, which reflects the dependence of $\Omega(\gamma,u)$ on the moduli $u\in\mathcal{M}_4$ at spatial infinity.

Precision counting of BPS black holes III

- When z crosses real codimension-1 walls in $\mathcal{M}_4$

$$W(\gamma_L, \gamma_R) = \{u \in \mathcal{M}_4, M(\gamma_L + \gamma_R) = M(\gamma_L) + M(\gamma_R)\}$$

where γ_L, γ_R are 1/2-BPS charge vectors, the contour $\mathcal{C}(\gamma, u)$ crosses a pole of $1/\Phi(\tau)$, so that the index Ω jumps according to the primitive wall-crossing formula

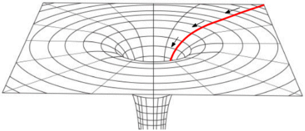
$$\Delta\Omega(\gamma_L + \gamma_R) = (-1)^{\langle\gamma_L,\gamma_R\rangle+1} \, |\langle\gamma_L, \gamma_R\rangle| \, \Omega(\gamma_L) \, \Omega(\gamma_R)$$

Denef Moore '07; Cheng, Verlinde '07; Sen '07-08

corresponding to contributions of bound states of two small 1/2-BPS black holes.

Precision counting of BPS black holes IV

- One may single out the contributions of single-centered black holes by evaluating $\Omega(\gamma, u)$ at the attractor point u_γ, where two-centered bound states are not allowed to form.



- The attractor indices $\Omega_*(\gamma) = \Omega(\gamma, u_\gamma)$ turn out to be Fourier coefficients of a vector-valued mock modular form. *[Dabholkar Murthy Zagier '12]*

Precision counting of $\mathcal{N}=2$ BPS black holes I

- In $\mathcal{N}=2$ string vacua, such as type II strings compactified on a CY threefold $\mathfrak{Y}$, the situation is far more complicated, due in part to the fact that $\mathcal{M}_4 = \mathcal{M}_V \times \mathcal{M}_H$ receives quantum corrections. The BPS mass $M(\gamma,u) = |Z(\gamma,u)|$ depends on the central charge function, a linear form in γ but complicated function of $u \in \mathcal{M}_V$.
- Unlike in $\mathcal{N} \geq 4$, BPS bound states can involve an arbitrary number of BPS constituents with charges $\{\gamma_i\}$ such that $\gamma = \sum_{i=1}^n \gamma_i$. In particular, across a wall where $Z(\gamma_L) \parallel Z(\gamma_R)$, all indices $\Omega(\gamma,u)$ with $\gamma = N_L\gamma_L + N_R\gamma_R$ may jump.
- Mathematically, the indices $\Omega(\gamma,u)$ are generalized Donaldson-Thomas invariants of the derived category of coherent sheaves on $\mathfrak{Y}$ (in type IIA), or the Fukaya category of Lagrangians in type IIB. They depend sensitively on the stability condition $Z(\gamma,u)$.

Precision counting of $\mathcal{N}=2$ BPS black holes II

- The jump $\Delta\Omega(N_L\gamma_L+N_R\gamma_R)$ was first computed by Joyce-Song and Kontsevich-Soibelman in the mathematics literature, and (re)derived physically from the SUSY quantum mechanics of multi-centered black holes.

Denef Moore '07; de Boer et al '08; Andriyash et al '10, Manschot BP Sen '10

- A natural chamber to consider is the attractor chamber, where stable two-particle bound states are ruled out. However, the attractor index $\Omega_*(\gamma)=\Omega(\gamma,u_\gamma)$ may still get contributions from bound states of $n\geq 3$ constituents allowing for scaling solutions.
- Eventually, one would like to extract the number $\Omega_S(\gamma)$ of single-centered black hole microstates, which is in principle computable recursively from $\Omega_*(\gamma)$ using the (conjectural) Coulomb branch formula.

Manschot BP Sen 2011-14

Precision counting of $\mathcal{N} = 2$ BPS black holes III

- A natural sector is to consider D4-D2-D0 branes wrapped on a divisor $\mathcal{D} \subset \mathfrak{Y}$. In M-theory on $\mathfrak{Y} \times S_1$, this configuration lifts to an M5-brane wrapping $\mathcal{D} \times S_1$, described at low energy by a (0,4) 'black string SCFT' with computable central charges.

Maldacena Strominger Witten '97

- One expects that the generating function of D4-D2-D0 indices

$$\chi_{p^a}(\tau, y) \sim \sum_{q_a, q_0} \Omega(0, p^a, q_a, q_0; u)\, e^{2\pi i(\tau q_0 + y^a q_a)}$$

is given by the modified elliptic genus of this SCFT and therefore should be a weak Jacobi form of fixed weight, index and multiplier system.

Gaiotto Strominger Yin '06, de Boer et al '06, Denef Moore '07

Precision counting of $\mathcal{N} = 2$ BPS black holes IV

- This strategy was applied successfully to compute BPS indices for a single D4-brane on the quintic, using modularity plus explicit computations at small D0-brane charge. *[Gaiotto et al '05-06]*
- However, this expectation breaks down for non-primitive D4-brane charge, or more generally when the D4-brane wraps a reducible divisor, due to wall-crossing.
- We shall be interested in the generating function of BPS indices $\Omega_{\rm MSW}(\gamma) = \Omega(\gamma, u_\infty(\gamma))$ at the large volume attractor point

$$u_\infty^a(\gamma) = \lim_{\lambda \to +\infty} (-q^a + \mathrm{i}\lambda p^a)\,, \qquad \begin{cases} q^a = \kappa^{ab} q_b \\ \kappa_{ab} = \kappa_{abc} p^c \end{cases}$$

Precision counting of $\mathcal{N} = 2$ BPS black holes V

- One reason is that the $\Omega_{\rm MSW}(\gamma)$'s are invariant under spectral flow

$$q_a \to q_a - \kappa_{abc} p^b \epsilon^a \,, \quad q_0 \mapsto q_0 - \epsilon^a q_a + \frac{1}{2} \kappa_{abc} p^a \epsilon^b \epsilon^c$$

  In contrast, $\Omega(\gamma, u)$ is not invariant unless $b^a \mapsto b^a + \epsilon^a$. This is key for having correct behavior under elliptic transformations, and obtaining a theta series decomposition.

  Manschot 09, Alexandrov Manschot BP 12

- Another more physical reason is that at the large volume attractor points, solutions with multiple AdS_3 throats are disallowed, and all states should be excitations in a single black string SCFT.

  de Boer et al 08, Andriyash 08

- Note that $\Omega_{\rm MSW}(\gamma)$ may differ from $\Omega_*(\gamma)$, since stable bound states at large volume may become unstable at finite volume.

Modularity from S-duality I

- To determine the precise modular properties of generalized DT invariants, one can focus on a particular BPS-saturated coupling in the low-energy action of type IIA/$\mathfrak{Y} \times S_1(R)$, which receives contributions from Euclidean BPS black holes wrapped on S_1.

  [Gunaydin Neitzke BP Waldron '05]

- Namely, in $D = 3$ the moduli space factorizes as $\mathcal{M}_3 = \widetilde{\mathcal{M}}_V \times \mathcal{M}_H$, where both factors are quaternion-Kähler manifolds. As $R \to \infty$,

$$\widetilde{\mathcal{M}}_V \sim \text{c-map}(\mathcal{M}_V) + \sum_{\gamma} \Omega(\gamma, u)\, e^{-RM(\gamma)+2\pi i\langle\gamma,c\rangle} + \dots$$

  Cecotti Ferrara Girardello '89, Ferrara Sabharwal '90; Alexandrov BP Vandoren '08

- Since IIA/$\mathfrak{Y} \times S_1 =$ M/$\mathfrak{Y} \times T^2$, $\widetilde{\mathcal{M}}_V$ must admit an isometric action of $SL(2,\mathbb{Z})$, which stays unbroken in the large volume limit.

Modularity from S-duality II

- Main point: *S-duality determines the modular behavior of the generating function of DT invariants*. In particular, it reproduces the naive MSW modularity constraints when the divisor $\mathcal{D}$ wrapped by the D4-brane is irreducible.
- When $\mathcal{D}$ is a sum of $n \geq 2$ irreducible divisors, the generating function acquires a specific modular anomaly: they are now mock modular forms of depth $n-1$. *[Alexandrov Banerjee Manschot BP '16, Alexandrov BP '18]*
- *Remark: $\widetilde{\mathcal{M}}_V$ is also the hypermultiplet moduli space $\mathcal{M}_H$ in type IIB string theory compactified on $\mathfrak{Y}$, with $SL(2,\mathbb{Z})$ being the usual type IIB S-duality in $D=10$. Counting D4-D2-D0 bound states is equivalent to computing D3-D1-D(-1) instanton corrections to $\mathcal{M}_H$.*

Alexandrov, Banerjee, Manschot, Persson, BP, Saueressig, Vandoren '08-18

Outline

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1. Introduction
2. D-instanton corrections in twistor space
3. From DT invariants to MSW invariants
4. Mock modularity of MSW invariants

Vector multiplet moduli space in $D = 3$ I

- The VM moduli space $\mathcal{M} = \widetilde{\mathcal{M}}_V$ in M-theory compactified on $\mathfrak{Y} \times T^2$ has dimension $4b_2 + 4$:
  - τ: complex structure of T^2
  - t^a: Kähler moduli of $\mathfrak{Y}$ on a basis γ^a, $a = 1 \dots b_2$ of $H_2(\mathfrak{Y}, \mathbb{Z})$
  - (b^a, c^a): period of the 3-form on $\gamma^a \times S_1$
  - $\tilde{c}_a$: period of 6-form on $\gamma_a \times T^2$, γ_a basis of $H_4(\mathfrak{Y}, \mathbb{Z})$
  - $(\tilde{c}_0, \psi)$: dual of the KK gravitons
- In the large volume limit $t^a \to \infty$, $\mathcal{M}$ reduces to the c-map of the special Kähler space $\mathcal{M}_V$ with prepotential

$$F(X) = -\frac{1}{6}\kappa_{abc}\frac{X^a X^b X^c}{X^0} , \qquad u^a = \frac{X^a}{X^0} = b^a + \mathrm{i} t^a$$

Notation: $(p_1 p_2 p_3) = \kappa_{abc} p_1^a p_2^b p_3^c$.

Vector multiplet moduli space in $D = 3$ II

- In the limit $t^a \to \infty$, $\mathcal{M}$ admits an isometric action of $SL(2,\mathbb{R})$:

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}\,, \qquad t^a \mapsto |c\tau + d|\, t^a, \qquad \begin{pmatrix} c^a \\ b^a \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} c^a \\ b^a \end{pmatrix},$$

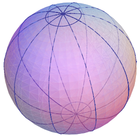
$$\tilde{c}_a \mapsto \tilde{c}_a\,, \qquad \begin{pmatrix} \tilde{c}_0 \\ \psi \end{pmatrix} \mapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} \tilde{c}_0 \\ \psi \end{pmatrix}$$

- $SL(2,\mathbb{R})$ is broken by worldsheet and D-instantons to $SL(2,\mathbb{Z})$

Robles-Llana Rocek Saueressig Theis Vandoren '05

- In absence of KK monopoles (or NS5-D5 instantons in IIB picture), the continuous isometries along $(\tilde{c}_0, \psi)$ are unbroken.

Twistorial description of instanton corrections I



- Instanton corrections to the QK metric are described in terms of a complex contact structure on the twistor space $\mathbb{P}^1 \to \mathcal{Z} \to \mathcal{M}$. The latter is specified by contact transformations relating local Darboux coordinates across BPS rays.

- In the large volume limit, the Darboux coordinates are encoded in a system of TBA equations for holomorphic Fourier modes satisfying $\mathcal{X}_\gamma \, \mathcal{X}_{\gamma'} = (-1)^{\langle \gamma,\gamma' \rangle} \, \mathcal{X}_{\gamma+\gamma'}$:

$$\mathcal{X}_\gamma(z) = \mathcal{X}^{\rm cl}_\gamma(z) \, \exp \left[\frac{1}{2\pi^2} \sum_{\gamma' \in \Gamma_+} \bar\Omega(\gamma', u) \int_{\ell_{\gamma'}} \mathrm{d}z' \, K_{\gamma\gamma'}(z,z') \, \mathcal{X}_{\gamma'}(z') \right]$$

Gaiotto Moore Neitzke 08, Alexandrov BP Saueressing Vandoren 08; Alexandrov BP 18

Twistorial description of instanton corrections II

$$\mathcal{X}_\gamma(z) = \mathcal{X}^{\rm cl}_\gamma(z)\, \exp\left[\frac{1}{2\pi^2}\sum_{\gamma'\in\Gamma_+}\bar\Omega(\gamma',u)\int_{\ell_{\gamma'}} \mathrm{d}z'\, K_{\gamma\gamma'}(z,z')\,\mathcal{X}_{\gamma'}(z')\right]$$

$$\mathcal{X}^{\rm cl}_\gamma = \sigma(\gamma)\, e^{-2\pi\tau_2(pt^2)(z^2-2z_\gamma)-\pi\tau_2(pt^2)+2\pi \mathrm{i} p^a(\tilde c_a - q_0\tau + c^a(q_a+\kappa_{abc}p^b b^c))}$$

$$z_\gamma = -\mathrm{i}\,\frac{(q_a+\kappa_{abc}p^b b^c)\, t^a}{(pt^2)}\ , \quad \bar\Omega(\gamma,u) = \sum_{d|\gamma}\frac{1}{d^2}\Omega(\gamma/d,u)\ ,$$

$$\ell_\gamma = \mathbb{R}+\mathrm{i}z_\gamma\ , \quad K_{\gamma\gamma'}(z,z') = 2\pi\left((tpp') + \frac{\mathrm{i}\langle\gamma,\gamma'\rangle}{z-z'}\right)$$

Twistorial description of instanton corrections III

- In addition, the space $\mathcal{M}$ carries a canonical function e^{Φ} known as the contact potential, related to the instanton generating function $\mathcal{G}$ through a covariant derivative operator:

$$\begin{aligned}\mathcal{G} =&\frac{1}{2\pi^2}\sum_{\gamma\in\Gamma_+}\bar\Omega(\gamma,u)\int_{\ell_\gamma}\mathrm{d}z\,\mathcal{X}_\gamma(z)\\ &-\frac{1}{8\pi^4}\sum_{\gamma,\gamma'\in\Gamma_+}\bar\Omega(\gamma,u)\,\bar\Omega(\gamma',u)\int_{\ell_{\gamma_1}}\mathrm{d}z\int_{\ell_{\gamma_2}}\mathrm{d}z'\,K_{\gamma\gamma'}(z,z')\,\mathcal{X}_\gamma(z)\,\mathcal{X}_{\gamma'}(z')\end{aligned}$$

- A key requirement for the existence of an isometric action of $SL(2,\mathbb{Z})$ on $\mathcal{M}$ is that the contact potential should transform as a modular form of weight $(-1,-1)$. and therefore $\mathcal{G}$ should transform as a modular form of weight $(-\frac{3}{2},\frac{1}{2})$

Twistorial description of instanton corrections IV

- A weak coupling $\tau_2 \to \infty$, the integral is dominated by a saddle point at $z = z_\gamma$, leading to $\mathcal{O}(e^{-\pi\tau_2 (pt^2)})$ corrections.
- The TBA equations can be solved iteratively,

$$\mathcal{G} = \sum_{\gamma} H^{\rm cl}_{\gamma} + \frac{1}{2} \sum_{\gamma_1,\gamma_2} K_{12} H^{\rm cl}_{\gamma_1} H^{\rm cl}_{\gamma_2} + \frac{1}{2} \sum_{\gamma_1,\gamma_2,\gamma_3} K_{12} K_{23} H^{\rm cl}_{\gamma_1} H^{\rm cl}_{\gamma_2} H^{\rm cl}_{\gamma_3} + \dots$$
$$+ \sum_{\gamma_1,\gamma_2,\gamma_3,\gamma_4} \left(\frac{1}{6} K_{12} K_{13} K_{14} + \frac{1}{2} K_{12} K_{23} K_{34} \right) H^{\rm cl}_{\gamma_1} H^{\rm cl}_{\gamma_2} H^{\rm cl}_{\gamma_3} H^{\rm cl}_{\gamma_4} + \dots$$

where $H^{\rm cl}_{\gamma} \equiv \frac{\bar\Omega(\gamma)}{2\pi^2} \mathcal{X}^{\rm cl}_{\gamma}$, $K_{ij} \equiv K_{\gamma_i\gamma_j}(z_i, z_j)$ and we omit the integrals.

Twistorial description of instanton corrections V

- To all orders, the expansion is given by

$$\mathcal{G} = \sum_{n=1}^{\infty} \prod_{i=1}^{n} \left(\sum_{\gamma_i \in \Gamma_+} \frac{\bar{\Omega}(\gamma_i, u)}{2\pi^2} \int_{\ell_{\gamma_i}} \mathrm{d}z_i \, \mathcal{X}^{\mathrm{cl}}_{\gamma_i}(z_i) \right) \sum_{\mathcal{T} \in \mathbb{T}_n} \frac{\prod_{e \in E_{\mathcal{T}}} K_{s(e)t(e)}}{|\mathrm{Aut}(\mathcal{T})|}$$

where $\mathbb{T}_n$ is the set of (unrooted) decorated trees with n vertices.

Gaiotto Moore Neitzke 08, Stoppa 11

- One may show that jumps of $\bar{\Omega}(\gamma_i, u)$ across walls of marginal stability cancel against contributions of poles due to exchanging contours ℓ_{γ_i}, in such a way that $\mathcal{G}$ is a smooth function on $\mathcal{M}$.
- *What are the conditions on $\bar{\Omega}(\gamma_i, u)$ such that $\mathcal{G}$ transforms as a modular form of weight $(-\frac{3}{2}, \frac{1}{2})$?*

Outline

From DT invariants to MSW invariants I

- The DT invariants $\Omega(\gamma, u)$ for $\gamma = (0, p^a, q_a, q_0)$ have complicated moduli dependence, even at large volume.
- The MSW invariants $\Omega_{\rm MSW}(\gamma)$ are defined as the value at the large volume attractor point

$$u^a_\infty(\gamma) = \lim_{\lambda\to+\infty} (-q^a + \mathrm{i}\lambda p^a)\,, \qquad \begin{cases} q^a = \kappa^{ab} q_b \\ \kappa_{ab} = \kappa_{abc} p^c \end{cases}$$

- Unlike $\Omega(\gamma, u)$, $\Omega_{\rm MSW}(\gamma)$ is invariant under the spectral flow

$$q_a \to q_a - \kappa_{abc} p^b \epsilon^a \,, \quad q_0 \mapsto q_0 - \epsilon^a q_a + \frac{1}{2} \kappa_{abc} p^a \epsilon^b \epsilon^c$$

From DT invariants to MSW invariants II

- This implies that $\Omega_{\rm MSW}(\gamma)$ depends only on $p^a, \mu_a, \hat{q}_0$ defined by

$$\hat{q}_0 = q_0 - \frac{1}{2}\kappa^{ab} q_a q_b \ , \quad q_a = \mu_a + \frac{1}{2}\kappa_{abc} p^b p^c + \kappa_{abc} p^b \epsilon^c$$

  For simplicity we shall often omit the residue class $\mu \in \Lambda/\Lambda^*$.

- We define the generating functions of MSW invariants

$$h_{p,\mu}(\tau) = \sum_{\hat{q}_0 \leq \hat{q}_{0,\rm max}} \Omega_{p,\mu}(\hat{q}_0)\, e^{-2\pi i \hat{q}_0 \tau}$$

  where $\bar{\Omega}_{p,\mu}(\hat{q}_0) = \bar{\Omega}_{\rm MSW}(0, p^a, \mu_a + \frac{1}{2}\kappa_{abc} p^b p^c, \hat{q}_0 + \frac{1}{2}\kappa^{ab} q_a q_b)$.

From DT invariants to MSW invariants III

- At first order in the multi-instanton expansion, ignoring the difference between $\bar{\Omega}(\gamma, u)$ and $\bar{\Omega}_{\rm MSW}(\gamma)$, $\mathcal{G}$ reduces to

$$\mathcal{G} \sim \frac{1}{2\pi\sqrt{\tau_2}} \sum_{p} e^{-\pi\tau_2(pt^2)+2\pi \mathrm{i} p^a \tilde{c}_a} \sum_{\mu\in\Lambda/\Lambda^*} h_{p,\mu}(\tau)\, \vartheta_{p,\mu}(\tau, y)$$

  where $\vartheta_{p,\mu}(\tau, y)$ is a Siegel theta series for the lattice Λ of signature $(1, b_2 - 1)$. Thus, $h_{p,\mu}$ should be a vector-valued modular form of weight $-\frac{1}{2}b_2 - 1$.
- The p-th Fourier coefficient of $\mathcal{G}$ wrt $\tilde{c}$ can be identified with the modified elliptic genus of the MSW SCFT on a divisor in class p.

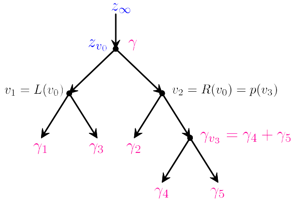
Alexandrov Manschot BP '12

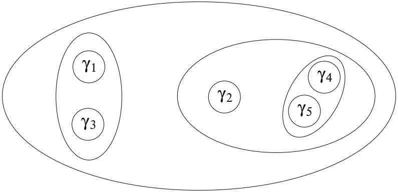
From DT invariants to MSW invariants IV

- If the divisor $p = \sum_{i=1}^{n} p_i$ is reducible, $\mathcal{G}$ will receive contributions from higher orders in the multi-instanton expansion, and $\Omega(\gamma, u)$ will differ from $\Omega_{\rm MSW}(\gamma)$ due to bound states of D4-branes with charge p_i.
- Our first task is to express the DT invariants $\Omega(\gamma, u)$ in terms of MSW invariants $\Omega_{\rm MSW}(\gamma)$ Assume for the moment that the latter coincide with attractor indices $\Omega_*(\gamma)$.

Tree flow formula I

- To express $\bar{\Omega}(\gamma, u)$ in terms of $\bar{\Omega}_*(\gamma_i)$, one may apply the split attractor flow conjecture, which posits that all BPS states can be constructed from nested two-particle bound states:

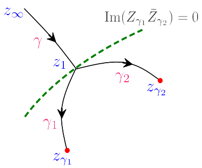




Denef '00; Denef Green Raugas '01; Denef Moore'07; Manschot 2010

Tree flow formula II

- Along each edge flowing into a vertex $\gamma \to \gamma_L + \gamma_R$, the moduli flow as in a spherically black hole, $\partial_r u^a = g^{a\bar{b}} \partial_{\bar{u}^b} |Z_\gamma(u)|$, until they hit the wall of marginal stability for the decay $\mathrm{Im} Z_{\gamma_L} \bar{Z}_{\gamma_R}(u_1) = 0$, and bifurcate into two flows with charges γ_L and γ_R.



- For the bound state to be stable, one requires at each vertex

$$\langle \gamma_{L(v)}, \gamma_{R(v)} \rangle \, \mathrm{Im}\left[Z_{\gamma_{L(v)}} \bar{Z}_{\gamma_{R(v)}}(u_{p(v)}) \right] > 0 \quad \& \quad \mathrm{Re}\left[Z_{\gamma_{L(v)}} \bar{Z}_{\gamma_{R(v)}}(u_v) \right] > 0$$

- Each stable tree contributes $\kappa(T) \prod_i \bar{\Omega}_*(\gamma_i)$ to $\bar{\Omega}(\gamma, u)$, where

$$\kappa(T) = \prod_{v \in V_T} (-1)^{\langle \gamma_{L(v)}, \gamma_{R(v)} \rangle + 1} |\langle \gamma_{L(v)}, \gamma_{R(v)} \rangle|$$

Tree flow formula III

- In the large volume limit, the same idea works if one restricts to constituents γ_i with D4-D2-D0 brane charge, and replaces the attractor index $\bar{\Omega}_*(\gamma_i)$ by the MSW index $\bar{\Omega}_{\rm MSW}(\gamma_i)$. The second stability condition $\mathrm{Re}\left[Z_{\gamma_{L(v)}} \bar{Z}_{\gamma_{R(v)}}(u_v)\right] > 0$ is then automatic.
- Remarkably, the first condition can be checked in terms of asymptotic stability parameters $c_i = \mathrm{Im} Z_{\gamma_i} \bar{Z}_{\sum \gamma_i}(u_\infty)$, without integrating the flow along each edge ! It suffices to apply the discrete attractor flow *[Alexandrov BP '18]*

$$c_{v,i} = c_{p(v),i} - \frac{\langle \gamma_v, \gamma_i \rangle}{\langle \gamma_v, \gamma_{L(v)} \rangle} \sum_{j=1}^{n} m^j_{L(v)} c_{p(v),j}$$

where m^i_v are integers such that $\gamma_v = \sum_{i=1}^n m^i_v \gamma_i$. Note that the condition $\sum_i c_i = 0$ is preserved at each step.

Tree flow formula IV

- Rather than summing over stable flow trees only, one may sum over all trees, but insert a factor

$$\Delta(T) = \frac{1}{2^{n-1}} \prod_{v \in V_T} \left[\mathrm{sgn}\left(\sum_i m^i_{L(v)} c_{v,i}\right) + \mathrm{sgn}(\gamma_{L(v)R(v)}) \right].$$

- The flow tree formula then states

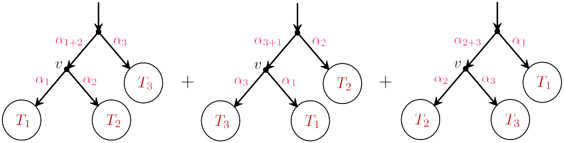
$$\boxed{\bar\Omega(\gamma, u) = \sum_{\gamma=\sum_{i=1}^n \gamma_i} \frac{g_{\mathrm{tr},n}(\{\gamma_i, c_i\})}{|\mathrm{Aut}\{\gamma_i\}|} \prod_{i=1}^n \bar\Omega_{\mathrm{MSW}}(\gamma_i)}$$

where $g_{\mathrm{tr},n}$ is the tree index

$$\boxed{g_{\mathrm{tr},n}(\{\gamma_i, c_i\}) = \sum_{T \in \mathcal{T}_n(\{\gamma_i\})} \Delta(T)\, \kappa(T)}$$

Tree flow formula V

- The tree flow formula is consistent with the wall-crossing formula across walls of marginal stability. Since it trivially holds in the (large volume) attractor chamber, it must hold everywhere.
- It appears to have additional discontinuities across fake walls associated to the inner bound states, but these cancel after summing over trees, due to $\gamma_{12}(\gamma_{13}+\gamma_{23})+\mathrm{cycl}=0$.



Tree flow formula VI

- After summing over trees and using sign identities such as

$$\mathrm{sgn}(x_1 + x_2) \times [\mathrm{sgn}(x_1) + \mathrm{sgn}(x_2)] = 1 + \mathrm{sgn}(x_1)\,\mathrm{sgn}(x_2)$$

  g_{tr} can be rewritten as a sum of products of sign functions whose arguments are linear both in γ_{ij} and c_i.

- *Remark: The tree flow formula is reminiscent of the Coulomb branch formula, where the tree index $g_{\mathrm{tr},n}$ is replaced by the Coulomb index g_C, and the attractor index $\bar{\Omega}_{\mathrm{MSW}}(\gamma_i)$ by the single-centered index $\bar{\Omega}_S(\gamma_i)$. But the Coulomb branch formula has additional terms in the presence of scaling solutions.*

Outline

Back to modularity I

- S-duality dictates that $\mathcal{G}$ should be modular of weight $(-\frac{3}{2},\frac{1}{2})$,

$$\mathcal{G}=\frac{1}{(2\pi)^2}\sum_{n=1}^{\infty}\left(\prod_{i=1}^{n}\sum_{\substack{\gamma_i\in\Gamma_+\\ \mathcal{T}\in\mathbb{T}_n}}\frac{\bar\Omega(\gamma_i,u)}{|\mathrm{Aut}(\mathcal{T})|}\int_{\ell_{\gamma_i}}\mathrm{d}z_i\,\sigma_{\gamma_i}\mathcal{X}^{\mathrm{cl}}_{\gamma_i}(z_i)\prod_{e\in E_{\mathcal{T}}}K_{s(e)t(e)}\right)$$

- Substituting $\bar\Omega(\gamma,u)=\sum_{\gamma=\sum_{i=1}^n\gamma_i}g_{\mathrm{tr}}(\{\gamma_i,c_i\})\prod_{j=1}^n\bar\Omega_{\mathrm{MSW}}(\gamma_i)$ and using invariance of $\bar\Omega_{\mathrm{MSW}}(\gamma_i)$ under spectral flow, one arrives as the theta series decomposition

$$\mathcal{G}=\frac{2^{-n/2}}{\pi\sqrt{2\tau_2}}\sum_{n=1}^{\infty}\left[\prod_{i=1}^{n}\sum_{p_i}h_{p_i}\right]\vartheta_{\mathbf{p},\mu}(\Phi_n)\,e^{-\pi\tau_2(pt^2)+2\pi\mathrm{i}p^a\tilde c_a}$$

where $\vartheta_{\mathbf{p}}(\Phi_n)$ is a theta series for a lattice $\Lambda=\oplus_i\Lambda_i$ of signature $n(b_2-1,1)$, with a complicated kernel Φ_n.

Vignéras theorem I

- $\vartheta_{\mathbf{p}}(\Phi_n)$ belongs to the class of non-holomorphic theta series

$$\vartheta(\Phi,\lambda) = \tau_2^{-\lambda} \sum_{q\in\Lambda} \Phi\left(\sqrt{2\tau_2}(q+b)\right)\, e^{-i\pi\tau Q(q+b)+2\pi i B(c,q+\frac{1}{2}b)}$$

  where (Λ, Q) is an even lattice of signature $(r, d-r)$, $b, c \in \mathbb{R}^d$, $\lambda \in \mathbb{R}$, and the kernel $\Phi(x)$ is such that $f(x) \equiv \Phi(x) e^{\frac{\pi}{2}Q(x)} \in L_1(\mathbb{R}^d)$

- Vignéras (1978): $\vartheta(\Phi,\lambda)$ is a modular form of weight $(\lambda + \frac{d}{2}, 0)$ provided [*]
  - $R(x)f(x), D(x)f(x) \in L_2(\mathbb{R}^d)$ for any polynomial $R(x)$ of degree ≤ 2 and any differential operator $D(x)$ of degree ≤ 2
  - $\boxed{V_\lambda \cdot \Phi = 0}$ where $\boxed{V_\lambda = \partial_x^2 + 2\pi(x\partial x - \lambda)}$
- $\vartheta(\Phi,\lambda)$ is holomorphic if $(x\partial_x - \lambda)\Phi = 0$. But if so, $f(x)$ will fail to be integrable: hence a tension between modularity and holomorphy !

Vignéras theorem II

- Example 1: $\Phi = e^{\pi Q(x_+)}$, where x_+ is the projection of x on a fixed plane of dimension r, satisfies [*] with $\lambda = -n$. $\vartheta(\Phi, -n)$ is then the usual (non-holomorphic) Siegel-Narain theta series.
- Example 2: In signature $(1, d-1)$, choose C, C' two vectors such that $Q(C), Q(C'), B(C, C') > 0$, then

$$\Phi(x) = \mathrm{Erf}\left(\frac{B(C,x)\sqrt{\pi}}{\sqrt{Q(C)}}\right) - \mathrm{Erf}\left(\frac{B(C',x)\sqrt{\pi}}{\sqrt{Q(C')}}\right)$$

  satisfies [*] with $\lambda = 0$. $\vartheta(\Phi, -n)$ is the Zwegers theta series.
- In general, $\vartheta(\Phi, 0)$ not holomorphic but modular, and its shadow $\partial_{\bar{\tau}}\vartheta(\Phi, 0)$ is proportional to $\vartheta(\Psi, -2)$ where

$$\Psi(x) = \frac{B(C,x)}{\sqrt{Q(C)}}\, e^{-\pi \frac{B(C,x)^2}{Q(C)}} - \frac{B(C',x)}{\sqrt{Q(C')}}\, e^{-\pi \frac{B(C',x)^2}{Q(C')}}$$

Vignéras theorem III

- For $r > 1$, one can construct solutions of Vignéras equation $V_0 \cdot \Phi = 0$, which asymptote to $\prod_i \operatorname{sgn}[B(C_i, x)]$ as $|x| \to \infty$: the generalized error functions of degree r

$$E_r(C_1, \dots C_r; x) = \int_{\langle C_1,\dots,C_r \rangle} \mathrm{d}x' \, e^{-\pi Q(x_+ - x')} \prod_i \operatorname{sgn}[B(C_i, x')]$$

  where x_+ is the projection of x on the plane $\langle C_1, \dots, C_r \rangle$.
- Taking suitable linear combinations of $E_r(C_1, \dots C_r; x)$, one can construct a kernel Φ which leads to a convergent, modular theta series $\vartheta(\Phi, 0)$: a mock theta functions of depth r.
- The shadow $\partial_{\bar{\tau}} \vartheta(\Phi, 0)$ is again a theta series with a kernel involving Gaussian factors and generalized error functions of degree $r - 1$.

Alexandrov Banerjee Manschot BP 2016; Nazaroglu 2016; Kudla Funke 2016-17

Back to modularity I

- Remarkably, one can show that $V_{n-2} \cdot \Phi_n = 0$ away from discontinuities of the kernel in charge space. Those coming from walls of marginal stability or fake walls cancel, but those coming from $\mathrm{sgn}\langle \gamma_{L(v)}, \gamma_{R(v)} \rangle$ don't ! Hence $h_p(\tau)$ cannot be modular !
- To characterize the modular anomaly, we look for functions $R_n(\{\gamma_i\}, \tau_2)$ such that

$$\widehat{h}_p = h_p + \sum_{n=2}^{\infty} \sum_{\gamma=\sum_{i=1}^n \gamma_i} e^{\mathrm{i}\pi\tau Q_n(\{\gamma_i\})} R_n(\{\gamma_i\}, \tau_2) \prod_{i=1}^{n} h_{p_i}$$

is no longer holomorphic but transforms as a modular form of weight $-\frac{1}{2} b_2 - 1$. Here $Q_n(\{\gamma_i\}) = \kappa_{ab} q^a q^b - \sum_{i=1}^n \kappa_i^{ab} q_{i,a} q_{i,b}$

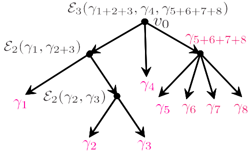
Back to modularity II

- Expressing $\mathcal{G}$ in terms of $\widehat{h}_p(\tau)$ leads to a theta series decomposition with a new kernel $\widehat{\Phi}_n$,

$$\mathcal{G} = \frac{2^{-n/2}}{\pi\sqrt{2\tau_2}} \sum_{n=1}^{\infty} \left[\prod_{i=1}^{n} \sum_{p_i} \widehat{h}_{p_i} \right] \vartheta_{\mathbf{p}}(\widehat{\Phi}_n)\, e^{-\pi\tau_2 (pt^2) + 2\pi \mathrm{i} p^a \tilde{c}_a}$$

- For the following choice of R_n, $\vartheta_{\mathbf{p}}(\widehat{\Phi}_n)$ satisfies the assumptions of Vignéras' theorem:

$$R_n = \mathrm{Sym} \left\{ \sum_{T \in \mathbb{T}_n^S} (-1)^{n_T - 1} \mathcal{E}_{v_0}^{(+)} \prod_{v \in V_T \setminus \{v_0\}} \mathcal{E}_v^{(0)} \right\}$$



where the sum runs over Schröder trees with n leaves, and $\mathcal{E}_v = \mathcal{E}_v^{(0)} + \mathcal{E}_v^{(+)}$ are certain generalized error functions.

Back to modularity III

- For $p = \sum_{i=1}^{n} p_i$ the sum of n irreducible divisors, h_p is then a mock modular form of depth $n-1$, with a computable shadow:

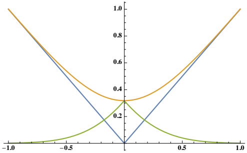
$$\partial_{\bar\tau} \widehat{h}_p = \sum_{n\geq 2} \sum_{\gamma=\sum_{i=1}^n \gamma_i} \mathrm{Sym} \left\{ \sum_{T\in\mathbb{T}_n^S} (-1)^{n_T-1} \partial_{\tau_2}\mathcal{E}_{v_0} \prod_{v\in V_T\setminus\{v_0\}} \mathcal{E}_v \right\} \prod_{i=1}^{n} \widehat{h}_{p_i}$$

- For example, for $n=2$,

$$R_2 = \frac{|\langle\gamma_1,\gamma_2\rangle|}{8\pi} \beta_{3/2}\left(\frac{2\tau_2\langle\gamma_1,\gamma_2\rangle^2}{(pp_1p_2)}\right)$$

where

$$\beta_{3/2}(x^2) = \frac{2}{|x|} e^{-\pi x^2} - 2\pi\,\mathrm{Erfc}\left(\sqrt{\pi}|x|\right)$$



Conclusion I

- By enforcing an isometric action of S-duality on the vector-multiplet moduli space in $D = 3$, we have determined the modularity constraints on generalized DT invariants counting D4-brane bound states in the large volume limit.
- For $p = \sum_{i=1}^{n} p_i$ the sum of n irreducible divisors, the generating function h_p of MSW invariants is a mock modular form of depth $n - 1$, with a computable shadow. From the knowledge of lowest lying coefficients, one should be able to reconstruct all invariants.
- The mock modularity will affect the growth of Fourier coefficients, hence the microscopic entropy of supersymmetric black holes. It should have an imprint on the macroscopic side as well.

Conclusion II

- An important question is to understand the physical origin of the non-holomorphic contributions, e.g. in terms of the MSW superconformal field theory. Presumably they come from a continuum of states with a spectral asymmetry.

  Troost 2010, BP 2015, Murthy BP 2018

- At finite volume, additional effects from D6-KKM contributions should also be consistent with $SL(2,\mathbb{Z})$. It would be interesting to determine the exact quantum corrected metric, and check consistency with mirror symmetry, string/string duality, etc.
- A similar tower of mock modular forms of higher depth should also appear in similar problems involving multi-particle bound states, e.g. in Vafa-Witten theory on a 4-manifold.

  Manschot 2017

Thanks for your attention !