Modular bootstrap for BPS indices on Calabi-Yau threefolds

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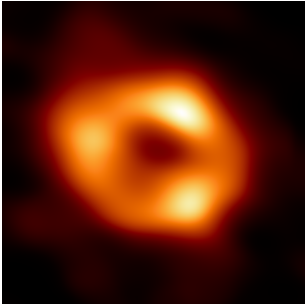
References

- "*"Indefinite theta series and generalized error functions"*,
with S. Alexandrov, S. Banerjee, J. Manschot, Selecta Math. 24 (2018) 3927 [arXiv:1606.05495]
- *"Black holes and higher depth mock modular forms"*,
with S. Alexandrov, Commun.Math.Phys. 374 (2019) 549 [arXiv:1808.08479]
- *"S-duality and refined BPS indices"*,
with S. Alexandrov and J. Manschot, Commun.Math.Phys. 380 (2020) 755 [arXiv:1910.03098]
- *"Modular bootstrap for D4-D2-D0 indices on compact Calabi-Yau threefolds"*, with S. Alexandrov, N. Gaddam, J. Manschot [arXiv:2204.02207]
- S. Alexandrov, S. Feyzbakhsh, A. Klemm, BP, T. Schimannek, in progress.

Introduction

- A driving force in high energy theoretical physics has been the quest for a microscopic explanation of the entropy of black holes. Providing a derivation of the Bekenstein-Hawking formula is a benchmark test of any theory of quantum gravity.

$$S_{BH} = \frac{1}{4G_N} A$$

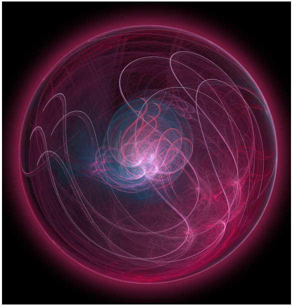


$$S_{BH} \stackrel{?}{=} \log \Omega$$

Sgr A, Event Horizon Telescope 2022*

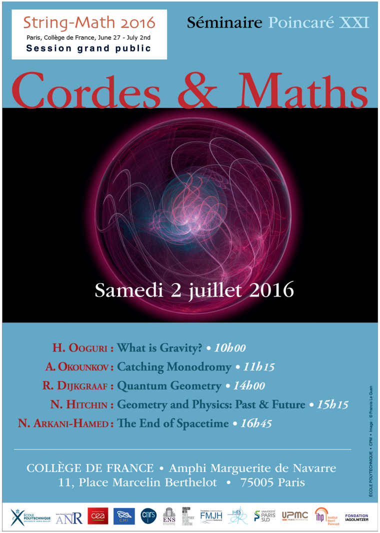
Black hole microstates as wrapped D-branes

- Back in 1996, Strominger and Vafa argued that String Theory passes this test with flying colors ♫, at least in the context of BPS black holes in vacua with extended supersymmetry: at weak coupling, BPS states are bound states of D-branes wrapped on minimal cycles of the internal Calabi-Yau manifold.



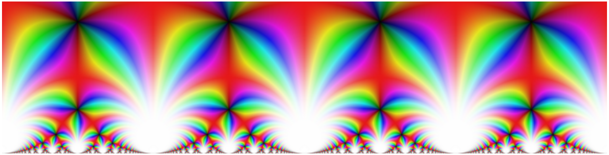
Calabi-Yau black hole, courtesy F. Le Guen

Trous noirs, Cordes et Maths



BPS black hole entropy from modularity

- D-brane bound states can often be understood as excitations of an effective black string, supporting a (0,4) superconformal field theory. BPS indices counting such states are encoded in the elliptic genus, and their asymptotic growth at large charge is governed by modularity.



- Recall that $f(\tau)=\sum_{n\geq 0} c(n) \mathrm{q}^{n-\Delta}$ (with $\mathrm{q}=e^{2\pi \mathrm{i}\tau}$, $\mathrm{Im}\tau>0$) is a *modular form* of weight k if $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$,

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k\, f(\tau) \quad \Rightarrow \quad c(n) \sim \exp\left(4\pi\sqrt{\Delta n}\right)$$

BPS indices and Donaldson-Thomas invariants

- In the context of type IIA strings compactified on a Calabi-Yau three-fold $\mathfrak{Y}$, bound states of D6-D4-D2-D0-branes are best understood as stable objects in the derived category $\mathcal{C} = D^b\mathrm{Coh}\mathfrak{Y}$.
- The problem becomes a question in enumerative geometry: for fixed electromagnetic charge $\gamma = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2, \mathrm{ch}_3)$, compute the Donaldson-Thomas invariant $\Omega_z(\gamma)$ counting stable objects in $\mathcal{C}$, with respect to a stability condition $z \in \mathrm{Stab}\,\mathcal{C}$, and determine its growth as $|\gamma| \to \infty$.
- Physical reasoning allows to make very non-trivial predictions about properties of DT invariants. In particular, the modular invariance of suitable generating series remains mysterious from mathematics viewpoint, and can only be verified a posteriori.

Precision counting of $\mathcal{N}=8$ BPS black holes

- For $\mathfrak{Y}=T^6$, the index $\Omega(\gamma)$ counting 1/8-BPS states depends only on a certain quartic polynomial $n=I_4(\gamma)$ in the charges, and is moduli independent. It is given by the Fourier coefficient $c(n)$ of a weak modular form,

$$\frac{\theta_3(2\tau)}{\eta^6(4\tau)} = \sum_{n\geq -1} c(n)\, \mathrm{q}^n = \frac{1}{\mathrm{q}} + 2 + 8\mathrm{q}^3 + 12\mathrm{q}^4 + 39\mathrm{q}^7 + 56\mathrm{q}^8 + \dots$$

Moore Maldacena Strominger 1999, BP 2005, Shih Strominger Yin 2005

Bryan Oberdieck Pandharipande Yin'15

- The Harder-Ramanujan-Hardy formula gives $c(n) \sim e^{\pi\sqrt{n}}$ as $n\to\infty$, in agreement with $S_{BH}(\gamma)=\frac{1}{4}A(\gamma)$ ☺
- The full Rademacher expansion can now be derived by localization in supergravity ☺☺☺*[Dabholkar Gomes Murthy'10, Iliesiu Turiaci Murthy'22]*

Precision counting of $\mathcal{N}=4$ BPS black holes

- For $\mathfrak{Y} = K_3 \times T_2$ (and orbifolds thereof preserving $\mathcal{N}=4$ SUSY), the BPS index counting 1/4-BPS states with charge $\gamma = (Q, P)$ is a Fourier coefficient of a meromorphic Siegel modular form,

$$\Omega_z(\gamma) = \oint_{\mathcal{C}(\gamma,z)} \frac{e^{\pi \mathrm{i}(\rho Q^2 + \sigma P^2 + 2 v P\cdot Q)}}{\Phi_{k-2}(\tau)} , \qquad \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix} \in \mathcal{H}_2$$

Dijkgraaf Verlinde Verlinde '96; David Jatkar Sen '05-06; ...

- The integration contour $\mathcal{C}(\gamma, z)$ depends on γ and on moduli $z \in \mathcal{M}_4 = \frac{SL(2)}{U(1)} \times \frac{O(6,2k-2)}{O(6)\times O(2k-2)}$. For large $|\gamma|$, a saddle-point computation gives $\log \Omega_z \sim \frac{1}{4} A(\gamma)$ ☺

Wall-crossing for $\mathcal{N} = 4$ BPS black holes

- When z crosses real codimension-1 walls

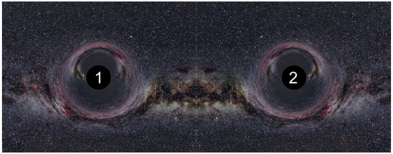
$$W(\gamma_1, \gamma_2) = \{z \in \mathcal{M}_4, M(\gamma_1 + \gamma_2) = M(\gamma_1) + M(\gamma_2)\}$$

where γ_1, γ_2 are 1/2-BPS charge vectors, the contour $\mathcal{C}(\gamma, z)$ crosses a pole of $1/\Phi_{k-2}(\tau)$, so that the index $\Omega_z(\gamma)$ jumps according to the primitive wall-crossing formula

$$\Delta\Omega(\gamma_1 + \gamma_2) = (-1)^{\langle\gamma_1,\gamma_2\rangle+1} |\langle\gamma_1, \gamma_2\rangle| \, \Omega(\gamma_1) \, \Omega(\gamma_2)$$

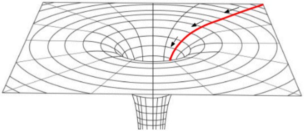
Denef Moore '07; Cheng, Verlinde '07; Sen '07-08

corresponding to contributions of bound states of two 1/2-BPS black holes.



Attractor indices and mock modular forms

- One may extract the contributions of single-centered black holes by evaluating $\Omega(\gamma, z)$ at the attractor point z_γ, where two-centered bound states are not allowed to form.



$$r^2 \frac{\mathrm{d}z^a}{\mathrm{d}r} = g^{ab}\partial_b M^2(\gamma, z)$$

- In this simple case, this fixes $\mathrm{Im}\rho, \mathrm{Im}v, \mathrm{Im}\sigma$ in terms of Q, P.
- The attractor indices $\Omega_*(\gamma) = \Omega_{z_\gamma}(\gamma)$ turn out to be Fourier coefficients of a (vector-valued) mock modular form.

Dabholkar Murthy Zagier '12

Mock modular forms

- A (depth one) mock modular form of weight w transforms inhomogeneously under $SL(2,\mathbb{Z})$,

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \left[f(\tau) - \int_{-d/c}^{\mathrm{i}\infty} \overline{g(-\bar\rho)}(\tau+\rho)^{-w} \mathrm{d}\rho \right]$$

where $g(\tau)$ is an ordinary modular form of weight $2-w$, known as the shadow. Equivalently, the non-holomorphic completion

$$\widehat{f}(\tau,\bar\tau) := f(\tau) + \int_{-\bar\tau}^{\mathrm{i}\infty} \overline{g(-\bar\rho)}(\tau+\rho)^{-w} \mathrm{d}\rho$$

transforms like a modular form of weight w, and satisfies the holomorphic anomaly equation

$$\tau_2^w \partial_{\bar\tau} \widehat{f}(\tau,\bar\tau) \propto \overline{g(\tau)}$$

Ramanujan'1920, Hirzebruch-Zagier'1973, Zwegers'02

Precision counting of Calabi-Yau black holes

- When $\mathfrak{Y}$ is a CY threefold of generic $SU(3)$ holonomy, life is more complicated. For one, the moduli space $\mathcal{M}_4$ is no longer a symmetric space. Instead, it factorizes into a product $\mathcal{M}_4 = \mathcal{M}_V \times \mathcal{M}_H$
  1. $\mathcal{M}_V$ parametrizes the Kähler structure of $\mathfrak{Y}$, and receives worldsheet instanton corrections weighted by GW/GV invariants
  2. $\mathcal{M}_H$ parametrizes the dilaton + complex structure of $\mathfrak{Y}$+ Ramond gauge fields, and receives D-instanton corrections (largely irrelevant for this talk)
- The BPS indices $\Omega_z(\gamma)$ are independent of $\mathcal{M}_H$, but have a complicated chamber structure on $\mathcal{M}_V$, due to the possibility of BPS bound states with an arbitrary number of constituents. The full wall-crossing formula for $\Delta\Omega(N_1\gamma_1 + N_2\gamma_2)$ is needed *[Kontsevich Soibelman'08, Joyce Song'08]*.

Instanton corrections from Euclidean black holes

- Upon reducing on a circle, $\mathcal{M}_H$ goes along for the ride but $\mathcal{M}_V$ extends to a larger quaternion-Kähler space $\widetilde{\mathcal{M}}_V$ parametrizing the radius R, Kähler moduli and Ramond gauge fields along S_1.
- At large R, $\widetilde{\mathcal{M}}_V$ is a flat torus bundle over $\mathbb{R}^+ \times \mathcal{M}_V$, but it receives $\mathcal{O}(e^{-RM(\gamma)})$ corrections from Euclidean black holes winding around S^1, weighted by the same DT invariants $\Omega_z(\gamma)$ counting black holes in $D = 4$.
- Since type IIA/S^1 is the same as M-theory on T^2, $\widetilde{\mathcal{M}}_V$ must have an isometric action of $SL(2,\mathbb{Z})$. This enforces modularity constraints on DT invariants. *[Alexandrov, Banerjee, Manschot, BP, Robles-Llana, Rocek, Saueressig, Theis, Vandoren '06-19]*
- By mirror symmetry, $\widetilde{\mathcal{M}}_V$ is also the hypermultiplet moduli space in type IIB on $\hat{\mathfrak{Y}}$, invariant under usual $SL(2,\mathbb{Z})$ S-duality.

S-duality constraints on BPS indices

Requiring that $\widetilde{\mathcal{M}}_V$ admits an isometric action of $SL(2,\mathbb{Z})$ near large volume, one can show that DT invariants $\Omega_z(\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2, \mathrm{ch}_3)$ satisfy

- For n D0-branes, $\Omega_z(0,0,0,n) = -\chi_{\mathfrak{Y}}$ (independent of n)
- For D2-branes supported on a curve of class $q_a\gamma^a \in \Lambda^* = H_2(\mathfrak{Y},\mathbb{Z})$, $\Omega_z(0,0,q_a,n) = N^{(0)}_{q_a}$ is given by the genus-zero GV invariant (independent of n)
- For D4-branes supported on an ample divisor $\mathcal{D}$ of class $p^a\gamma_a \in \Lambda = H_4(\mathfrak{Y},\mathbb{Z})$, the generating series

$$h_{p^a,q_a}(\tau) := \sum_n \Omega_\star(0,p^a,q_a,n)\,\mathrm{q}^{n-\frac{1}{2}q_a\kappa^{ab}q_b}$$

 should be a vector-valued weakly holomorphic modular form of weight $w = -\frac{1}{2}b_2(\mathfrak{Y}) - 1$ and prescribed multiplier system.

S-duality constraints on D4-D2-D0 indices

$$h_{p^a,q_a}(\tau) = \sum_n \Omega_\star(0, p^a, q_a, n)\, \mathrm{q}^{n-\frac{1}{2}q_a\kappa^{ab}q_b}$$

- Here, κ^{ab} is the inverse of the quadratic form $\kappa_{ab} = \kappa_{abc}p^c$ with Lorentzian signature $(1, b_2(\mathfrak{Y}) - 1)$, and $\Omega_\star(\gamma)$ is the index in the large volume attractor chamber

$$\Omega_*(\gamma) = \lim_{\lambda\to+\infty} \Omega_{(-\kappa^{ab}q_b+\mathrm{i}\lambda p^a)}(\gamma)$$

- In particular, $\Omega_\star(0, p^a, q_a, n)$ is invariant under spectral flow (tensoring with a line bundle on the divisor $\mathcal{D}$)

$$q_a \to q_a - \kappa_{ab}\epsilon^b \,, \quad n \mapsto n - \epsilon^a q_a + \frac{1}{2}\kappa_{ab}\epsilon^a\epsilon^b$$

Thus, the D2-brane charge q_a can be restricted to the finite set Λ^*/Λ, of cardinal $|\det(\kappa_{ab})|$.

D4-D2-D0 indices from elliptic genus

- Summing over all D2-brane charges and using spectral flow invariance, one gets

$$\begin{aligned} Z_p(\tau, v) &:= \sum_{q\in\Lambda,n} \Omega_\star(0, p^a, q_a, n)\, \mathrm{q}^{n-\frac{1}{2}q_a\kappa^{ab}q_b} e^{2\pi \mathrm{i} q_a v^a} \\ &= \sum_{q\in\Lambda^*/\Lambda} h_{p,q}(\tau)\Theta_q(\tau, v) \end{aligned}$$

  where $\Theta_q(\tau, v)$ is the (non-holomorphic) Siegel theta series for the indefinite lattice (Λ, κ_{ab}). S-duality then requires that Z_p should transform as a (non-holomorphic) Jacobi form.
- The Jacobi form Z_p can be interpreted as the elliptic genus of the $(0, 4)$ superconformal field theory obtained by wrapping an M5-brane on the divisor $\mathcal{D}$ *[Maldacena Strominger Witten '98]*.

D4-D2-D0 indices from polar coefficients

- A weak modular form $h(\tau) = \sum_{n\geq 0} c(n) \mathrm{q}^{n-\Delta}$ of weight $w < 0$ is uniquely determined by polar terms with $n - \Delta < 0$. The existence of cusp forms in dual weight $2 - w$ may impose constraints on polar coefficients *[Bantay Gannon'07, Manschot Moore'07]*
- Provided the leading polar coefficient is non-zero, the Hardy-Ramanujan-Cardy formula gives

$$\log \Omega_\star(\gamma) \sim 4\pi\sqrt{|\Delta| n} \sim 2\pi\sqrt{\frac{n}{6}\kappa_{abc}p^a p^b p^c}$$

  in precise agreement with the Bekenstein-Hawking entropy. ☺
- I will discuss later how to compute polar indices in some simple CY3 manifolds. For now, let me continue with the general story.

Mock modularity constraints on D4-D2-D0 indices

- For γ supported on a reducible divisor $\mathcal{D} = \sum_{i=1}^{n\geq 2} \mathcal{D}_i$, the generating series h_p (omitting q index for simplicity) is no longer expected to be modular. Rather, it should be a vector-valued mock modular form of depth $n-1$ and same weight/multiplier system.

Alexandrov Banerjee Manschot BP '16-19

- There exists explicit non-holomorphic theta series such that

$$\widehat{h}_p(\tau,\bar\tau) = h_p(\tau) + \sum_{p=\sum_{i=1}^{n\geq 2} p_i} \Theta_n(\{p_i\},\tau,\bar\tau) \prod_{i=1}^{n} h_{p_i}(\tau)$$

transforms as a modular form of weight $-\frac{1}{2}b_2(\mathfrak{Y}) - 1$. Moreover the completion satisfies an explicit holomorphic anomaly equation,

$$\partial_{\bar\tau}\widehat{h}_p(\tau,\bar\tau) = \sum_{p=\sum_{i=1}^{n\geq 2} p_i} \widehat{\Theta}_n(\{p_i\},\tau,\bar\tau) \prod_{i=1}^{n} \widehat{h}_{p_i}(\tau,\bar\tau)$$

Crash course on Indefinite theta series

- Θ_n and $\widehat{\Theta}_n$ belongs to the class of indefinite theta series

$$\vartheta_{\Phi,q}(\tau,\bar\tau) = \tau_2^{-\lambda} \sum_{k\in\Lambda+q} \Phi\left(\sqrt{2\tau_2}k\right)\, e^{-\mathrm{i}\pi\tau Q(k)}$$

  where (Λ, Q) is an even lattice of signature $(r, d-r)$, $q \in \Lambda^*/\Lambda$, $\lambda \in \mathbb{R}$. The series converges if $f(x) \equiv \Phi(x) e^{\frac{\pi}{2}Q(x)} \in L_1(\Lambda\otimes\mathbb{R})$.
- Theorem (Vignéras, 1978): $\{\vartheta_{\Phi,q}, q \in \Lambda^*/\Lambda\}$ transforms as a vector-valued modular form of weight $(\lambda + \frac{d}{2}, 0)$ provided
  - $R(x)f, R(\partial_x)f \in L_2(\Lambda\otimes\mathbb{R})$ for any polynomial $R(x)$ of degree ≤ 2
  - $\left[\partial_x^2 + 2\pi(x\partial_x - \lambda)\right]\Phi = 0$ [*]
- The relevant lattice $\Lambda = H^2(\mathfrak{Y},\mathbb{Z})^{\oplus n-1}$ has signature $(r, d-r) = (n-1)(1, b_2(\mathfrak{Y}) - 1)$.

Indefinite theta series

- Example 1 (Siegel): $\Phi = e^{\pi Q(x_+)}$, where x_+ is the projection of x on a fixed plane of dimension r, satisfies [*] with $\lambda = -n$. ϑ_Φ is then the usual (non-holomorphic) Siegel-Narain theta series.
- Example 2 (Zwegers): In signature $(1, d-1)$, choose C, C' two vectors such that $Q(C), Q(C'), (C, C') > 0$, then

$$\widehat{\Phi}(x) = \mathrm{Erf}\left(\frac{(C,x)\sqrt{\pi}}{\sqrt{Q(C)}}\right) - \mathrm{Erf}\left(\frac{(C',x)\sqrt{\pi}}{\sqrt{Q(C')}}\right)$$

satisfies [*] with $\lambda = 0$. As $|x| \to \infty$, or if $Q(C) = Q(C') = 0$,

$$\widehat{\Phi}(x) \to \Phi(x) := \mathrm{sgn}(C, x) - \mathrm{sgn}(C', x)$$

- The theta series $\Theta_2(\{p_1, p_2\})$, $\widehat{\Theta}_2(\{p_1, p_2\})$ fall in this class. The generalization to $n > 2$ involves generalized error functions.

Alexandrov Banerjee Manschot BP 2016; Nazaroglu 2016

Modularity for one-modulus compact CY

- We now specialize to compact CY threefolds with $b_2(\mathfrak{Y}) = 1$ and $p = N[\mathcal{D}]$ where $\mathcal{D}$ is an ample divisor with $[\mathcal{D}]^3 := \kappa$.
- We focus on smooth complete intersections in weighted projective space (CICY), $\mathfrak{Y} = X_{d_i}(w_j)$ with $\sum d_i = \sum w_j$. There are 13 such models, with Kähler moduli space $\mathbb{P}^1 \backslash \{0, 1, \infty\}$, with a large volume point at $z = 0$ and a conifold singularity at $z = 1$.
- The central charge $Z_z(\gamma)$ is expressed in terms of hypergeometric functions, and GV invariants $N_q^{(g)}$ are known up to high genus *[Huang Klemm Quackenbush'06]*
- I will concentrate on $N = 1$, and discuss $N = 2$ if time permits.

Gaiotto Strominger Yin '06-07; Alexandrov Gaddam Manschot BP'22

CICY	$\chi(\mathfrak{Y})$	κ	$c_2(T\mathfrak{Y})$	$\chi(\mathcal{O}_\mathcal{D})$	n_1	C_1
$X_5(1^5)$	-200	5	50	5	7	0
$X_6(1^4,2)$	-204	3	42	4	4	0
$X_8(1^4,4)$	-296	2	44	4	4	0
$X_{10}(1^3,2,5)$	-288	1	34	3	2	0
$X_{4,3}(1^5,2)$	-156	6	48	5	9	0
$X_{4,4}(1^4,2^2)$	-144	4	40	4	6	1
$X_{6,2}(1^5,3)$	-256	4	52	5	7	0
$X_{6,4}(1^3,2^2,3)$	-156	2	32	3	3	0
$X_{6,6}(1^2,2^2,3^2)$	-120	1	22	2	1	0
$X_{3,3}(1^6)$	-144	9	54	6	14	1
$X_{4,2}(1^6)$	-176	8	56	6	15	1
$X_{3,2,2}(1^7)$	-144	12	60	7	21	1
$X_{2,2,2,2}(1^8)$	-128	16	64	8	33	3

Computing the polar terms

- For $N = 1$, the generating series

$$h_{1,q} = \sum_{n\in\mathbb{Z}} \Omega(0,1,q,n)\, \mathrm{q}^{n+\frac{q^2}{2\kappa}+\frac{q}{2}-\frac{\chi(\mathcal{D})}{24}}$$

  should transform as a vector-valued modular form of weight $-\frac{3}{2}$ in the Weil representation of $(\mathbb{Z}, m \mapsto \kappa m^2)$. In particular $q \in \mathbb{Z}/\kappa\mathbb{Z}$.

- An overcomplete basis is given for κ even by

$$\frac{E_4^a E_6^b}{\eta^{4\kappa+c_2}} D^\ell(\vartheta_q^{(\kappa)}) \quad \text{with} \quad \vartheta_q^{(\kappa)} = \sum_{k\in\mathbb{Z}+\frac{q}{\kappa}+\frac{1}{2}} \mathrm{q}^{\frac{1}{2}\kappa k^2}$$

  where $D = \mathrm{q}\partial_{\mathrm{q}} - \frac{w}{12} E_2$, is the Serre derivative (Alternatively, one may use Rankin-Cohen brackets).

- For κ odd, the same works with an extra insertion of $(-1)^{\kappa k}\, k^2$.

Computing the polar terms

- $h_{1,q}$ is uniquely determined by the polar terms $n < \frac{\chi(\mathcal{D})}{24} - \frac{q^2}{2\kappa} - \frac{q}{2}$, but the dimension $d_1 = n_1 - C_1$ of the space of modular forms may be smaller than the number n_1 of polar terms !
- Physically, we expect that polar coefficients arise as bound states of D6-brane and anti D6-branes.
- For the most polar terms, only states with $[D6] = \pm 1$ ought to contribute *[Denef Moore'07]*.

Computing the polar terms

- For a single D6-brane, the DT-invariant $DT(q,n) = \Omega(1,0,q,n)$ at large volume can be computed via the GV/DT relation

$$
\begin{aligned}
\Psi_{\rm top} &= M(-\mathrm{p})^{\chi_{\mathfrak{Y}}/2} \sum_{q,n} DT(q,n)\, \mathrm{p}^n v^q \\
&= M(-\mathrm{p})^{\chi_{\mathfrak{Y}}} \prod_{q,g,\ell} \left(1-(-\mathrm{p})^{g-\ell-1} v^q\right)^{(-1)^{g+\ell}\binom{2g-2}{\ell} N_q^{(g)}}
\end{aligned}
$$

Maulik Nekrasov Okounkov Pandharipande'06

- Pandharipande-Thomas invariants $PT(q,n)$ satisfy the same relation without Mac-Mahon factor $M(-\mathrm{p}) = \prod_{n\geq 1}(1-(-\mathrm{p})^n)^{-n}$.

A naive Ansatz for the polar terms

- Earlier studies *[Gaiotto Strominger Yin'06]* suggest that only bound states of the form $(D6 - qD2 - nD0, \overline{D6(-1)})$ contribute. If so:

$$\boxed{\Omega(0,1,q,n) = (-1)^{\sharp} \left(\chi(\mathcal{O}_{\mathcal{D}}) - q - n\right) DT(q,n)\, PT(0,0)}$$

  with $PT(0,0) = 1$ *[Alexandrov Gaddam Manschot BP'22]*

- Remarkably, there exists a modular form with integer Fourier coefficients matching these polar terms for all models ☺
  – except $X_{4,2}, X_{3,2,2}, X_{2,2,2,2}$ ☹
- In particular, the Ansatz above satisfies the modular constraints on polar terms for $X_{3,3}$ and $X_{4,4}$, and reproduces earlier results by *[Gaiotto Yin]* for X_5, X_6, X_8, X_{10} and $X_{3,3}$ ☺

Modular predictions for D4-D2-D0 indices (naive)

- X_5 (Quintic in $\mathbb{P}^4$):

$$h_{1,0} = \mathrm{q}^{-\frac{55}{24}} \left(\underline{5 - 800q + 58500\mathrm{q}^2} + 5817125\mathrm{q}^3 + \dots \right)$$
$$h_{1,\pm 1} = \mathrm{q}^{-\frac{55}{24}+\frac{3}{5}} \left(\underline{0 + 8625q} - 1138500\mathrm{q}^2 + 3777474000\mathrm{q}^3 + \dots \right)$$
$$h_{1,\pm 2} = \mathrm{q}^{-\frac{55}{24}+\frac{2}{5}} \left(\underline{0 + 0q} - 1218500\mathrm{q}^2 + 441969250\mathrm{q}^3 + \dots \right)$$

- X_{10} (Decantic in $W\mathbb{P}^{5,2,1,1,1}$):

$$h_{1,0} \overset{?}{=} \frac{541 E_4^4 + 1187 E_4 E_6^2}{576\, \eta^{35}}$$
$$= \mathrm{q}^{-\frac{35}{24}} \left(\underline{3 - 576q} + 271704\mathrm{q}^2 + 206401533\mathrm{q}^3 + \cdots \right)$$

Rank 0 DT invariants from GV invariants

- Our Ansatz for polar terms was just an educated guess. Fortunately, recent progress in Donaldson-Thomas theory allows to compute D4-D2-D0 indices in a rigorous fashion, and compare with modular predictions.

  Bayer Macri Toda'11; Toda'11; Feyzbakhsh Thomas'20-22

- The key idea is to consider a (non-physical) slice in the space Stab $\mathcal{C}$ of Bridgeland stability conditions, called tilt stability, with degenerate central charge

$$Z_{b,t}(E) = \frac{\mathrm{i}}{6} t^3 \,\mathrm{ch}_0(E) - \frac{1}{2} t^2 \,\mathrm{ch}_1^b(E) - \mathrm{i}\, t \,\mathrm{ch}_2^b(E) + 0 \,\mathrm{ch}_3^b(E)$$

with $\mathrm{ch}_k^b = \int_{\mathfrak{Y}} H^{3-k} e^{-bH} \mathrm{ch}$. The heart $\mathcal{A}$ is given by length-two complexes $\mathcal{E} \to \mathcal{F}$ with $\mathrm{ch}_1^b(\mathcal{E}) \leq 0, \mathrm{ch}_1^b(\mathcal{F}) > 0$.

Rank 0 DT invariants from GV invariants

- Tilt stability agrees with physical stability at large volume, but the chamber structure is much simpler: walls are nested half-circles in the Poincaré upper half-plane spanned by $z = b + \mathrm{i}\frac{t}{\sqrt{3}}$.
- Most importantly, for any tilt-stable object E there is a conjectural inequality on Chern classes $C_i := \int H^{3-i}\,\mathrm{ch}_i(E)$ *[Bayer Macri Toda'11; Bayer Macri Stellari'16]*

$$(C_1^2 - 2C_0C_2)|z|^2 + (3C_0C_3 - C_1C_2)b + (2C_2^2 - 3C_1C_3) \geq 0$$

  The BMT bound is known to hold for $X_5, X_6, X_8, X_{4,2}$ *[Li'19,Koseki'20]*.
- By studying wall-crossing between the empty chamber provided by BMT bound and large volume, *[Feyzbakhsh Thomas]* show that D4-D2-D0 indices can be computed from rank 1 DT or PT invariants, which are in turn related to GV invariants.

Rank 0 DT invariants from GV invariants

- In particular for $\gamma = (0, 1, q, n)$ and (q, n) large enough, the PT invariant counting states with charge $(-1, 0, q, n)$ is given by

$$PT(q,n) = (-1)^{\langle \overline{D6(1)},\gamma\rangle+1} \langle \overline{D6(1)},\gamma\rangle \, \Omega(\gamma)$$

  Using spectral flow invariance, one obtains for m large enough

$$\boxed{\Omega(\gamma) = \frac{(-1)^{\langle \overline{D6(1-m)},\gamma\rangle}+1}{\langle \overline{D6(1-m)},\gamma\rangle} PT(q',n')} \qquad \begin{cases} q' = q+\kappa m \\ n' = n - mq - \frac{\kappa}{2} m(m+1) \end{cases}$$

- PT invariants can be computed from GV invariants via

$$\sum_{q,n} PT(q,n)\, \mathrm{p}^n v^q = \prod_{q,g,\ell} \left(1-(-\mathrm{p})^{g-\ell-1} v^q\right)^{(-1)^{g+\ell}\binom{2g-2}{\ell} N_q^{(g)}}$$

Modular predictions for D4-D2-D0 (rigorous)

- Using this idea, we have computed most of the polar terms (and many non-polar ones) for all models except $X_{3,2,2}, X_{2,2,2,2}$ – for those the required GV invariants are currently out of reach.

  Alexandrov, Feyzbakhsh, Klemm,, BP, Schimannek, to appear

- We find that our educated guess is correct for $X_5, X_6, X_8, X_{3,3}, X_{4,4}, X_{6,6}$ ☺ , but (as anticipated by *[van Herck Wyder'09]*) misses some $\mathcal{O}(1)$ contributions for $X_{10}, X_{6,2}, X_{6,4}, X_{4,3}$ ☹ E.g. for X_{10},

$$h_{1,0} = \frac{203 E_4^4 + 445 E_4 E_6^2}{216\,\eta^{35}} = \mathrm{q}^{-\frac{35}{24}} \left(\underline{3 - 575q} + 271955\mathrm{q}^2 + \cdots \right)$$

  In all cases, modularity holds with flying colors ! ☼ ♫ ☺

- Note that *[Toda'13, Feyzbakhsh'22]* also prove a version of our $D6 - \overline{D6}$ ansatz, but under very restrictive conditions which are only satisfied by the most polar terms.

Mock modularity for non-Abelian D4-D2-D0 indices

- Finally, let us discuss D4-D2-D0 indices with $N = 2$ units of D4-brane charge. In that case, $\{h_{2,q}, q \in \mathbb{Z}/(2\kappa\mathbb{Z})\}$ should transform as a vv mock modular form with modular completion

$$\widehat{h}_{2,q}(\tau,\bar{\tau}) = h_{2,q}(\tau) + \sum_{q_1,q_2=0}^{\kappa-1} \delta^{(\kappa)}_{q_1+q_2-q}\, \Theta^{(\kappa)}_{q_2-q_1+\kappa}\, h_{1,q_1}\, h_{1,q_2}$$

where

$$\Theta^{(\kappa)}_q = \frac{(-1)^q}{8\pi} \sum_{k\in 2\kappa\mathbb{Z}+q} |k|\, \beta\left(\frac{\tau_2 k^2}{\kappa}\right) e^{-\frac{\pi i \tau}{2\kappa} k^2},$$

and $\beta(x^2) = 2|x|^{-1} e^{-\pi x^2} - 2\pi \mathrm{Erfc}(\sqrt{\pi}|x|)$.

- For $\kappa = 1$, the series $\Theta^{(1)}_q$ is the one appearing in the modular completion of rank 2 Vafa-Witten invariants on $\mathbb{P}^2$!

Mock modularity for non-Abelian D4-D2-D0 indices

- The series $\Theta_q^{(\kappa)}$ is convergent but not modular invariant. *Suppose there exists a holomorphic function $g_q^{(\kappa)}$ such that $\Theta_q^{(\kappa)} + g_q^{(\kappa)}$ transforms as a vv modular form.* Then

$$\widetilde{h}_{2,q}(\tau,\bar{\tau}) = h_{2,q}(\tau) - \sum_{q_1,q_2=0}^{\kappa-1} \delta^{(\kappa)}_{q_1+q_2-q}\, g^{(\kappa)}_{q_2-q_1+\kappa}\, h_{1,q_1}\, h_{1,q_2}$$

  will be an ordinary weakly holomorphic vv modular form, hence uniquely determined by its polar part.

- To construct $g_q^{(\kappa)}$, notice that for κ prime, $\Theta_q^{(\kappa)}$ is obtained from $\Theta_q^{(1)}$ by acting with the Hecke-type operator *[Bouchard Creutzig Diaconescu Doran Quigley Sheshmani'16]*

$$(\mathcal{T}_\kappa[\phi])_q(\tau) = \frac{1}{\kappa} \sum_{\substack{a,d>0\\ ad=\kappa}} \left(\tfrac{\kappa}{d}\right)^{w+\frac{1}{2}} \delta_\kappa(q,d) \sum_{b=0}^{d-1} e^{-\pi \mathrm{i}\, \frac{b}{a}\, q^2}\, \phi_{dq}\left(\tfrac{a\tau+b}{d}\right),$$

  with $q \in \Lambda^*/\Lambda(\kappa)$ and $\delta_\kappa(q,d) = 1$ if $q \in \Lambda^*/\Lambda(d)$ and 0 otherwise.

Mock modularity for non-Abelian D4-D2-D0 indices

- For $\kappa = 1$, a candidate for $g_q^{(1)}$ is well-known: the generating series of Hurwitz class numbers *[Hirzebruch Zagier 1973]*

$$H_0(\tau) = -\frac{1}{12} + \frac{1}{2}\mathrm{q} + \mathrm{q}^2 + \frac{4}{3}\mathrm{q}^3 + \frac{3}{2}\mathrm{q}^4 + \dots$$
$$H_1(\tau) = \mathrm{q}^{\frac{3}{4}} \left(\frac{1}{3} + \mathrm{q} + \mathrm{q}^2 + 2\mathrm{q}^3 + \mathrm{q}^4 + \dots \right)$$

  For any κ, we can thus choose $g_q^{(\kappa)} = \mathcal{T}_\kappa(H)_q$.

- The vv modular form $\widetilde{h}_{2,q}$ is uniquely specified by its polar terms but those must satisfy constraints for such a form to exist, and integrality is not guaranteed !
- Mathematical results by Feyzbakhsh in principle allow to compute polar terms from DT/PT invariants, hence GV invariants, but the required degree and genus is prohibitive so far.

Mock modularity for non-Abelian D4-D2-D0 indices

CICY	χ	κ	c_2	$\chi(\mathcal{O}_{2\mathcal{D}})$	n_2	C_2
$X_5(1^5)$	-200	5	50	15	36	1
$X_6(1^4,2)$	-204	3	42	11	19	1
$X_8(1^4,4)$	-296	2	44	10	14	1
$X_{10}(1^3,2,5)$	-288	1	34	7	7	0
$X_{4,3}(1^5,2)$	-156	6	48	16	42	0
$X_{4,4}(1^4,2^2)$	-144	4	40	12	25	1
$X_{6,2}(1^5,3)$	-256	4	52	14	30	1
$X_{6,4}(1^3,2^2,3)$	-156	2	32	8	11	1
$X_{6,6}(1^2,2^2,3^2)$	-120	1	5	2	5	0
$X_{3,3}(1^6)$	-144	9	54	21	78	3
$X_{4,2}(1^6)$	-176	8	56	20	69	3
$X_{3,2,2}(1^7)$	-144	12	60	26	117	0
$X_{2,2,2,2}(1^8)$	-128	16	64	32	185	4

Quantum geometry from stability and modularity

- Conversely, we can use our knowledge of Abelian D4-D2-D0 invariants to compute GV invariants and push the direct integration method to higher genus !

CICY	χ	κ	type	ρ	g_{integ}	g_{avail}
$X_5(1^5)$	-200	5	F	5	53	51
$X_6(1^4,2)$	-204	3	F	6	48	31
$X_8(1^4,4)$	-296	2	F	8	60	32
$X_{10}(1^3,2,5)$	-288	1	F	10	50	32
$X_{4,3}(1^5,2)$	-156	6	F	3	20	**24**
$X_{6,4}(1^3,2^2,3)$	-156	2	F	4	14	**17**
$X_{6,6}(1^2,2^2,3^2)$	-120	1	K	6	18	**22**
$X_{4,4}(1^4,2^2)$	-144	4	K	4	26	**33**
$X_{3,3}(1^6)$	-144	9	K	3	29	**33**
$X_{4,2}(1^6)$	-176	8	C	4	50	43

Conclusion

- The existence of an isometric action of S-duality on the vector-multiplet moduli space in $D=3$, leads to strong modularity constraints on rank 0 DT invariants in the large volume limit.
- For $p=\sum_{i=1}^{n} p_i$ the sum of n irreducible divisors, the generating function h_p is a mock modular form of depth $n-1$ with an explicit shadow, thus it is uniquely determined by its polar coefficients.
- While modularity is clear physically, its mathematical origin is mysterious. Perhaps Noether-Lefschetz theory or VOAs can help *[Bouchard Creutzig Diaconescu Doran Quigley Sheshmani'16]*
- Using modularity and GV/DT/PT relations, we can not only compute D4D2-D0 indices, but also push $\Psi_{\rm top}$ to higher genus !
- Mock modularity affects the growth of Fourier coefficients, hence the microscopic entropy of supersymmetric black holes. It should have an imprint on the macroscopic side as well...

Thanks for your attention !