

Protected couplings and BPS black holes

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with Guillaume Bossard and Charles Cosnier-Horeau*

Precision counting of BPS black holes I

- Since Strominger and Vafa's seminal 1995 work, a lot of work has gone into performing **precision counting of BPS black hole micro-states** in various string vacua with extended SUSY, and detailed comparison with macroscopic supergravity predictions.
- For string vacua with 16 or 32 supercharges, exact degeneracies are given by **Fourier coefficients of (classical, or Jacobi, or Siegel) modular forms**, giving access to their large charge behavior, and enabling detailed **comparison with the Bekenstein-Hawking formula** and its refinements.

Precision counting of BPS black holes II

- An important complication in $\mathcal{N} \leq 4$ string vacua in $D = 4$ is that **multi-centered black hole solutions** exist, and correspondingly, the spectrum of BPS states is subject to **wall-crossing**. Microstates of **single-centered** black holes are counted by **mock** modular forms, which affects the growth of Fourier coefficients.

Dabholkar Murthy Zagier 2012

- In string vacua with 8 supersymmetries, such as Calabi-Yau vacua, precision counting is much more difficult, as it involves detailed properties of the internal manifold (Gromov-Witten invariants, generalized **Donaldson-Thomas invariants**, etc), and complicated structure of walls of marginal stability.

Maldacena Strominger Witten 1998; Denef 2000; Denef Moore 2007

Counting black holes via protected couplings I

- For several years, I have advocated to approach the problem of precision counting of BPS states in $D + 1$ -dimensional string vacua by considering **protected couplings in the low energy effective action** in D dimensions **after compactifying on a circle** of radius R .

Gunaydin Neitzke BP Waldron 2005

- Indeed, a **finite energy** stationary solution in dimension $D + 1$ produce a **finite action** solution in D Euclidean dimensions. States breaking k supercharges lead to instantons with k fermionic zero-modes, contributing vertices with more than k fermions (or $k/2$ derivatives) in the LEEA.
- The simplest example of this phenomenon are 't Hooft-Polyakov monopoles in $D = 4$, which induce a scalar potential in 3D QED with compact $U(1)$, explaining confinement [*Polyakov 1977*].

Counting black holes via protected couplings II

- Couplings in the LEEA in dimension D are functions $f^{(D)}(R, z^a, \phi^I)$ of the radius R , moduli z^a in dimension $D + 1$, and holonomies ϕ^I of the $D + 1$ -dimensional gauge fields along the circle:

$$\mathcal{M}_D = \mathbb{R}^+ \times \mathcal{M}_{D+1} \times \mathcal{T}$$

- Any coupling has a Fourier expansion w.r.t the torus \mathcal{T} ,

$$f^{(D)}(R, z^a, \varphi^I) = \sum_{Q \in \Lambda^+} \mathcal{F}_Q(R, z^a) e^{2\pi i \langle Q, \phi \rangle} + \text{cc}$$

where Λ_+ is a positive cone in the charge lattice Λ .

Counting black holes via protected couplings III

- For BPS saturated couplings, and for Q primitive, $\mathcal{F}_Q(R, z^a)$ is expected to receive contributions from BPS states of charge Q in dimension $D + 1$, **exponentially suppressed** as $R \rightarrow \infty$ and weighted by a suitable **BPS index** $\Omega_k(Q)$, or **helicity supertrace**,

$$\mathcal{F}_Q(R, z^a) = \Omega_k(Q) \mathcal{K}_Q(R, z^a), \quad \mathcal{K}_Q(R, z^a) \sim e^{-2\pi R \mathcal{M}(Q)}$$

- If Q is not primitive, i.e. $Q = \sum_{i=1}^n Q_i$ with $Q_i \in \Lambda^+$, $n > 1$, $\Omega_k(Q)$ may jump as a function of z^a , but contributions from **multi-particle states** of charge Q_i ensure that $\mathcal{F}_Q(R, z^a)$ is **smooth across walls of marginal stability**.

Alexandrov Moore Neitzke BP 2013

Counting black holes via protected couplings IV

- Thus, $f^{(D)}(R, z^a, \varphi^I)$ plays the rôle of a **thermodynamical black hole partition function** at temperature $T = 1/R$, chemical potentials φ^I , for fixed values $z^a \in \mathcal{M}_{D+1}$ of the moduli at spatial infinity.
- In contrast, the constant term $\mathcal{F}_0(R, z^a)$ typically grows as a power of R as $R \rightarrow \infty$, and matches terms in the LEEA in dimension $D + 1$.

Counting black holes via protected couplings V

- For $D + 1 = 4$, the moduli space \mathcal{M}_3 also includes the **NUT potential** σ , dual to the KK gauge field, and valued in a circle bundle over \mathcal{T} . The Fourier expansion includes **non-Abelian Fourier coefficients**

$$f^{(3)}(R, z^a, \varphi^I, \sigma) = \sum_{Q \in \Lambda} \mathcal{F}_Q(R, z^a) e^{2\pi i \langle Q, \phi \rangle} + \sum_{k \neq 0} \mathcal{F}_k(R, z^a, \phi^I) e^{i\pi k \sigma}$$

where $\mathcal{F}_k(R, z^a, \phi^I)$ is a section of a circle bundle \mathcal{L}^k over \mathcal{T} . It receives contributions from **Taub-NUT instantons of charge k** , suppressed as $e^{-\pi R^2 / \ell_P^2}$ as $R \rightarrow \infty$.

- In that case, the black hole partition function is the constant term of $f^{(3)}(R, z^a, \varphi^I, \sigma)$ with respect to σ .

Counting black holes via protected couplings VI

- For vacua with $\mathcal{N} \geq 4$ supersymmetries, the moduli space is a symmetric space $\mathcal{M}_D = G_D/K_D$, exact at tree-level, and $f^{(D)}$ is an automorphic function under the U-duality group, an arithmetic subgroup $G_D(\mathbb{Z}) \subset G_D$.

Hull Townsend 1994; Witten 1995

- BPS indices in dimension $D + 1$ thus arise as Fourier coefficients \mathcal{F}_Q of an automorphic form under $G_D(\mathbb{Z})$. They are automatically invariant under the U-duality group $G_{D+1}(\mathbb{Z})$ in dimension $D + 1$, while $G_D(\mathbb{Z})$ plays the role of a spectrum generating symmetry.

Breitenlohner Mason Gibbons 1988; Gunaydin Neitzke BP Waldron 2005

- In the remainder of this talk, I will discuss protected couplings in $D = 3$ string vacua with 32 and 16 supercharges, and demonstrate their relationship to BPS indices in $D = 4$.

- 1 From BPS indices to BPS-saturated couplings
- 2 Protected couplings in $\mathcal{N} = 8$ string vacua
- 3 Protected couplings in $\mathcal{N} = 4$ string vacua

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Protected couplings in $\mathcal{N} = 8$ string vacua I

- In type II string compactified on a torus T^d , the LEEA is expected to be invariant under the **U-duality group** $E_{d+1}(\mathbb{Z})$, which extends both the T-duality group $SO(d, d, \mathbb{Z})$ and global diffeomorphisms of the M-theory torus T^{d+1} .

$$\begin{array}{ccccccc} & & & \circ_2 & & & \\ & & & | & & & \\ \circ_1 & - & \circ_3 & - & \circ_4 & - & \circ_5 & - \cdots - & \circ_{d+1} \end{array}$$

- Supersymmetric Ward identities and known perturbative contributions uniquely determine the \mathcal{R}^4 and $\nabla^4 \mathcal{R}^4$ couplings to be given by **Eisenstein series**,

$$f_{\mathcal{R}^4}^{(D)} = 2\zeta(3) \mathcal{E}_{\frac{3}{2}\Lambda_1}^{E_{d+1}(\mathbb{Z})}, \quad f_{\nabla^4 \mathcal{R}^4}^{(D)} = \zeta(5) \mathcal{E}_{\frac{5}{2}\Lambda_1}^{E_{d+1}(\mathbb{Z})}$$

Protected couplings in $\mathcal{N} = 8$ string vacua II

- In the limit where a circle decompactifies, one expects these couplings to receive contributions from 1/2-BPS and 1/4-BPS states in dimension $D + 1$, respectively, weighted by the helicity supertraces $\Omega_8(Q)$ and $\Omega_{12}(Q)$, where

$$\Omega_n(Q) = \frac{(-1)^{n/2}}{n!} \text{Tr}'_Q (-1)^{2J_3} (2J_3)^n$$

- In the perturbative spectrum of type II strings compactified on T^d ,
 - 1 1/2-BPS states arise from ground states $N = \bar{N} = m_i w^i = 0$ and have $\Omega_8 = 1$;
 - 2 1/4-BPS states arise from $N = m_i w^i > 0, \bar{N} = 0$, or $\bar{N} = -m_i w^i > 0, N = 0$. There is an exponentially large number of 1/4-BPS states, but large cancellations in the helicity supertrace: $\Omega_{12} = \sigma_3(N)$.

Protected couplings in $\mathcal{N} = 8$ string vacua III

- More generally, for general primitive charges Q ,

$$\Omega_8(Q) = \begin{cases} 1 & (Q \times Q = 0) \\ 0 & (Q \times Q \neq 0) \end{cases}$$

$$\Omega_{12}(Q) = \begin{cases} \sigma_3[\text{gcd}(Q \times Q)] & (I'_4(Q) = 0, Q \times Q \neq 0) \\ 0 & (I'_4(Q) \neq 0) \end{cases}$$

where $Q \times Q$ is the Jordan quadratic product and $I'_4(Q)$ is the Freudenthal cubic product, which coincides with the gradient of the quartic invariant $I_4(Q)$.

- Does this match the large radius expansion of $f_{\mathcal{R}^4}^{(D)}$ and $f_{\nabla^4 \mathcal{R}^4}^{(D)}$?

Protected couplings in $\mathcal{N} = 8$ string vacua IV

- The Fourier expansion of general Eisenstein series with respect to non-minimal parabolic subgroups is not known in general, but here it can be obtained by **analyzing the decompactification limit of the perturbative terms**, and **covariantizing the result under U-duality**.
- For the \mathcal{R}^4 coupling, the result is known from previous work:

$$f_{\mathcal{R}^4}^{(D)} = R^{\frac{6}{8-d}} \left(f_{\mathcal{R}^4}^{(D+1)} + 4\pi \xi(d-2) R^{d-3} + 4\pi R^{\frac{d-3}{2}} \sum_{Q \times Q=0}^{\prime} \sigma_{d-3}(Q) \frac{K_{\frac{d-3}{2}}(2\pi R|Z(Q)|)}{|Z(Q)|^{\frac{d-3}{2}}} e^{2\pi i \langle Q, a \rangle} \right)$$

exhibiting the expected contributions of 1/2-BPS states with mass $\mathcal{M} = |Z(Q)|$, $\Omega_8 = 1$ (for Q primitive). NB: $\xi(s) \equiv \pi^{-\frac{s}{2}} \Gamma(s/2) \zeta(s)$.

Kazhdan BP Waldron 2001; Kazhdan Polishchuk 2002;

BP 2010; Green Russo Vanhove 2010

Protected couplings in $\mathcal{N} = 8$ string vacua V

- For the $\nabla^4 \mathcal{R}^4$ coupling,

$$\begin{aligned}
 f_{\nabla^4 \mathcal{R}^4}^{(D)} &= R^{\frac{10}{8-d}} \left(f_{\nabla^4 \mathcal{R}^4}^{(D+1)} + 2\xi(d-4)R^{d-5} f_{\mathcal{R}^4}^{(D+1)} + 8\pi\xi(4)\xi(d+2)R^{d+1} \right. \\
 &+ 16\pi\xi(4)R^{\frac{d+1}{2}} \sum_{Q \times Q=0}^{\prime} \sigma_{d+1}(Q) \frac{K_{\frac{d+1}{2}}(2\pi R|Z(Q)|)}{|Z(Q)|^{\frac{d+1}{2}}} e^{2\pi i \langle Q, a \rangle} \\
 &+ 16\pi\xi(3)R^{\frac{d-5}{2}} \sum_{Q \times Q=0}^{\prime} \frac{\sigma_{d-5}(Q) \mathcal{E}_{\frac{3}{2}\Lambda_1}^{E_{d-1}(\mathbb{Z})}(g_Q)}{(\gcd Q)^{\frac{6}{d-10}}} \frac{K_{\frac{d-5}{2}}(2\pi R|Z(Q)|)}{|Z(Q)|^{\frac{d-5}{2} + \frac{6}{10-d}}} e^{2\pi i \langle Q, a \rangle} \\
 &+ 16\pi R^{d-2} \sum_{\substack{l_4'(Q)=0, \\ Q \times Q \neq 0}} \sum_{n|Q} n^{d+1} \sigma_3\left(\frac{Q \times Q}{n^2}\right) \frac{B_{\frac{d-2}{2}, \frac{3}{2}}(R^2|Z(Q)|^2, R^2\sqrt{\Delta(Q)})}{\Delta(Q)^{\frac{3}{4}}} e^{2\pi i \langle Q, a \rangle} \left. \right) \\
 &+ \dots
 \end{aligned}$$

Protected couplings in $\mathcal{N} = 8$ string vacua VI

- The result exhibits contributions from **1/2-BPS states** with $Q \times Q = 0$, mass $\mathcal{M}_{1/2} = |Z(Q)|$, weighted by divisor sums and by **Eisenstein series for the stabilizer $E_{d-1}(\mathbb{Z}) \subset E_d(\mathbb{Z})$** of the charge vector Q ;
- In addition, there are contributions from **1/4-BPS states** with $Q \times Q \neq 0$, $l'_4(Q) = 0$, with mass $\mathcal{M}_{1/4} = \sqrt{|Z(Q)|^2 + 2\sqrt{\Delta(Q)}}$, as follows from the asymptotics of the "double Bessel function"

$$B_{s,\nu}(x, y) = \int_0^\infty \frac{dt}{t^{1+s}} e^{-\pi t - \frac{\pi x}{t}} K_\nu(2\pi y/t) \sim \frac{e^{-2\pi\sqrt{x+2y}}}{2y^{1/2}(x+2y)^{s/2}}$$

- For $D \geq 4$, the previous result is complete. For $D = 3$, it misses non-Abelian Fourier coefficients. For $s \neq \frac{5}{2}$, additional Fourier coefficients with $l'_4(Q) \neq 0$ will appear.

Protected couplings in $\mathcal{N} = 8$ string vacua VII

- Similarly, one may expect that the $\nabla^6 \mathcal{R}^4$ coupling in dimension D exhibits contributions from 1/8-states, weighted by the helicity supertrace $\Omega_{14}(Q)$.
- For Q primitive, $\Omega_{14}(Q)$ is given by a Fourier coefficient of a weak Jacobi form,

$$\Omega_{14}(Q) = c(l_4(Q)) , \quad -\frac{\theta_1^2(z, \tau)}{\eta^6} = \sum_{N, \ell} c(4N - \ell^2) q^N y^\ell$$

At large Q , $\Omega_{14}(Q) \sim e^{\pi\sqrt{l_4(Q)}}$, in agreement with the Bekenstein-Hawking entropy formula.

Maldacena Moore Strominger 1999; Shih Strominger Yin 2005; BP 2005

Protected couplings in $\mathcal{N} = 8$ string vacua VIII

- It is clear however that $f_{\nabla^6 \mathcal{R}^4}^{(D)}$ is not simply an Eisenstein series, indeed SUSY Ward identities require

$$\left(\Delta_{E_{d+1}} - \frac{6(D-6)(14-D)}{D-2} \right) f_{\nabla^6 \mathcal{R}^4}^{(D)} = -[f_{\mathcal{R}^4}^{(D)}]^2$$

up to additional linear source terms in dimension $D = 4, 5, 6$ where the local and non-local parts of the 1PI effective action mix.

Green Vanhove 2005, Green Russo Vanhove 2010; BP 2015; Bossard Verschinin 2015

Protected couplings in $\mathcal{N} = 8$ string vacua IX

- Based on these Ward identities and the known perturbative contributions up to 3 loops, one can show that the exact $\nabla^6 \mathcal{R}^4$ coupling for $D = 6$ should be given by

$$f_{\nabla^6 \mathcal{R}^4}^{(6)} = \pi \text{R.N.} \int_{\mathcal{F}_2} d\mu_2 \Gamma_{5,5,2} \varphi_{KZ} + \frac{8}{189} \mathcal{E}_{4\Lambda_5}^{SO(5,5,\mathbb{Z})}$$

where $\Gamma_{5,5,2}$ is the partition function of the ‘non-perturbative’ Narain lattice at genus-two, and φ_{KZ} is the **Kawazumi-Zhang invariant**, which appears naturally in the integrand of the two-loop $\nabla^6 \mathcal{R}^4$ coupling.

d’Hoker Green 2014; BP 2015

- Another non-perturbative completion has been proposed, which involves a two-loop amplitude in exceptional field theory...

Bossard Kleinschmidt 2015

Protected couplings in $\mathcal{N} = 8$ string vacua X

- The large radius expansion is in principle computable from the Fourier expansion of φ_{KZ} , which follows from the **theta lift representation**

$$\varphi_{KZ}(\Omega) = -\frac{1}{2} \int_{\mathcal{F}_1} d\mu_1 \left[\Gamma_{3,2,1}^{(0)} D_\tau h_0 + \Gamma_{3,2,1}^{(1)} D_\tau h_1 \right]$$

where h_i are the coefficients of the theta series decomposition

$$\frac{\theta_1^2(z, \tau)}{\eta^6} = h_0(\tau) \theta_3(2z, 2\tau) + h_1(\tau) \theta_2(2z, 2\tau) .$$

BP 2015

- Remarkably, $\varphi_{KZ}(\Omega)$ knows about degeneracies of 1/8-BPS black holes ! More work is needed to make this connection precise.

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Protected couplings in $\mathcal{N} = 4$ string vacua I

- in $D = 4$ string vacua with 16 supercharges, the moduli space is

$$\mathcal{M}_4 = \frac{SL(2)}{U(1)} \times \frac{O(r-6, 6)}{O(r-6) \times O(6)}$$

The highest rank $r = 28$ is attained in **Het/ T^6** or its dual **type II/ $K3 \times T^2$** . A large set of **CHL models** with reduced rank can be obtained as freely acting \mathbb{Z}_N orbifolds. The $SL(2)/U(1)$ factor corresponds to the **heterotic axiodilaton** $S = a + i/g_4^2$.

Chaudhury Hockney Lykken 1995

- These 4D models are believed to be invariant under $G_4(\mathbb{Z})$, an arithmetic subgroup of $SL(2) \times O(r-6, 6)$ preserving the charge lattice $\Lambda_e \oplus \Lambda_m$ (extended by "Fricke S-duality", which mixes the two factors)

Font Ibanez Lüst Quevedo 1990; Sen 1994; Persson Volpato 2015

Protected couplings in $\mathcal{N} = 4$ string vacua II

- Degeneracies of 1/4-BPS dyons are given by Fourier coefficients of a **meromorphic Siegel modular form** of weight $-k = \frac{8-r}{2}$:

$$\Omega_6(Q, P, z^a) = (-1)^{Q \cdot P} \int_{\mathcal{C}} d\rho d\sigma d\nu \frac{e^{i\pi(\rho Q^2 + \sigma P^2 + 2\nu Q \cdot P)}}{\Phi_k(\rho, \sigma, \nu)}$$

where \mathcal{C} is a suitable **contour**, depending on $z^a \in \mathcal{M}_4$.

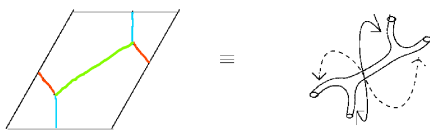
Dijkgraaf Verlinde Verlinde 1996; David Jatkar Sen 2005-06; Cheng Verlinde 2007

- Across walls of marginal stability, $\Omega_6(Q, P, z^a)$ jumps due to poles of $1/\Phi_k$ on the separating divisor $\nu = 0$ (and its images), corresponding to **bound states of two 1/2-BPS dyons**.
- In particular, the BPS indices $\Omega_4(Q, 0)$ and $\Omega_4(0, P)$ for purely electric or magnetic states are Fourier coefficients of $1/f_1(\rho)$ and $1/f_2(\sigma)$, such that $\Phi_k(\rho, \sigma, \nu) \sim \nu^2 f_1(\rho) f_2(\sigma)$ as $\nu \rightarrow 0$.

Protected couplings in $\mathcal{N} = 4$ string vacua III

- For $r = 28$, i.e. heterotic on T^6 or type II string on $K3 \times T^2$, Φ_{10} is the weight 10 Igusa cusp form under $Sp(4, \mathbb{Z})$, and $f_1 = f_2 = 1/\Delta$.
- Invariance under $G_4(\mathbb{Z}) = SL(2, \mathbb{Z}) \times O(\Lambda_e)$ is manifest, thanks to $SL(2, \mathbb{Z}) \subset Sp(4, \mathbb{Z})$, but **the physical origin of the $Sp(4, \mathbb{Z})$ symmetry is obscure**.
- Gaiotto and Dabholkar proposed that 1/4-BPS dyons can be interpreted as **heterotic strings wrapped on a genus-two Riemann surface Σ_2** , or M5-branes wrapped on $K3 \times \Sigma_2$, but left many questions unanswered (e.g. why higher genera are not allowed).

Gaiotto 2005; Dabholkar Gaiotto 2006



Protected couplings in $\mathcal{N} = 4$ string vacua IV

- After compactification on a circle, the moduli space extends to

$$\mathcal{M}_3 = \frac{O(r-4, 8)}{O(r-4) \times O(8)} \supset \begin{cases} \mathbb{R}_R^+ \times \mathcal{M}_4 \times \mathbb{R}^{2r+1} \\ \mathbb{R}_{1/g_3^2}^+ \times \frac{O(r-5, 7)}{O(r-5) \times O(7)} \times \mathbb{R}^{r+2} \end{cases}$$

and the U-duality group enhances to an arithmetic subgroup $G_3(\mathbb{Z}) \subset O(r-4, 8)$, containing both $G_4(\mathbb{Z})$ and the T-duality group in $D = 3$.

Markus Schwarz 1983, Sen 1994

- For $r = 28$, $G_3(\mathbb{Z})$ is the automorphism group of the non-perturbative Narain lattice $\tilde{\Lambda} = \Lambda_e \oplus \Lambda_{2,2}$. For CHL orbifolds, noting that $\Lambda_e = \Lambda_m^* = \Lambda_m[M]$, it is natural to propose that $\tilde{\Lambda} = \Lambda_m \oplus \Lambda_{1,1} \oplus \Lambda_{1,1}[M]$.

Cosnier-Horeau, Bossard, BP, 2017

Protected couplings in $\mathcal{N} = 4$ string vacua V

- The 4-derivative and 6-derivative couplings in the LEEA

$$F_{abcd}(\Phi) \nabla\Phi^a \nabla\Phi^b \nabla\Phi^c \nabla\Phi^d + G_{ab,cd}(\Phi) \nabla(\nabla\Phi^a \nabla\Phi^b) \nabla(\nabla\Phi^c \nabla\Phi^d)$$

are expected to satisfy non-renormalization theorems and get contributions from 1/2-BPS and 1/4-BPS instantons, respectively.

- Indeed, they satisfy **supersymmetric Ward identities**

$$\mathcal{D}_{ef}^2 F_{abcd} = c_1 \delta_{ef} F_{abcd} + c_2 \delta_{e(a} F_{bcd)(f} + c_3 \delta_{(ab} F_{cd)ef} ,$$

$$\begin{aligned} \mathcal{D}_{ef}^2 G_{ab,cd} = & c_4 \delta_{ef} G_{ab,cd} + c_5 [\delta_{e(a} G_{b)(f,cd} + \delta_{e)(c} G_{d)(f,ab}] \\ & + c_6 [\delta_{ab} G_{ef,cd} + \delta_{cd} G_{ef,ab} - 2\delta_{a)(c} G_{ef,d)(b}] \\ & + c_7 [F_{abk(e} F_{f)cd}{}^k - F_{c)ka(e} F_{f)b(d}{}^k] , \end{aligned}$$

$$\mathcal{D}_{[e} [\hat{e} \mathcal{D}_{f]} \hat{f}] F_{abcd} = 0 , \quad \mathcal{D}_{[e} [\hat{e} \mathcal{D}_f \hat{f} \mathcal{D}_g \hat{g}] G_{ab,cd} = 0 .$$

Bossard, Cosnier-Horeau, BP, 2016



Exact $(\nabla\Phi)^4$ coupling in $\mathcal{N} = 4$ string vacua I

- The coupling $(\nabla\Phi)^4$ is a 3D version of the F^4 and \mathcal{R}^2 couplings which were analyzed in the past. The F^4 coupling is **one-loop exact on the heterotic side** in $D \geq 4$, while the \mathcal{R}^2 coupling is **one-loop exact on the type II side** in $D = 4$.

Lerche Nilsson Schellekens Warner 1988; Harvey Moore 1996

- Requiring invariance under U-duality, it is natural to conjecture that **the exact coefficient of the $(\nabla\Phi)^4$ in $D = 3$ is** [*Obers BP 2000*]

$$F_{abcd}^{(r-4,8)} = \int_{\mathcal{F}_1(N)} \frac{d\rho_1 d\rho_2}{\rho_2^2} \frac{\Gamma_{r-4,8,1}[P_{abcd}]}{\Delta_{k+2}}$$

where Δ_{k+2} is a weight $k+2$ modular form, and $\Gamma_{r-4,8}$ is the Narain partition function of the lattice $\tilde{\Lambda}$ with polynomial insertion,

$$\Gamma_{r-4,8}[P_{abcd}] = \rho_2^4 \sum_{Q \in \tilde{\Lambda}} P_{abcd}(Q_L) e^{i\pi Q_L^2 \rho - i\pi Q_R^2 \bar{\rho}}$$

Obers BP, 2000

Exact $(\nabla\Phi)^4$ coupling in $\mathcal{N} = 4$ string vacua II

- This Ansatz satisfies the Ward identities and has the correct perturbative expansion on the heterotic side:

$$F_{\alpha\beta\gamma\delta}^{(r-4,8)} = \frac{c_0}{16\pi g_3^4} \delta_{(\alpha\beta}\delta_{\gamma\delta)} + \frac{F_{\alpha\beta\gamma\delta}^{(r-5,7)}}{g_3^2} + 4 \sum_{\ell=1}^3 \sum_{Q \in \Lambda_{r-5,7}} P_{\alpha\beta\gamma\delta}^{(\ell)} \\ \times \bar{c}(Q) g_3^{2\ell-9} |\sqrt{2}Q_R|^{\ell-\frac{7}{2}} K_{\ell-\frac{7}{2}} \left(\frac{2\pi}{g_3^2} |\sqrt{2}Q_R| \right) e^{-2\pi i a' Q_I}$$

exhibiting the **tree-level** and **one-loop** contribution and an infinite sum of **NS5-brane and KK5-brane instantons**. Here $P_{\alpha\beta\gamma\delta}^{(\ell)}$ are degree $6 - 2k$ polynomials in Q_L , and

$$\bar{c}(Q) = \sum_{d|Q} d c\left(-\frac{|Q|^2}{2d}\right), \quad \frac{1}{\Delta_k} = \sum_{N \geq 1} c(N) q^N.$$

Exact $(\nabla\Phi)^4$ coupling in $\mathcal{N} = 4$ string vacua III

- In the large radius limit, one finds instead

$$F_{\alpha\beta\gamma\delta}^{(r-4,8)} = R^2 \left(f_{\mathcal{R}^2}(\mathcal{S}) \delta_{(\alpha\beta} \delta_{\gamma\delta)} + F_{\alpha\beta\gamma\delta}^{(r-6,6)} \right) + 4 \sum_{\ell=1}^3 R^{5-\ell} \sum_{\tilde{Q} \in \Lambda_{r-6,6}} \sum_{j,p} c\left(-\frac{|\tilde{Q}|^2}{2}\right) P_{\alpha\beta\gamma\delta}^{(\ell)} K_{\ell-\frac{7}{2}}\left(\frac{2\pi R|p\mathcal{S}+j|}{\sqrt{\mathcal{S}_2}} |\sqrt{2}\tilde{Q}_R|\right) e^{-2\pi i(ja^1+pa^2)\cdot\tilde{Q}} + \mathcal{O}(e^{-R^2})$$

exhibiting the exact \mathcal{R}^2 and F^4 couplings in $D = 4$, along with $\mathcal{O}(e^{-R})$ terms from **1/2-BPS dyons** with charge $(Q, P) = (j, p)\tilde{Q}$, with measure

$$\mu(Q, P) = \sum_{d|(Q,P)} c\left(-\frac{\gcd(Q^2, P^2, Q\cdot P)}{2d^2}\right) \stackrel{\text{primitive}}{=} \Omega_4(Q, P).$$

The non-Abelian $\mathcal{O}(e^{-R^2})$ terms come from **Taub-NUT instantons**.

Exact $(\nabla\Phi)^4$ coupling in $\mathcal{N} = 4$ string vacua IV

- These expansions are easily obtained using the **unfolding trick**: for $\Gamma_{p,q} \rightarrow \Gamma_{p-1,q-1}$, the sum in $\Gamma_{1,1} = R \sum_{(\tilde{m},n)} e^{-\pi R^2 |\tilde{m}-n\rho|^2/\rho_2}$ can be restricted to $n=0$ provided it is integrated on the strip $\mathcal{S} = \mathcal{H}_1/\mathbb{Z}$.
- For $\Gamma_{p,q} \rightarrow \Gamma_{p-2,q-2}$, the sum over (dual momenta, windings) in $\Gamma_{2,2}$ has three orbits:

$$\begin{pmatrix} \tilde{m}_1 & n_1 \\ \tilde{m}_2 & n_2 \end{pmatrix} /_{SL(2,\mathbb{Z})} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \begin{pmatrix} j & 0 \\ p & 0 \end{pmatrix}; \begin{pmatrix} j & k \\ p & 0 \end{pmatrix} \right\}$$

$(j,p) \neq (0,0) \quad 0 \leq j < k, p \neq 0$

integrated over $\mathcal{F}_1, \mathcal{H}_1/\mathbb{Z}, 2\mathcal{H}_1$, respectively. These produce the **powerlike**, **Abelian** and **non-Abelian** Fourier coefficients, respectively.

Dixon Kaplunovsky Louis 1990; Harvey Moore 1995

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $\mathcal{N} = 4$ string vacua I

- Similarly, it is natural to conjecture that the exact coefficient of the $\nabla^2(\nabla\Phi)^4$ in $D = 3$ is given by

$$G_{ab,cd}^{(r-4,8)} = \int_{\mathcal{F}_2(N)} \frac{d^3\Omega_1 d^3\Omega_2}{|\Omega_2|^3} \frac{\Gamma_{r-4,8,2}[R_{ab,cd}]}{\Phi_k}$$

where Φ_k is a cusp form of weight k under a suitable level N subgroup of the Siegel modular group, and $\Gamma_{24,8,2}$ is the genus-two partition function of the non pert. Narain lattice $\tilde{\Lambda}$,

$$\Gamma_{24,8,2}[R_{ab,cd}] = |\Omega_2|^4 \sum_{Q^i \in \tilde{\Lambda}^{\otimes 2}} R_{ab,cd}(Q_L) e^{i\pi(Q_L^i \Omega_{ij} Q_L^j - Q_R^i \bar{\Omega}_{ij} Q_R^j)}$$

- Again, this ansatz satisfies the correct Ward identity, including the quadratic source term originating from the pole of $1/\Phi_k$ in the separating degeneration.

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $\mathcal{N} = 4$ string vacua II

- At weak heterotic coupling, it reproduces the known perturbative contributions,

$$G_{\alpha\beta,\gamma\delta}^{(r-4,8)} = \frac{G_{\alpha\beta,\gamma\delta}^{(r-5,7)}}{g_3^4} - \frac{\delta_{\alpha\beta} G_{\gamma\delta}^{(r-5,7)} + \delta_{\gamma\delta} G_{\alpha\beta}^{(r-5,7)} - 2\delta_{\gamma(\alpha} G_{\beta)\delta}^{(r-5,7)}}{12g_3^6} - \frac{1}{2\pi g_3^8} [\delta_{\alpha\beta}\delta_{\gamma\delta} - \delta_{\alpha(\gamma}\delta_{\delta)\beta}] + \mathcal{O}(e^{-1/g_3^2})$$

exhibiting the **two-loop** [d'Hoker Phong 2005], **one-loop** [Sakai Tani 1987],

$$G_{ab}^{(r-5,7)} = \int_{\mathcal{F}_1(N)} \frac{d\rho_1 d\rho_2}{\rho_2^2} \frac{\widehat{E}_2 \Gamma_{r-5,7}[P_{ab}]}{\Delta_k},$$

tree-level, and NS5/KK5-brane instantons.

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $\mathcal{N} = 4$ string vacua III

- In the large radius limit, we find instead, schematically,

$$G_{\alpha\beta,\gamma\delta}^{(r-4,8)} = R^4 \left[G_{\alpha\beta,\gamma\delta}^{(r-6,6)} - f_{\mathcal{R}^2}(\mathcal{S}) \left(\delta_{\alpha\beta} G_{\gamma\delta}^{(r-6,6)} + \delta_{\gamma\delta} G_{\alpha\beta}^{(r-6,6)} - 2\delta_{\gamma(\alpha} G_{\beta)\delta}^{(r-6,6)} \right) \right. \\ \left. + [f_{\mathcal{R}^2}(\mathcal{S})]^2 (\delta_{\alpha\beta}\delta_{\gamma\delta} - \delta_{\alpha(\gamma}\delta_{\delta)\beta}) \right] + G_{\alpha\beta,\gamma\delta}^{(1)} + \boxed{G_{\alpha\beta,\gamma\delta}^{(2)}} + G_{\alpha\beta,\gamma\delta}^{(\text{KKM})}$$

exhibiting the exact $\nabla^2 F^4$ and $\mathcal{R}^2 F^2$ couplings in $D = 4$.

- The Abelian Fourier coefficients $G^{(1)}$ and $G^{(2)}$ are both $\mathcal{O}(e^{-R})$, and come from 1/2-BPS and 1/4-BPS states in $D = 4$.
- The non-Abelian Fourier coefficient $G^{(\text{KKM})}$ is $\mathcal{O}(e^{-R^2})$ and comes from Taub-NUT instantons.

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $\mathcal{N} = 4$ string vacua IV

- These expansions follow again from the unfolding trick: for $\Gamma_{p,q} \rightarrow \Gamma_{p-1,q-1}$, the sum over non-zero (dual momenta, windings) unfolds onto $\mathbb{R}^+ \times \mathcal{F}_1(N) \times T^{2+1}$.
- For $\Gamma_{p,q} \rightarrow \Gamma_{p-2,q-2}$, the sum has (for $N = 1$) 4 orbits:

$$\left\{ 0, \begin{pmatrix} 0 & m_1 & 0 & 0 \\ 0 & m_2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} k & 0 & 0 & 0 \\ j & p & 0 & 0 \end{pmatrix}, \begin{pmatrix} j_1 & j_2 & 0 & p \\ 0 & k & 0 & 0 \end{pmatrix} \right\}$$

$(m_1, m_2) \neq (0, 0) \quad 0 \leq j < p, k \neq 0 \quad 0 \leq j_1, j_2 < p, k \neq 0$

integrated over $\mathbb{R}^+ \times \mathcal{F}_1 \times T^{2+1}$, $\mathcal{P}_2 \times T^3$, $\mathcal{P}_2 \times \mathbb{R}^3$. These produce the powerlike, 1/2-BPS Abelian, 1/4-BPS Abelian and non-Abelian Fourier coefficients, respectively.

- Due to the pole in $1/\Phi_k$, the naive unfolding procedure misses a crucial contact term corresponding to $\nabla^2 F^4$ amplitude in $\mathcal{N} = 4$ supergravity.

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $\mathcal{N} = 4$ string vacua V

- We focus on the **Abelian rank-two orbit** $G^{(2)}$, integrated over $\mathcal{P}_2 \times T^3$. The integral over Ω_1 in T^3 extracts the Fourier coefficient

$$C \left[\begin{array}{cc} -\frac{1}{2}|Q_1|^2 & -Q_1 \cdot Q_2 \\ -Q_1 \cdot Q_2 & -\frac{1}{2}|Q_2|^2 \end{array} ; \Omega_2 \right] = \int_{[0,1]^3} d\rho_1 d\sigma_1 d\nu_1 \frac{e^{i\pi(\rho Q_1^2 + \sigma Q_2^2 + 2\nu Q_1 \cdot Q_2)}}{\Phi_k(\rho, \sigma, \nu)}$$

which is a **locally constant** function of Ω_2 .

- For large R , the integral is dominated by a **saddle point** at

$$\Omega_2^* = \frac{R}{\mathcal{M}(Q, P)} A^T \left[\frac{1}{S_2} \begin{pmatrix} 1 & S_1 \\ S_1 & |S|^2 \end{pmatrix} + \frac{1}{|P_R \wedge Q_R|} \begin{pmatrix} |P_R|^2 & -P_R \cdot Q_R \\ -P_R \cdot Q_R & |Q_R|^2 \end{pmatrix} \right] A.$$

where $\begin{pmatrix} Q \\ P \end{pmatrix} = A \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$, $A = \begin{pmatrix} k & 0 \\ j & \rho \end{pmatrix}$, $|P_R \wedge Q_R| = \sqrt{(P_R^2)(Q_R^2) - (P_R \cdot Q_R)^2}$.

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $\mathcal{N} = 4$ string vacua VI

- Approximating $C[; \Omega_2]$ by its saddle point value, we find

$$G_{\alpha\beta,\gamma\delta}^{(2)} = 2R^7 \sum_{Q,P \in \Lambda'_{r-6,6}} \sum_{\ell=1}^3 P_{\alpha\beta,\gamma\delta}^{(\ell)} e^{-2\pi i(a^1 Q + a^2 P)}$$

$$\times \frac{\mu(Q, P)}{|2P_R \wedge Q_R|^{\frac{4-\ell}{2}}} B_{\frac{1}{2}, \frac{4-\ell}{2}} \left[\frac{2R^2}{S_2} \begin{pmatrix} 1 & S_1 \\ 0 & S_2 \end{pmatrix} \begin{pmatrix} |Q_R|^2 & P_R \cdot Q_R \\ P_R \cdot Q_R & |P_R|^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ S_1 & S_2 \end{pmatrix} \right]$$

where

$$\mu(Q, P) = \sum_{\substack{A \in M_2(\mathbb{Z})/GL(2,\mathbb{Z}) \\ A^{-1} \begin{pmatrix} Q \\ P \end{pmatrix} \in \Lambda_{r-6,6}^{\otimes 2}}} |A| C \left[A^{-1} \begin{pmatrix} -\frac{1}{2}|Q|^2 & -Q \cdot P \\ -Q \cdot P & -\frac{1}{2}|P|^2 \end{pmatrix} A^{-T}; \Omega_2^* \right]$$

and B is the same "double Bessel function" encountered in $\nabla^4 \mathcal{R}^4$,

$$B_{\nu,\delta}(Z) = \int_0^\infty \frac{dt}{t^{1+s}} e^{-\pi t - \frac{\pi \text{Tr} Z}{t}} K_\delta \left(\frac{2\pi}{t} \sqrt{|Z|} \right)$$

Exact $\nabla^2(\nabla\Phi)^4$ coupling in $\mathcal{N} = 4$ string vacua VII

- In the limit $R \rightarrow \infty$, using $B_{\nu,\delta}(Z) \sim e^{-2\pi\sqrt{\text{Tr}Z+2\sqrt{|Z|}}}$, one finds that the contributions are suppressed as $e^{-2\pi R\mathcal{M}(Q,P)}$.
- In ‘primitive’ cases where only $A = 1$ contributes, $\mu(Q, P)$ agrees with the helicity supertrace $\Omega_6(Q, P; z^a)$, evaluated with the correct contour prescription. It also refines earlier proposals for counting dyons with ‘non-primitive’ charges.

Cheng Verlinde 2007; Banerjee Sen Srivastava 2008; Dabholkar Gomes Murthy 2008

- There are exponentially suppressed corrections due to the discrepancy between $C[; \Omega_2]$ and its saddle point value. Presumably these terms give the instanton/anti-instanton effects sourced by the square of the $(\nabla\Phi)^4$ coupling in the Ward identity.

Conclusion - Outlook I

- $\nabla^2(\nabla\Phi)^4$ couplings in $D = 3, \mathcal{N} = 4$ string vacua nicely incorporate degeneracies of 1/4-BPS dyons in $D = 4$, and explain their hidden modular invariance. They give a precise implementation of Gaiotto's idea that **1/4-BPS dyons are (U-duals of) heterotic strings wrapped on genus-two Riemann surfaces**.
- A similar story presumably relates $\nabla^6\mathcal{R}^4$ couplings in $\mathcal{N} = 8$ string vacua and degeneracies of 1/8-BPS dyons, but details remain to be worked out.

BP 2015; Bossard Kleinschmidt 2015

Conclusion - Outlook II

- In $D = 4, \mathcal{N} = 2$ string vacua, the appropriate coupling capturing degeneracies of 1/2-BPS black holes is the metric on the **vector-multiplet moduli space** \mathcal{M}_V after compactification on a circle. It is related by T-duality to the **hypermultiplet moduli space** \mathcal{M}_H . Hopefully, progress on understanding \mathcal{M}_V and \mathcal{M}_H will allow new precision tests of BPS black holes, and provide new ways of computing Donaldson-Thomas invariants...

Alexandrov BP Vandoren 2008, Alexandrov Banerjee Manschot BP 2016

- From a mathematical viewpoint, higher-genus modular integrals are an interesting source of new automorphic objects beyond Eisenstein series, which satisfy Poisson-type equations with sources.