Attractor invariants for local Calabi-Yau threefolds

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Séminaire Groupes, Représentations et Géométrie
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based on arXiv:2004.14466 with Guillaume Beaujard, Jan Manschot

and arXiv:2012.14358 with Sergey Mozgovoy

From math to physics and back

- Given a compact Calabi-Yau threefold X, one associates an infinite set of rational numbers n_β, called (genus zero) Gromov-Witten invariants, which count rational curves in homology class $\beta \in H_2(X)$. They are invariant under complex deformations of X, computable using mirror symmetry, and provide a deformation of the usual intersection product on $H_2(X)$.

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- The metric $G_{ab}(z)$ determines the low-energy effective action

$$S[g, z, \dots] = \int_{\mathbb{R}^{3,1}} \sqrt{-\det g}\, \mathrm{d}^4 x \left[R(g) + G_{ab}(z) g^{\mu\nu} \partial_\mu z^a \partial_\nu z^b + \dots \right]$$

where $g_{\mu\nu}$ is a Lorentzian metric on $\mathbb{R}^{3,1}$, and $z : \mathbb{R}^{3,1} \to \mathcal{M}_K$.

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- The (genus zero) Gopakumar-Vafa invariants defined by the multi-cover formula $N_\beta = \sum_{m|\beta} \frac{1}{m^3} n_{\beta/m}$ are conjecturally integer. In string theory, they count BPS states originating from D2-branes wrapped on curves in homology class β.

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- More generally, BPS states in string theory arise by wrapping D0/D2/D4/ D6-branes on a point/curve/divisor/X. Mathematically, they correspond to stable objects E in the derived category of coherent sheaves $D(X)$, and are counted by the generalized Donaldson-Thomas invariants $\Omega_z(\gamma)$.

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- $\Omega_z(\gamma)$ depends on the Chern character $\gamma = \mathrm{ch}(E) \in K(X)$ and on the central charge function $Z \in \mathrm{Hom}(K(X), \mathbb{C})$, which is itself determined by the Kähler moduli $z \in \mathcal{M}_K$. For sheaves supported on curves, $\Omega_z(\beta) = N_\beta$. For skyscraper sheaves, $\Omega_z(\delta) = -\chi_X$.

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- For example, they expect that for large $|\gamma|$, stable objects in $D(X)$ correspond to BPS black hole solutions to $\mathcal{N}=2$ supergravity. Moreover, $\log|\Omega_{z_\gamma}(\gamma)| \sim \frac{1}{4}\mathcal{A}$ where $\mathcal{A}$ is the BH horizon area (measured in Planck units), and z_γ is the attractor point, which extremizes $|Z_\gamma(z)|$ locally in $\mathcal{M}_K$.

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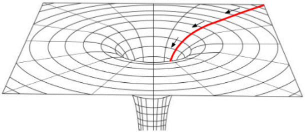
- Indeed, in a spherically symmetric black hole, the Kähler moduli have a non-trivial radial profile which interpolates from z at $r=\infty$ to z_γ at the horizon:



$$r^2\frac{dz^a}{dr} = 2e^U G^{ab}\partial_{z^b}|Z_\gamma(z)|$$
$$r^2\frac{dU}{dr} = e^U|Z_\gamma(z)|$$

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- In general, for $z \neq z_\gamma$, there are also multi-centered black hole solutions, which appear/disappear across codimension-one walls in Kähler moduli space. The location of these walls is exactly where stable objects in $D(X)$ become unstable.

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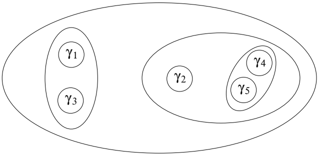
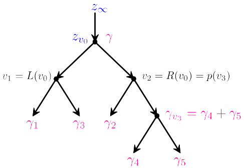
- The wall-crossing formulae of Kontsevich-Soibelman and Joyce-Song (or formulae equivalent to those) can be derived by solving the quantum mechanics of n BPS black holes *[Manschot BP Sen 2010]*.
- Since multi-centered solutions are ruled out for $z = z_\gamma$ (with the exception of scaling solutions), the attractor DT invariants defined by $\Omega_\star(\gamma) := \Omega_{z_\gamma}(\gamma)$ are expected to be simpler than the DT invariants for generic z.

- Multi-centered solutions turn out to have a hierarchical structure labelled by attractor trees:





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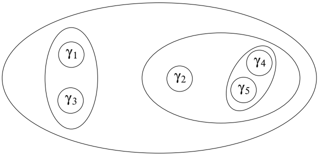

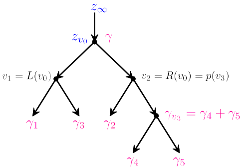

- This physical picture leads to 'attractor flow formulae' expressing $\Omega_z(\gamma)$ as a sum of products of attractor indices $\Omega_\star(\gamma_i)$.

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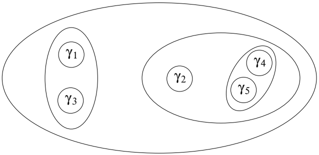

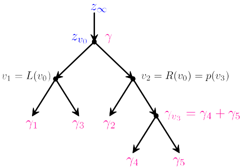

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- Thus, an interesting goal is to compute attractor DT invariants for the category $D(X)$ of coherent sheaves on a CY threefold.

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- If true, this conjecture determines the entire set of DT invariants $\Omega_\theta(d)$ via the attractor flow tree formulae, which are now theorems due to Argüz, Bousseau and Mozgovoy.

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$$h_{d,\delta}(\tau) = \sum_{n=0}^{\infty} \Omega_{\theta=\langle d,-\rangle}(d + n\delta)\, q^{n+\Delta} , \qquad q = e^{2\pi i\tau}$$

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- A natural question is whether this can be understood from the action of some VOA on the cohomology of quiver moduli...

Definitions and notations [slide courtesy of S. Mozgovoy]

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- Define the Jacobian algebra $J(Q, W) = \mathbb{C}Q/(\partial W/\partial a : a \in Q_1)$.
- Define a cut to be a subset $I \subset Q_1$ such that every term of W contains exactly one arrow in I. Setting $Q' = Q \backslash I$, let $J_I(Q, W) = \mathbb{C}Q'/(\partial W/\partial a : a \in I)$.

Brane tilings

- The quiver (Q, W) associated to a singular toric CY3 X is conveniently encoded in a brane tiling, i.e. a bipartite graph G embedded in a 2-dimensional (real) torus $\mathcal{T}$. Each vertex is black or white, edges connect only vertices of different colors.

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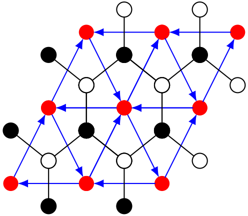

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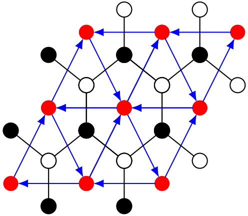

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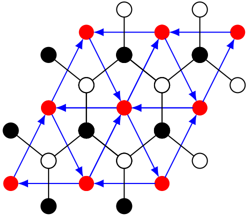

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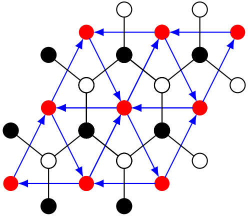

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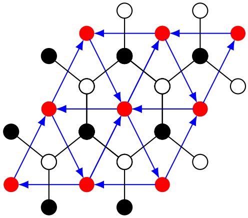
- For every $i \in Q_0$, as many arrows come in as come out.



1. Let $Q_2 = Q_2^+ \cup Q_2^-$ be the set of white and black vertices of G, or equivalently the set of faces of Q
2. For any face $F \in Q_2$, let w_F be the cycle obtained by going along the arrows of F (defined up to a cyclic shift).
3. The potential is $W = \sum_{F\in Q_2^+} w_F - \sum_{F\in Q_2^-} w_F$

- The quiver (Q, W) can be derived from a tilting sequence on X. Conversely, the toric diagram of X can be read off from zig-zag paths on the brane tiling. X arises as the moduli space of representations of (Q, W) with dimension vector $\delta = (1, 1, \dots)$.

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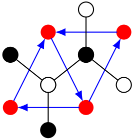

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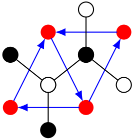

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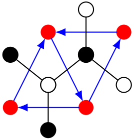

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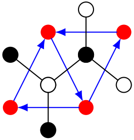

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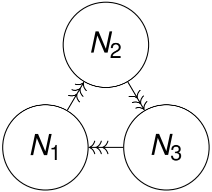
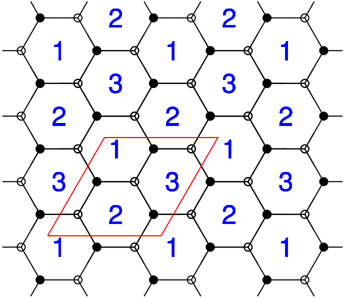
- For $I = \{z\}$, $J_I(Q, W) = \mathbb{C}[x, y]$ is the coordinate ring of $\mathbb{C}^2$.

Example 2: $\mathbb{C}^3/\mathbb{Z}_3 \sim K_{\mathbb{P}^2}$





$$W = \sum_{i,j,k} \epsilon_{ijk} \Phi^i_{12} \Phi^j_{23} \Phi^k_{31}$$

- For $I = \{\Phi^1_{31}, \Phi^2_{31}, \Phi^3_{31}\}$, the quiver $Q' = Q \backslash I$ with relations $\sum_{j,k} \epsilon_{ijk} \Phi^j_{12} \Phi^k_{23} = 0$ is the familiar Beilinson quiver describing the category of coherent sheaves on $\mathbb{P}^2$ [Drézet Le Potier '85]

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- Let $R_Z(J, d) \subset R(J, d)$ be the subspace of semistable representations and $M_Z(J, d) = R_Z(J, d)/\!\!/G_d$ the GIT quotient.

Stacky invariants

- Given a cut $I \subset Q_1$, we define the generating series of stacky DT invariants by

$$\mathcal{A}(x) = \sum_{d \in \mathbb{N}^{Q_0}} (-y)^{\chi_Q(d,d)+2\gamma_I(d)} \frac{[R(J_I, d)]}{[G_d]} x^d$$

where $[X] = \sum_n \dim H^n(X)(-y)^n$ for smooth projective X,

$$\chi_Q(d,d') = \sum_{i \in Q_0} d_i d'_i - \sum_{a \in Q_1} d_{s(a)} d'_{t(a)}, \quad \gamma_I(d) = \sum_{a \in I} d_{s(a)} d_{t(a)},$$

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- $\Omega_Z(d, y)$ is expected to be a Laurent polynomial with integer coefficients.

Wall-crossing

- Define the quantum torus

$$\mathbb{A} = \bigoplus_{d\in\mathbb{N}^{Q_0}} \mathbb{Q}(y) x^d \,, \quad x^d \circ x^{d'} = (-y)^{\langle d,d'\rangle} x^{d+d'}$$

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- Wall-crossing formula (Kontsevich-Soibelman 2008): for any stability function Z,

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- Joyce has given a formula expressing $\bar{\Omega}_Z(d, y)$ in terms of $\bar{\Omega}_{Z'}(d', y)$ for all $0 < d' \leq d$.

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- For example, for $X=\mathbb{C}^3$

$$\mathcal{A}(x)=\mathrm{Exp}\left(\frac{-y^3\sum_{n\geq 1}x^n}{y^{-1}-y}\right)\qquad\Rightarrow\Omega(n,y)=-y^3$$

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- Similarly, for dimension vectors d associated to coherent sheaves supported on curves C which do not intersect the compact divisors, $\Omega_Z(d, y)$ is independent of Z, and coincides (in unrefined limit $y \to 1$) with the genus-zero Gopakumar-Vafa invariant N_β.

Attractor invariants

- Given a dimension vector $d \in \mathbb{N}^{Q_0}$, consider $\theta = \langle -, d \rangle : \mathbb{Z}^{Q_0} \to \mathbb{R}$ and let θ' be a generic perturbation. Theorem *[MP 2020, Gross Hackinng Keel Konstevich 2014]*: $\bar{\Omega}_{\theta'}(d, y)$ is independent of the perturbation.

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- Define the attractor DT invariant as $\bar{\Omega}_\star(d, y) = \bar{\Omega}_{\theta'}(d, y)$ and similarly for $\Omega_\star(d, y)$ and $\mathcal{A}_\star(d, y)$. The latter coincide with the notion of initial data for scattering diagrams.

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- Theorem (easy): If Q is acyclic, then $\Omega_\star(d) = 1$ for $d = e_i$ and zero otherwise. More generally, if the support of d is not strongly connected, then $\Omega_\star(d) = 0$.

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- Theorem (easy): If Q is acyclic, then $\Omega_\star(d) = 1$ for $d = e_i$ and zero otherwise. More generally, if the support of d is not strongly connected, then $\Omega_\star(d) = 0$.
- Using the wall-crossing formulas, the DT invariants $\bar{\Omega}_Z(d, y)$ for any stability parameter Z can be recursively expressed in terms of attractor invariants.

Attractor flow tree formulae

- More directly, the attractor tree formula allows to express $\bar{\Omega}_\theta(\gamma, y)$ in terms of the attractor indices $\bar{\Omega}_\star(\alpha_i, y)$:

$$\bar{\Omega}_\theta(\gamma, y) = \sum_{\gamma=\sum \alpha_i} \frac{g_\theta(\{\alpha_i\}, y)}{|\mathrm{Aut}(\{\alpha_i\})|} \prod_i \bar{\Omega}_\star(\alpha_i, y)$$

Manschot'10, Alexandrov BP '18; Argüz Bousseau '21

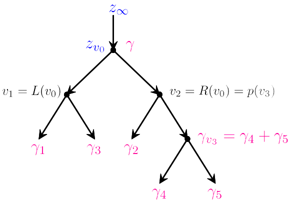
where

$$g_\theta(\{\alpha_i\}, y) = \sum_{T \in \mathcal{T}_\theta} \prod_{v \in V_T} (-1)^{\gamma_{LR}} \frac{y^{\gamma_{LR}} - y^{-\gamma_{LR}}}{y - 1/y}$$

Here T runs over all θ-stable flow trees ending on the leaves $\alpha_1, \dots, \alpha_n$, v runs over all vertices and $\gamma_{LR} = \langle \gamma_{L(v)}, \gamma_{R(v)} \rangle$.

Attractor flow and attractor indices

- To define stability, decorate each vertex v with a dimension vector γ_v and stability parameter θ_v, such that $\gamma_v = \alpha_i$ for the i-th leaf, $\theta_{v_0} = \theta$ for the root vertex, and for any v distinct from the root and the leaves, with parent $p(v)$ and descendants $L(v), R(v)$,



$$\begin{aligned} \gamma_v &= \gamma_{L(v)} + \gamma_{R(v)} \\ \theta_v &= \theta_{p(v)} + \frac{\langle \gamma_v, - \rangle}{\langle \gamma_{L(v)}, \gamma_v \rangle} \theta_{p(v)}(\gamma_{L(v)}) \end{aligned}$$

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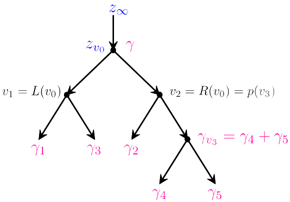

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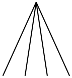
- The flow tree is θ-stable if $\langle \gamma_{L(v)}, \gamma_{R(v)} \rangle \times \theta_v(\gamma_{L(v)}) > 0$ for all v (after perturbing $\langle -, - \rangle$ or θ).

Attractor flow and attractor indices

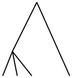
- There is a different formula called flow tree formula which does not require any perturbations. It involves a sum over rooted plane trees with vertices of arbitrary valency, produces numerous cancellations and its physical interpretation is obscure. *[Alexandrov BP Manschot '19, Mozgovoy BP '20; Mozgovoy '20]*

- For example for Kronecker quiver K_m, $d = (1,3)$,



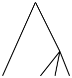
$\gamma_1\gamma_2\gamma_2\gamma_2$

$a : -\frac{1}{4}(-y)^{3m}$



$\gamma_1\gamma_2\gamma_2\gamma_2$

$b : \frac{1}{4}(-y)^{3m}$

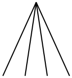


$\gamma_1\gamma_2\gamma_2\gamma_2$

$c : \frac{1}{12}(-y)^{3m}$

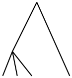
$\gamma_1\gamma_2\gamma_2\gamma_2$

$d : -\frac{1}{4}(-y)^{3m}$



$\gamma_2\gamma_1\gamma_2\gamma_2$

$e : \frac{1}{4}(-y)^{m}$



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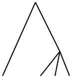
$f : -\frac{1}{4}(-y)^{m}$



$\gamma_2\gamma_1\gamma_2\gamma_2$

$g : -\frac{1}{4}(-y)^{m}$

$\gamma_2\gamma_1\gamma_2\gamma_2$

$h : \frac{1}{4}(-y)^{m}$

$\gamma_2\gamma_1\gamma_2\gamma_2$

$i : \frac{1}{4}(-y)^{m}$

$\gamma_2\gamma_1\gamma_2\gamma_2$

$j : \frac{1}{4}(-y)^{m}$

$$+\left(y \to \frac{1}{y}\right) = -\frac{1}{6}\left((-y)^m - (-y)^{-m}\right)^3$$

Attractor conjecture

- Let $\widetilde{X}$ be the crepant resolution of an isolated toric CY3 singularity X with $i > 0$ compact divisors, and (Q, W) the associated the quiver with potential. Then $\Omega_\star(d, y) = 0$ unless $d = e_i$ or $d = n\delta$ where $\delta = (1, 1, \dots, 1)$, in which case

$$\Omega_\star(e_i, y) = 1 \ , \quad \Omega_\star(n\delta, y) = (-y)^3[\widetilde{X}] = -y^3 - (i + b - 3)y - iy^{-1}$$

where i (resp. b) are the number of internal (resp. boundary) lattice points on the toric diagram. *[Mozgovoy BP '20]*

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- If X is a non-isolated singularity toric CY3 with $i > 0$, then $\Omega_\star(d, y) = 0$ unless $d = e_i$ or $\langle d, -\rangle = -0$. The value of $\Omega_\star(n\delta, y)$ is as above, but there are other trajectories of the form $d = d_0 + n\delta$ with $\langle d_0, -\rangle = -0$ such that $\Omega_\star(d, y) = -y$ *[Descombes '21]*

Time for a pause !

Note the workshop on Sep 6-10 at Institut Henri Poincaré:
`https://indico.in2p3.fr/event/24629/`

Tentative list of speakers:
Pierrick Bousseau (Orsay and ETH Zurich), Ben Davison (Edinburgh), Michele del Zotto (Uppsala), Soheyla Feyzbakhsh (Imperial), Albra Grassi (Geneva)*, Amihay Hanany (Imperial College), Dominic Joyce (Oxford), Albrecht Klemm (Bonn)*, Maxim Kontsevich (IHES), Wei Li (CAS Beijing), Pietro Longhi (ETH Zurich), Sergey Mozgovoy (Trinity College Dublin), Markus Reineke (Bochum), Sakura Schaefer-Nameki (Oxford), Hendrik Suess (Manchester), Alessandro Tanzini (SISSA), Alessandro Tomasiello (Milano Biccoca)

Attractor indices from stacky invariants

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- One way to determine the attractor invariants $\Omega_\star(d, y)$ is to compute the stacky invariants for trivial stability $\mathcal{A}(d, y)$ and apply the wall-crossing formula.
- For quivers associated to brane tilings, $\mathcal{A}(d, y)$ can be computed using double dimensional reduction. Let I and I' be two disjoint cuts, and let $Q' = Q\backslash I$, $Q'' = Q\backslash(I \cup I')$. There is a forgetful map $\pi : R(J_I, d) \to R(Q'', d)$ with linear fibers. Thus $\mathcal{A}(d, y)$ can be deduced from the set of indecomposable representations $\mathcal{R}$ of Q'':

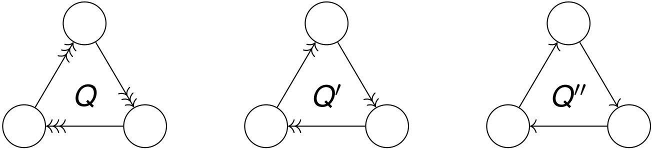
$$\mathcal{A}(x) = \sum_{m:\mathcal{R}\to\mathbb{N}} \frac{(-y)^{-\sum_{M,N\in\mathcal{R}} m_M m_N \sigma(M,N)}}{\prod_{M\in\mathcal{R}}[GL(m_M)]} x^{\sum_{m\in\mathcal{R}} m_M \dim M}$$

$$\sigma(M, N) = 2\dim \mathrm{Hom}(M, N) - \phi(M, N) - \chi_Q(M, N) - 2\gamma_I(M, N)$$

where $\phi(M, N)$ is the quad. form such that $\phi(M, M) = \dim \pi^{-1}(M)$.

Attractor indices from stacky invariants

- Consider $K_{\mathbb{P}^2}$:



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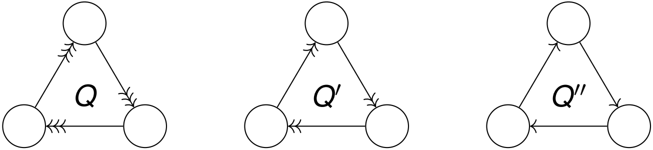

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$$\Omega_*(e_i, y) = 1, \qquad \Omega_*(n\delta, y) = -y^3 - y - y^{-1}$$

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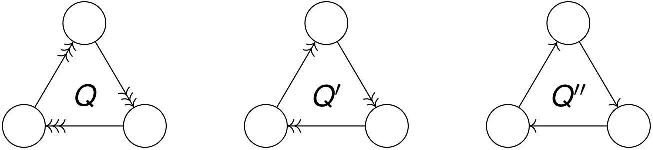

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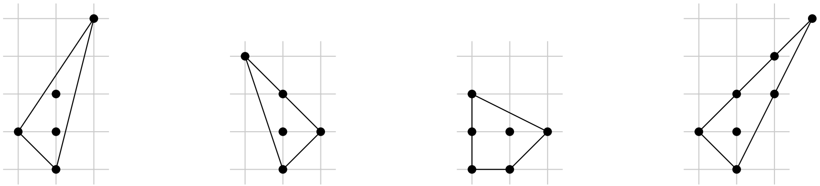
- The same type of computation for $K_{\mathbb{F}_0}, K_{\mathbb{F}_1}, K_{dP_2}, \mathbb{C}^3/\mathbb{Z}_5$, …support the conjecture for isolated toric CY3 singularities:

$$\Omega_\star(n\delta, y) = (-y)^3[X] = -y^3 - (i + b - 3)y - iy^{-1}$$

where i (resp. b) are the number of internal (resp. boundary) lattice points on the toric diagram.

Attractor indices from stacky invariants

- For non-isolated toric singularities, such that the boundary of the toric diagram contains lattice points beyond the corners, we find $\Omega_\star(d + n\delta, y) = -y$ for some d in the kernel of $\langle -, - \rangle$. See *[Descombes (2021)]* for a precise conjecture covering all brane tilings.



$\mathbb{C}^3/\mathbb{Z}_5$ $\mathbb{F}_2$ PdP_2 $\mathbb{C}^3/\mathbb{Z}_6(1,2,3)$

Attractor invariants from NCDT invariants

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- Let $J^{\mathrm{f}} = J(Q^{\mathrm{f}}, W)$, $d^{\mathrm{f}} = (d, 1)$ and $R^{\mathrm{f}}(J, d) = R(J^{\mathrm{f}}, d^{\mathrm{f}})$. Let $R^{\mathrm{f},\mathrm{NC}}(J, d) \subset R^{\mathrm{f}}(J, d)$ to be the subspace of cyclic representations M (i.e. satisfying $N \subset M, N_\infty \neq 0 \Rightarrow N = M$).

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- We define the generating function of unrefined NCDT invariants

$$Z_{\mathrm{f,NC}}(x) = \sum_{d \in \mathbb{N}^{Q_0}} (-1)^{\chi_Q(d,d) - \mathrm{f}\cdot d} e\left(R^{\mathrm{f,NC}}(J, d)/G_d\right) x^d$$

Attractor invariants from NCDT invariants

- NCDT invariants are related to (unframed, unrefined) DT invariants by wall-crossing. The formula is simplest for symmetric quivers,

$$Z_{\mathrm{f},\mathrm{NC}}(x) = \bar{S}_{\mathrm{f}}\,\mathrm{Exp}\left(- \sum_{d\in\mathbb{N}^{Q_0}} (\mathrm{f}\cdot d)\Omega(d,1)x^d \right) \,, \quad S_{\mathrm{f}}(x^d) = (-1)^{\mathrm{f}\cdot d} x^d$$

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- NCDT invariants can be computed using toric localization, which amounts to counting molten crystals.

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- Theorem *[Mozgovoy Reineke (2008)]*:

$$Z_{e_i,\mathrm{NC}}(x) = \sum_{\mathcal{I} \subset \Delta_i, d = \dim \mathcal{I}} (-1)^{\chi_Q(d,d)+d_i} x^d$$

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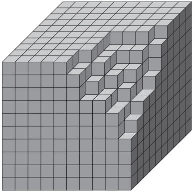
  where $\dim \mathcal{I} = \sum_{u \in \mathcal{I}} e_{t(u)} \in \mathbb{Z}^{Q_0}$.
- Z_f for other framing vectors $f \in \mathbb{N}^{Q_0}$ follows from $Z_{f+f'} = Z_f Z_{f'}$.

Attractor invariants from molten ctrystals



Example: For $X = \mathbb{C}^3$, with Jacobian algebra $J(Q, W) = \mathbb{C}[x, y, z]$, one can identify the poset Δ_1 with $\mathbb{N}^3$, and ideals with plane partitions.

- The generating function of NCDT invariants is *[MacMahon 1916]*

$$Z_1(-x) = \prod_{k=1}^{\infty}(1-x^k)^{-k} = 1+x+3x^2+6x^3+13x^4+24x^5+48x^6+\dots$$

consistent with the unrefined indices $\Omega(n, y=1) = -1$ for all $n > 0$.

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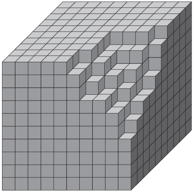

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- Using this approach we have confirmed the Attractor Conjecture for all brane tilings in the unrefined limit. The computation of refined NCDT invariants by toric localization is much harder, but confirms the conjecture. *[Descombes (2021)]*

Attractor invariants from Vafa-Witten invariants

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- Denoting by $e_i = \mathrm{ch}\, E_i$ the Chern character of the exceptional sheaf corresponding to the node $i \in Q_0$, the dimension vector for coherent sheaves with Chern character γ is $d = \sum_{i \in Q_0} d_i e_i$. The stability parameters follow as usual from

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- Remarkably, for the canonical polarization $J \propto c_1(S)$, $\theta \propto \langle d, - \rangle$ corresponds to the anti-attractor stability condition !

Attractor invariants from Vafa-Witten invariants

- The VW invariants $c_J^S([N, \mu, n])$ for any rational surface S, polarization J, rank N, first Chern class μ and second Chern class n can be computed by combining blow up and wall-crossing formulae *[Yoshioka (1996), Goettsche (1999), Manschot (2011-14)]*.

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- Agreement persists for non-toric Fano surfaces $dP_{4\leq n\leq 8}$.

Anti-attractor invariants and modularity

- For fixed rank N and first Chern class μ, the generating series $h^S_{N,\mu,J}(\tau,z)$ is expected to be quasi-invariant under $SL(2,\mathbb{Z})$ transformations: $\tau \to \frac{a\tau+b}{c\tau+d}, z \to \frac{z}{c\tau+d}$. This follows from Montonen-Olive S-duality of $\mathcal{N}=4$ super Yang-Mills theory.

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- This anomalous transformation properties can be repaired at the cost of adding non-holomorphic corrections, determined in terms of mock Jacobi forms of lower depth.

Anti-attractor invariants and modularity

- Translating into the language of quivers, this suggests that the generating function of anti-attractor DT invariants

$$h_{d,\delta}(\tau) = \sum_{n=0}^{\infty} \Omega_{\langle d,-\rangle}(d+n\delta)\, q^{n+\Delta}\ , \qquad q = e^{2\pi i\tau}$$

for δ a primitive vector such that $\langle \delta, - \rangle = 0$ and a suitable $\Delta \in \mathbb{Q}$, should be a vector-valued mock modular form.

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- Can one construct some VOA acting on cohomology of quiver moduli which would explain such modular invariance ?

Another occurence of modularity

- Recall the generating series $\mathcal{A}(x,y) = \sum_{d\in\mathbb{N}^{Q_0}} \mathcal{A}_d(y)\, x^d$ of stacky invariants. Define $\overline{\mathcal{A}(x,y)} = \sum_{d\in\mathbb{N}^{Q_0}} \mathcal{A}_d(y)\, x^{-d}$. Let

$$T(\tau,z) = (q)_\infty^r \,\mathrm{Tr}\left[\overline{\mathcal{A}(x,y)}\mathcal{A}(x,y)\right]\;,\quad y^2 = e^{2\pi\mathrm{i}\tau}, x = e^{2\pi\mathrm{i}z}$$

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- Conjecture (Cordova Shao 2015): T is a character of a VOA
- Examples: for K_1, $T(\tau) = \sum_{n\geq 0} \frac{q^{n^2+n}}{(q)_n}$ is the Rogers-Ramanujan function. For K_2, $T(\tau) = \sum_{n\geq 0} q^{4k^2+2k}$ is an ordinary theta series. False theta functions also occur...

Thank you for your attention, and mind the wall !